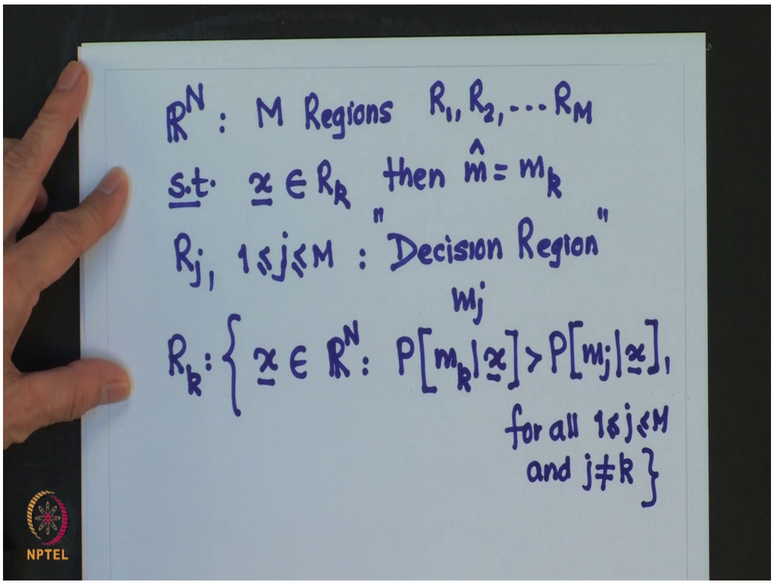
Principles of Digital Communications
Prof. Shabbir N. Merchant
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture - 22
Probability of Error for Optimum Receiver

We have derived the optimum map detector for AWGN channel and also studied its implementation based on correlation and match filter. The next task is basically is to determine the probability of error of this optimum receiver. And in order to do this we need to determine what is known as decision regions in the signal space.

So, any detector including the map and ML detector; what it does basically it partitions the N dimensional signal space into M regions which we indicate by R 1, R 2, R m; capital M denotes the number of message signals.

(Refer Slide Time: 00:58)

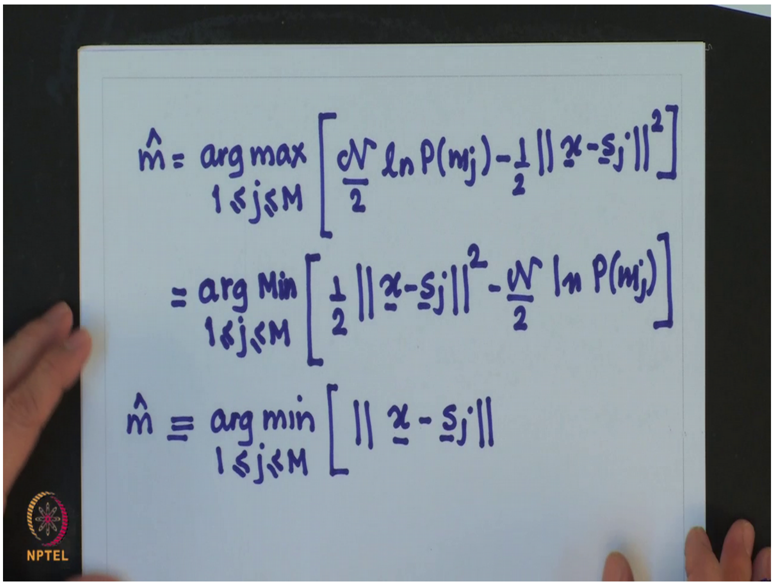


This partition is done in such a way that the vector x which is the projection of the signal x t receive signal onto the N dimensional space belongs to R k. If this condition is satisfied then we take the decision that m k was transmitted. Now, this R j for j equal to 1 to M is called the decision region for message m j and R j is the set of all outputs of the channel that are mapped into message m j by the detector. This R js are chosen to minimize the probability of error.

So, how is this how does the optimum receiver set this R js. So, if you are using a map detector then R js constitute the optimal decision regions resulting in the minimum probability of error and for a map detector this region R k would consist of all the points in N dimensional space such that probability of the message m k given.

We have observed the vector x is greater than probability of the message m j, given we have observed the vector x for all j from 1 to m, but j not equal to k correct.

