Principles of Digital Communications
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Lecture – 17
Vector Representation of a Random Process

In Digital Communications we are concerned with the transmission of one of the M messages generated by an information source. This message gets transmitted by using one of the M message signals of waveforms. Now, at the receiver what we receive is basically a noisy version of this message signal, the goal of the receiver is basically to detect which one of the M message signal was transmitted in the presence of noise.

Now, this signal detection problem and determination of the optimum receiver gets simplified, if we could represent signals in terms of vectors. Now, if we assume the additive noise model of the channel, then your received signal is equal to the message signal plus the noise signal.

Now, we know that given a set of message signals, we can always generate a complete orthonormal basis signals for this set. So, I can represent each of the message signals in terms of a vector. So, now, if you could represent the noise signal also in term of vectors, then our detection problem will get simplified ok.

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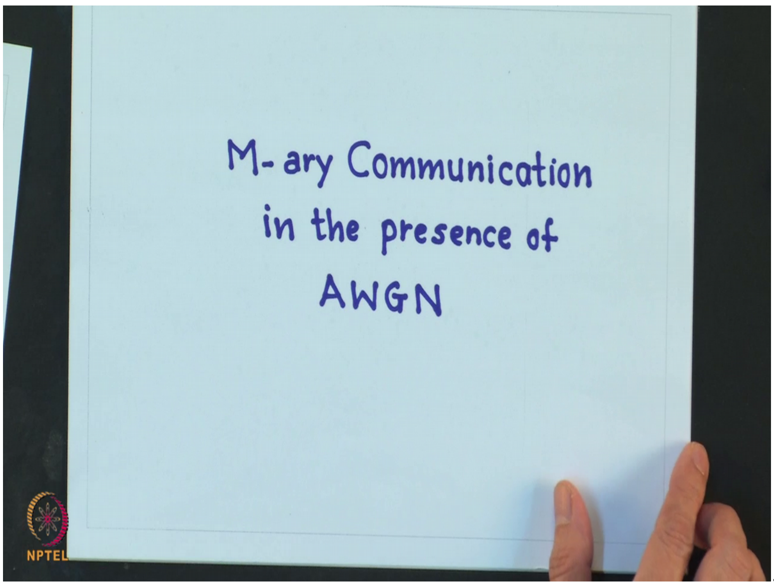

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So, now we start our study with M-ary Communication in the presence of Additive White Gaussian Noise. And the first thing we will do is basically is to look at the vector representation of a random process fine.

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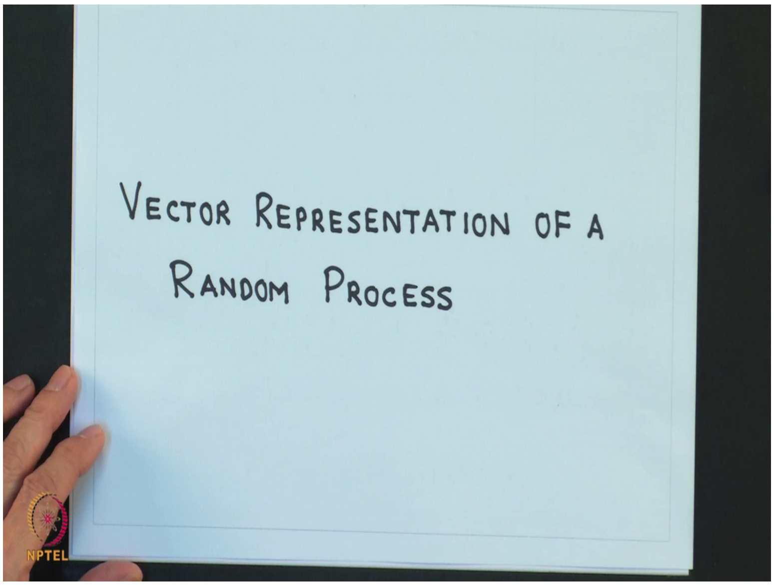

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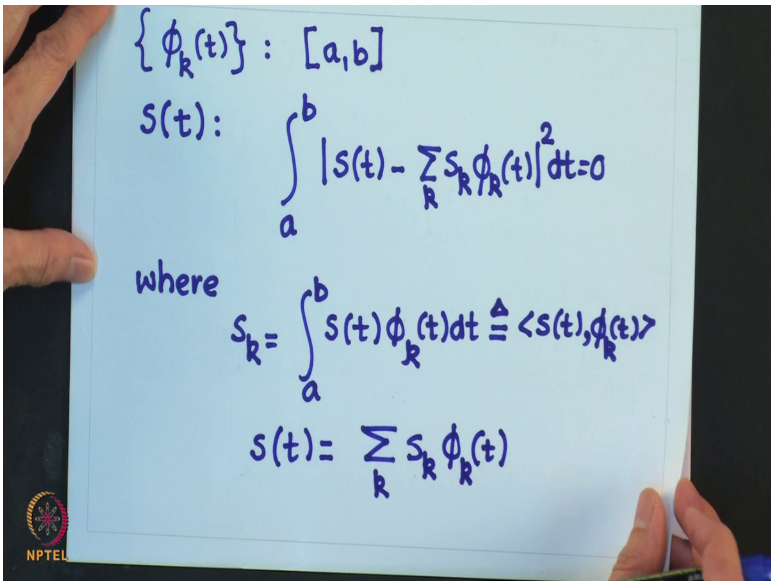

So, let us consider a complete orthonormal set of basis signals, phi k t for a signal space defined over the interval a b correct, then any deterministic signal S t in this signal space will satisfy the following condition, where your S k is the production of S t over the this is signal phi k t, this is also denoted as follows ok.