(Refer Slide Time: 03:53)

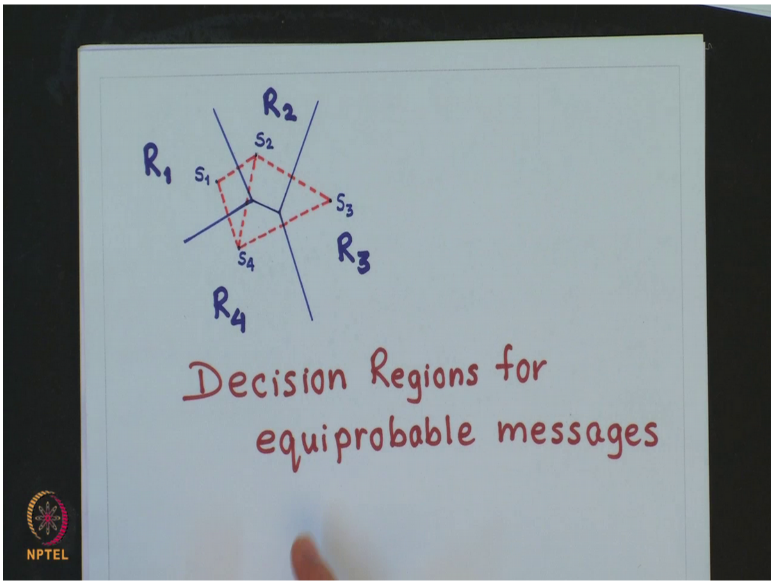


So, for the case of additive white Gaussian noise channel map decision function is given by the following expression and this can be re-written as argument minimum of this quantity. So, for the case where the message signals are equiprobable this will reduce to argument minimum of the norm of the difference of the vector between x and S j.

Now, this is the distance of the vector x from the signal vector S j. So, the decision is made in favor of that signal which is closest to the vector x correct. So, in the case of Gaussian noise this is qualitatively expected because it has a spherical symmetry.

So, for equiprobable messages the boundary of say the region R j and R k will be the set of points that are equidistance from two vectors S j and S k which implies that it will be the perpendicular bisector of the line joining the two signal points S j and S k right ok.

(Refer Slide Time: 06:31)



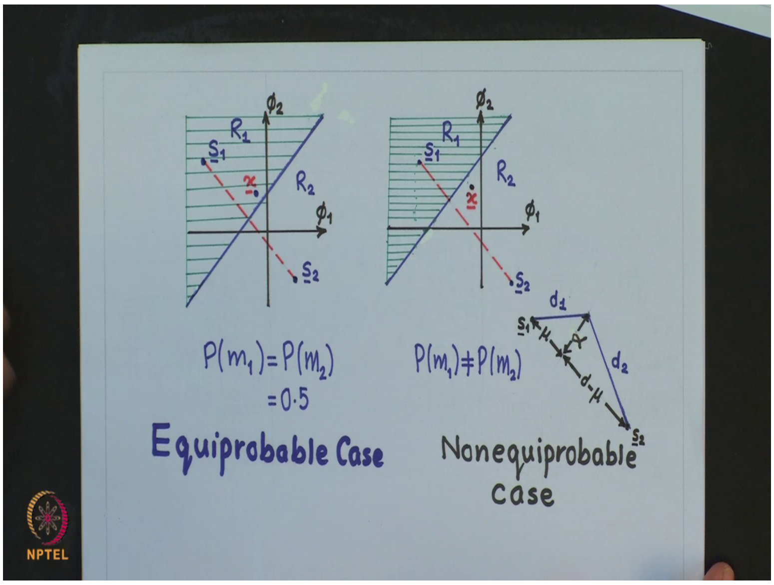
So, taking a simple case for n equal to 2 and m equal to 4; If you have equiprobable messages S 1, S 2, S 3, S 4 then the decision regions would be obtained by taking the bisectors of the lines joining the signal points.

So S 1, S 4 you take the bisector S 1, S 2 you take the bisector correct and this is how you obtain the regions for decision region for equiprobable message for n equal to 2 and m equal to 4. Now, this kind of a visualization would become difficult if we have the dimensionality to be larger than 2 correct and this kind of reasoning also will not hold good when you have non-equiprobable messages.

So, in that case we can draw some broad conclusions in the sense that, if a particular message m j is more likely than the others it will be safer in deciding more often in favor of m j.

So, what will happen that there will be some kind of a bias of weighted decision regions in favor of a particular message signal m j which has a higher probability correct and this is also reflected by the appearance of the term log of probability of m j in the decision function for the AWGN channel. Now, let us take a simple case to understand what we are doing.

(Refer Slide Time: 08:25)

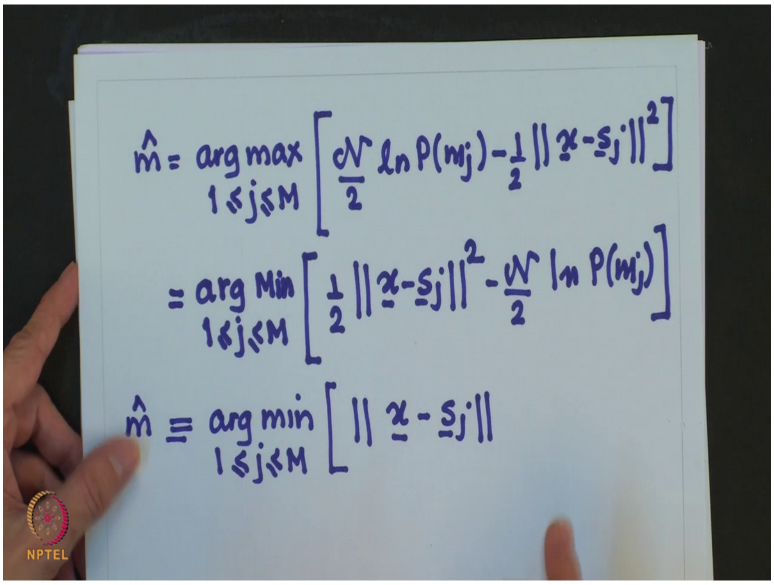


So, here I show a simple example where I have two signals S 1 and S 2 and assume that probability of message m 1 is same as probability of message m 2. In this case the decision region basically can be found out very easily and it is a line which is a bisector of the line joining S 1, S 2.

So, all the points on this side of this line are region R 1 and all the points of this side of the line is R 2. This will not hold good if the probabilities are unequal. So, here I have shown a case where the probability of message m 2 is higher. So, the decision region for R 2 is biased in the favor of S 2. So, you see that this line has got shifted away from the center of the line joining S 1 and S 2 towards S 1. So, this region has become larger; that is R 2 region has become larger ok.

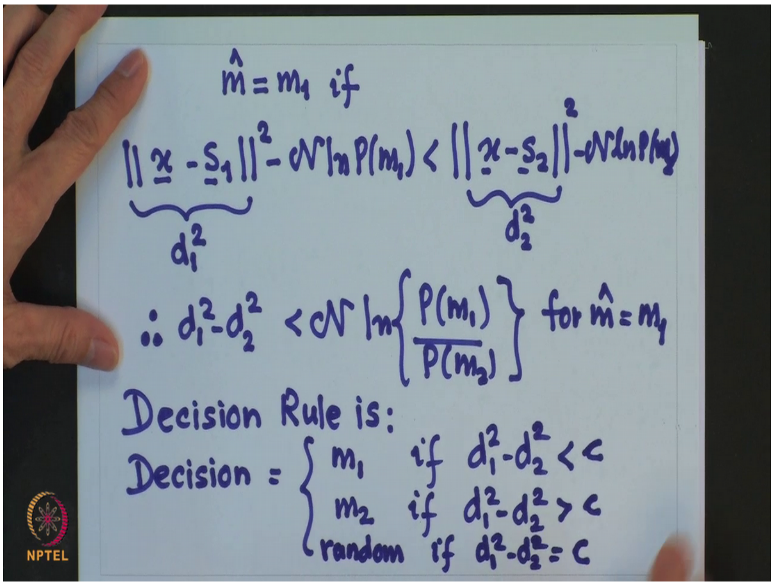
So, let us try to formally analyze this example and see where this line perpendicular line will prove that this a perpendicular line located in the signal space. Now, we will use this relation sorry.

(Refer Slide Time: 10:12)



We will use this decision function to partition the signal space in two regions R 1 and R 2 and this will be done as follows.

(Refer Slide Time: 10:26)



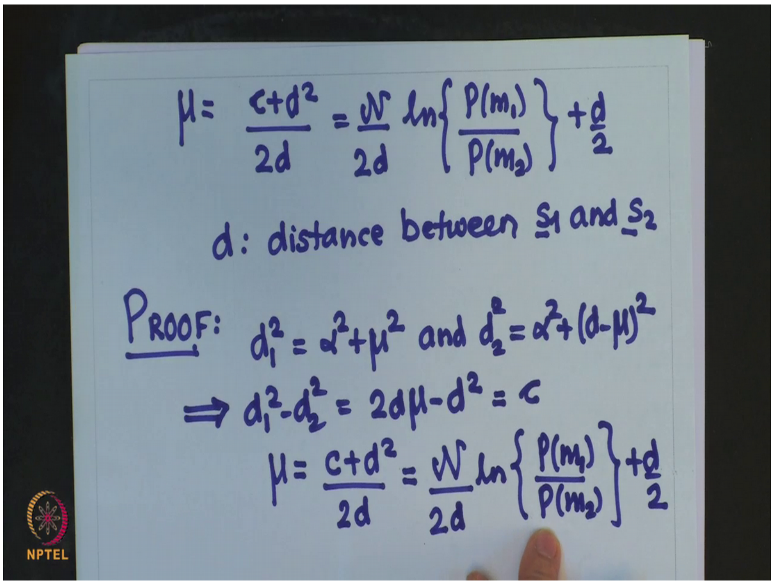
So, without loss of generality our decision will be equal to m 1 if this condition is satisfied. Now, this basically is nothing, but the distance of vector x from S 1 square of it.

So, let me denote this as a d 1 square and this is d 2 squared correct. So, if this condition is not satisfied if it is greater than you will it will be m hat will be m 2 ok. Therefore, from this we get this relationship.

So, we can say that the decision rule is equal to m 1; if this is less than this quantity for given P m 1 and P m 2 is a constant and let me call that constant to be equal to c. This is equal to m 2; if d 1 square minus d 2 square is greater than c and if it is equal we will toss a coin and randomly decide the decision.

. So, the boundary of the decision is given by this equation d 1 squared minus d 2 squared is equal to c and we will now show that this boundary is given by a straight line perpendicular to the line joining the two points S 1 and S 2 and passing through this line at a distance from point S 1; where mu is given by c plus d squared by 2 d which is equal to this expression; where your d is the distance between point S 1 and S 2.

(Refer Slide Time: 13:40)



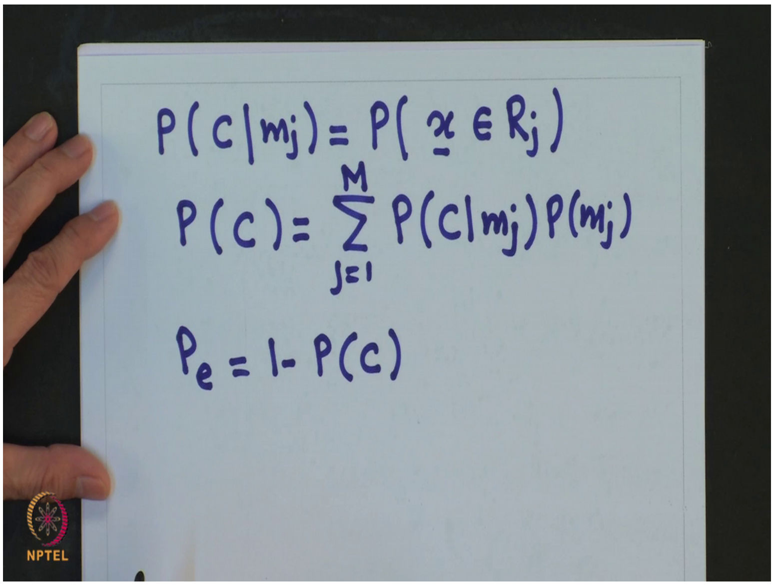
So, the proof for this follows. So, I have just redrawn S 1 and S 2 in this figure out here. This point S 1 and S 2 and now if you take any line perpendicular to this line S 1, S 2 joining it correct, any point on it and let me indicate this is a point and let the distance be alpha from this line.

Then, I can write the relationship between d 1 alpha mu d minus mu correct as follows. So, from this it is very clear that d 1 squared is equal to alpha squared plus mu squared and d 2 squared is equal to alpha squared plus d minus mu whole squared.

So, if you take the difference between the two I get it as 2 d mu minus d squared; which is a constant correct. So, it implies that this constant is equal to your c correct because this is the constant which I get from my decision function correct. So, and from this I get my value of mu to be equal to c plus d square by 2 d and plugging in the value of c; I get it as this quantity which was the desired result to be proved correct.

So the boundary, boundary is a straight line perpendicular to the line joining S 1 and S 2 and passing through the line joining S 1 and S 2 at a distance which is this is a distance mu from S 1 and that mu which I have indicated here correct; same is equal to given by this quantity fine. So, now, if you have N dimensional signal space then your R js will be also N dimensional, but corresponding to m messages you will have m regions correct.

(Refer Slide Time: 17:17)

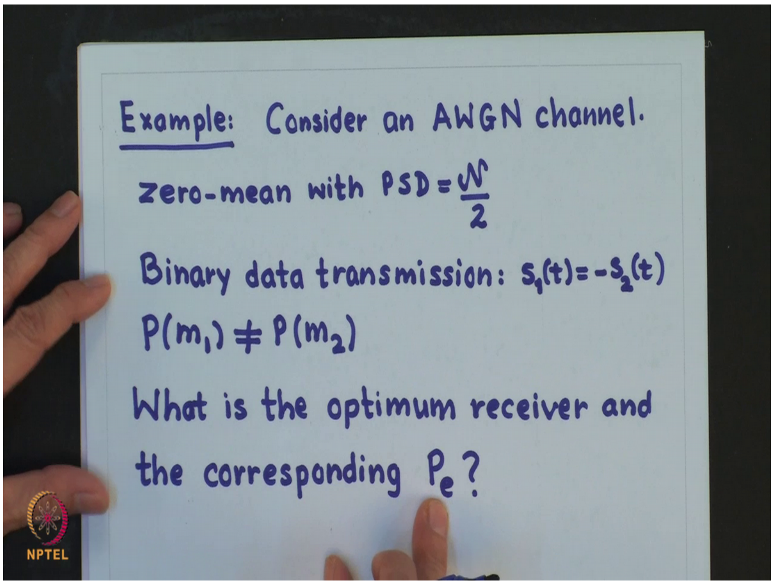


So, if you were to calculate the probability of error then what we would be required is to calculate the probability of correct decision given that I have transmitted the message m j correct.

So, this is the probability that your vector x belongs to the region R j correct. So, the unconditional probability of correct detection would be equal to the summation of the conditional probabilities and the probability of error would be equal to 1 minus probability of correct detection. So, given the decision regions it would be easier for us to calculate this kind of probability of errors correct.

So, let us take another example to show you how to calculate this probability of error.

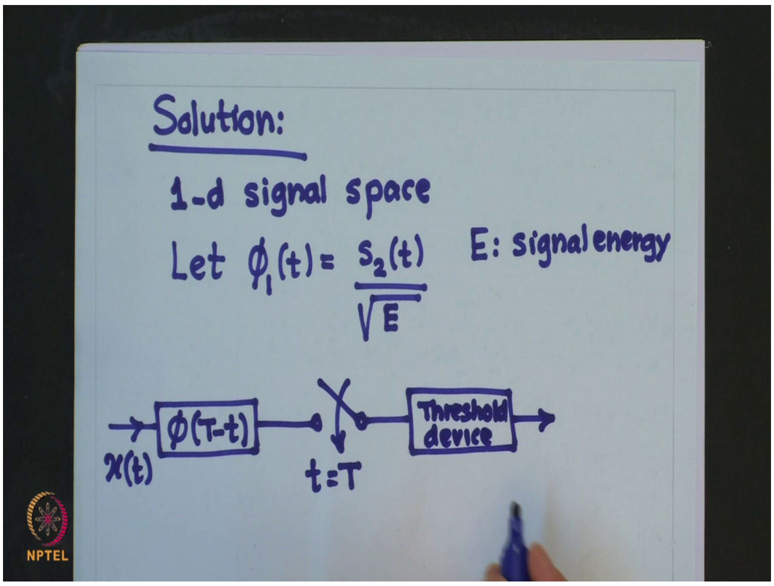
(Refer Slide Time: 18:24)



Let us consider that I have an AWGN channel and the noise is zero mean with power spectral given by italic N by 2. We assume that we have two messages to be transmitted m 1 and m 2 and we use the message signal as S 1 t and S 2 t for this messages m 1 and m 2 and the message signals are related like this; that means, they are basically antipodal and this a non-equiprobable case.

The goal is to find the optimum receiver and for that optimum receiver find the corresponding probability of error fine.

(Refer Slide Time: 19:14)

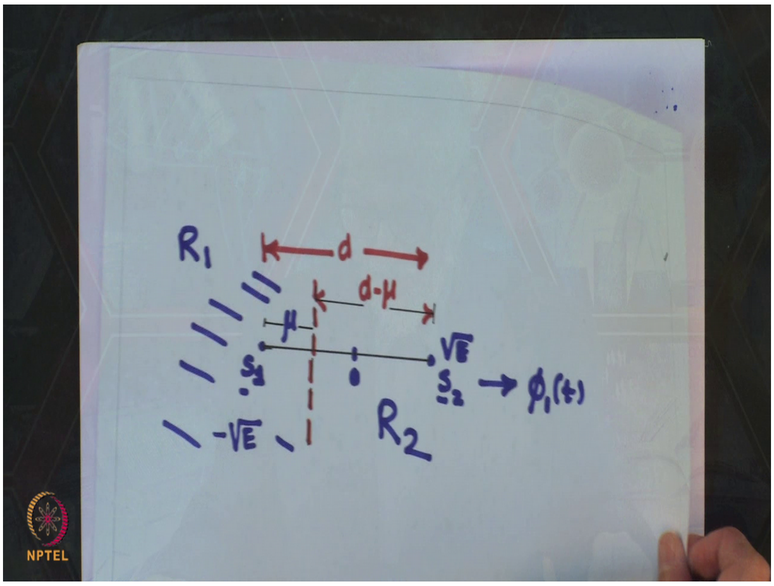


So, the solution follows. It is simple to realize that in this case it is a 1-dimensional signal space. So, I need only 1 basis signal let that basis signal be phi 1 t equal to S 2 t normalized by the energy of S 2 t which I call it E. In this case the energy for both the signal S 1 t and S 2 t is same which is equal to E. So, for the implementation point of view I can just use this block schematic. Let me implement using it match filter.

So, I will have a match filter corresponding to the basis signal; this is my signal received x t then I need to sample it because I am using a match filter sample it exactly at t equal to capital T and then I can put it to a threshold device and get the output.

And here basically we have this problem basically is very similar to what we have discussed earlier. So, let me just show you the how the two signals will look geometrically in the vector space.

(Refer Slide Time: 21:13)

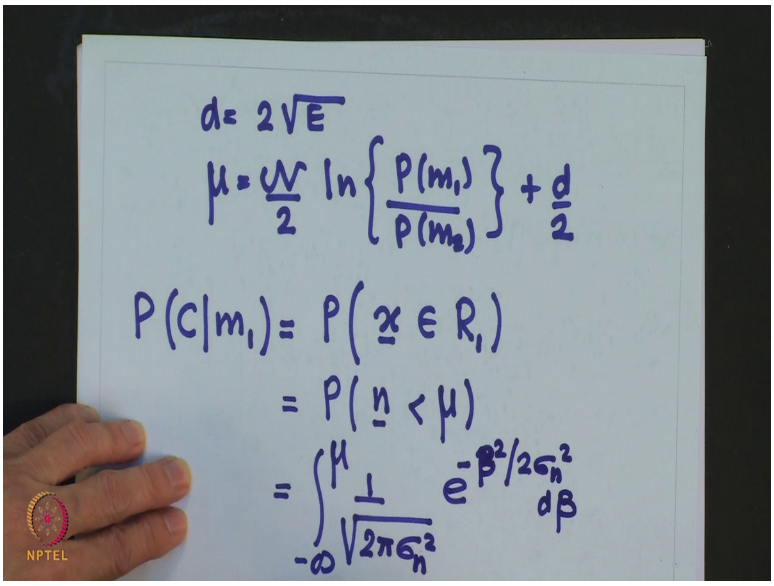


So I have S 1, I have S 2; let me in indicate this as a origin so, this is located. So, I require only 1 basis signal phi 1 t. So, this will be root E this point will be minus root E. Let me indicate this distance to be mu this is the partition which I am going to form.

We have just studied that because the probabilities are unequal and I have drawn this assuming that probability of m 2 is larger than probability of m 1. So, let me indicate this mu this distance is d between the two signal points. So, this is equal to d minus mu distance correct.

Now, this problem is very similar to what we did earlier. So, we can calculate quickly what is the value of mu; So, in d in this case is going to be 2 root E, mu is going to be italic N by 2 log of P m 1 by P m 2.

(Refer Slide Time: 22:30)

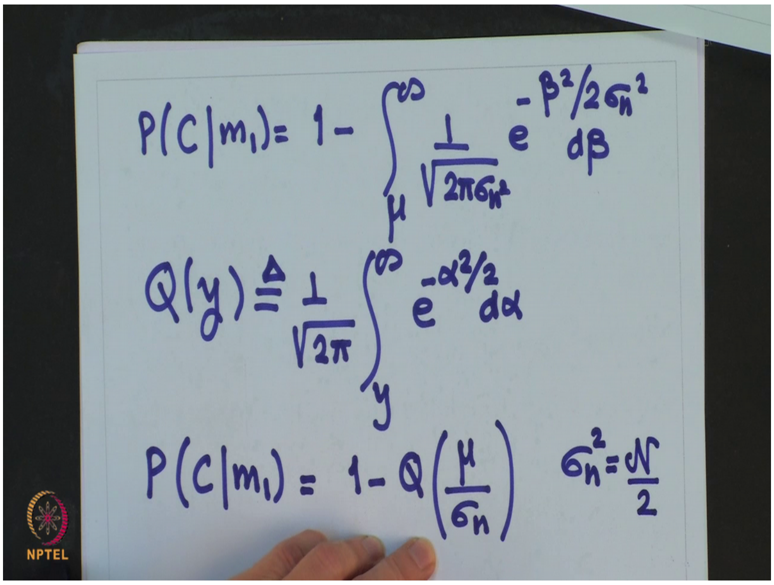


So, now, probability of correct decision is the probability the vector x lies in region R 1. So, remember this is our all this side is region R 1 and this side it is going to be region R 2 correct.

So, now, we assume the noise to be Gaussian. So, we are without loss of generality let us assume that we have transmitted message m 1. So, this being your center the noise is going to be distributed in a Gaussian shape around this. So, what you should do is basically, if you want correct decision the noise should not be that large that point lands up here; vector x correct. So, your vector x should be lying only in this region R 1 correct. So, this can be easily calculate I want to find out what is the probability of my vector x lying in this region.

So, assuming Gaussian distribution for the noise I have to calculate this so, this means basically what is the probability of my noise being less than mu. Here when I am writing this is assume that noise origin is at S 1 fine. So, this I can write it as assuming Gaussian distribution to be pdf of my noise 2 pi there is a sigma n squared is a variance of the noise e raised to minus say x sorry. I will write it as a beta squared 2; I have to integrate this quantity.

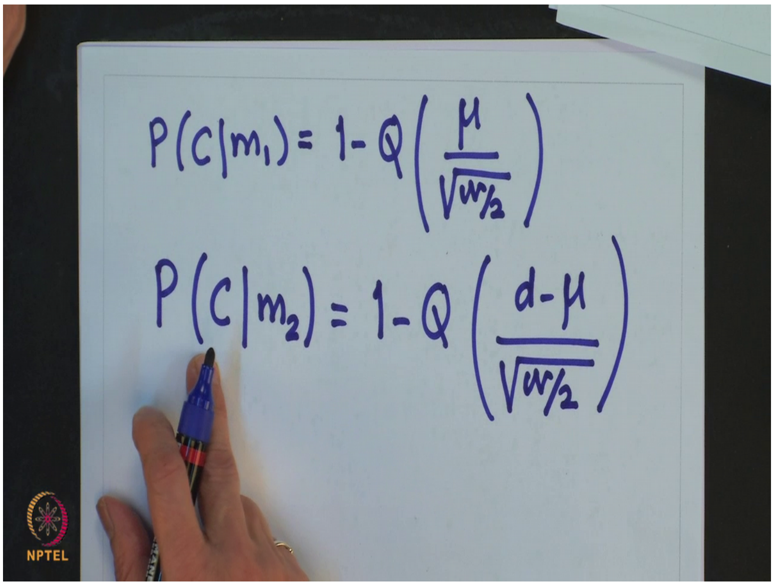
(Refer Slide Time: 25:06)



Now, this quantity I can rewrite it as now, if I define a Q function of this form then probability of correct decision given m 1 is equal to 1 minus Q times mu by sigma n correct. Let us just change your variables we can show this.

Now, this is the square root of the variance. So, variance for the noise in our case is going to be it is projected on the basis signal with unit energy. So, this will be equal to italic N by 2 because the energy in the impulse response which is equivalent to the basis signal is equal to 1 fine. So, I get this quantity.

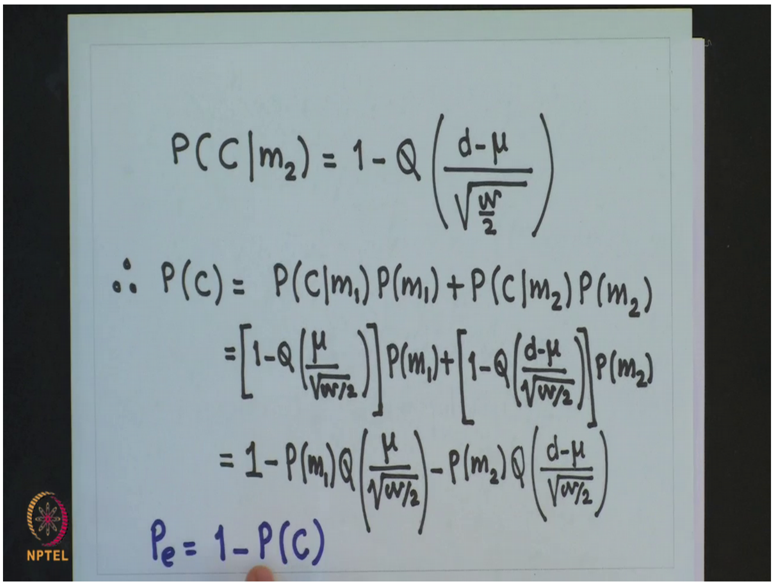
(Refer Slide Time: 26:41)



So, from this using this I get my probability of correct detection given message m 1 is equal to 1 minus Q times mu root of italic N by 2 correct. So, this is what I get.

Now, so similarly you can find out what is the probability of correct detection given m 2; there the only the distance will change. It will become d minus mu rest of the things will remain the same; I will get this quantity and then I can write what is the probability of correct detection right.

(Refer Slide Time: 27:28)



So, if I do this basically I get this probability of correct detection it is a unconditional probability. I just plug in the values which I have just recently calculated and I expand it I get and then knowing that probability of m 1 and probability of m 2 is equal to 1. I get equal to this quantity and this can be simplified and rewritten like this.

Now, if you assume that both the probabilities are equal then in that case your mu will be equal to d by 2 and the probability of error will turn out to be this expression; just plug in these values out there and it is not very difficult to show that this is what I am going to get correct.

So, for the simple case I have shown you what is the optimum receiver and that is done by partitioning the signal space in two regions. Once I have done that it becomes easier for me to visualize and write the probability of correct detection and from there I can find out what is the probability of error.

But if the dimension of the signal space is very high then this kind of visualization may be very difficult and calculating the probability of error may be non-trivial task.

We will discuss this in the next class.

Thank you.