Now, this implies that for any t in the interval between a and b, we have the equality S t is equal to correct. Now, for random processes defined over the interval a to b this statement is generally not true. So, we require certain modifications and let us see what are those.

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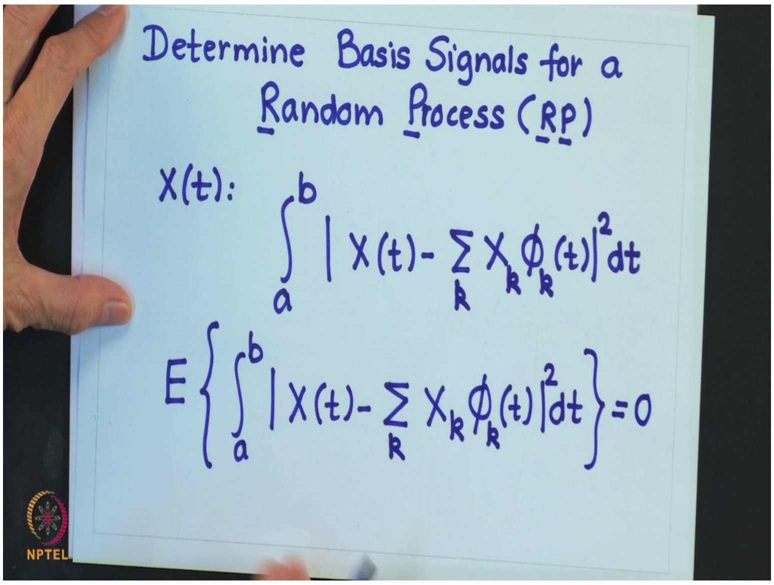

So, our goal is now to determine basis function for a random process or basis signals ok. So, first of all a general random process X t cannot strictly satisfy the following equation, correct instead a proper convergence requirement is in the mean square sense, what it means that it satisfies this equation..

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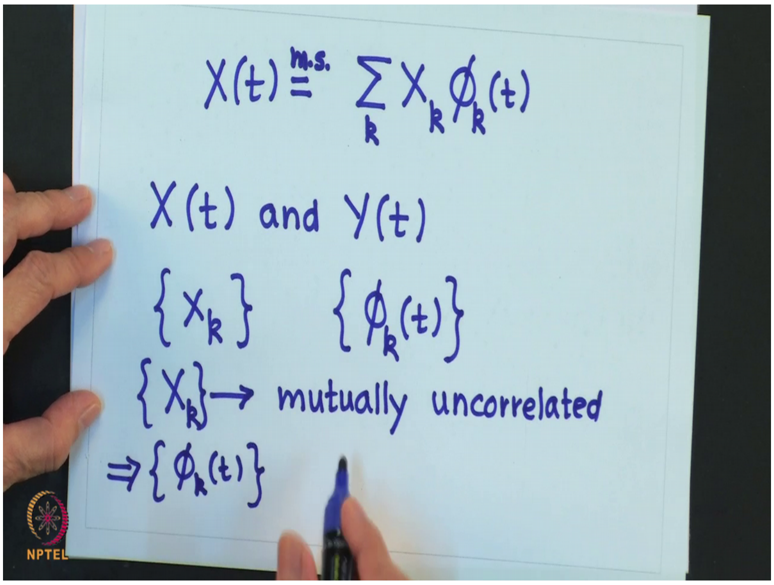

Now, this equality can be denoted as X t is equal to X k phi k t over all k, but this is equality in the mean square sense. So, if you have two random processes X t and Y t and if we say that these two random processes are equal in the mean square sense, then practically the difference between X t and Y t random processes have 0 energy correct.

So, as far as we are concerned in communications signals, or signal differences with 0 energy have no physical effect and can be considered to be 0 ok. For set of deterministic signal the basis signals, can be derived why are the Gram Schmidt orthogonal procedure which we have studied earlier correct.

However, Gram Schmidt procedure is invalid for random processes correct. A random process is an ensemble of signals. The basis signals for this random process will depend on the characteristic of the random process. There are many ways in which a random process can be expanded in terms of a sequence of random variables and orthonormal basis phi k t.

Now, if we require the additional condition that the random variable X k be mutually uncorrelated, then the orthonormal basis, have to be solution of an Eigen function problem given by an integral equation whose kernel is the autocorrelation function of the random process and, it will be given as follows.

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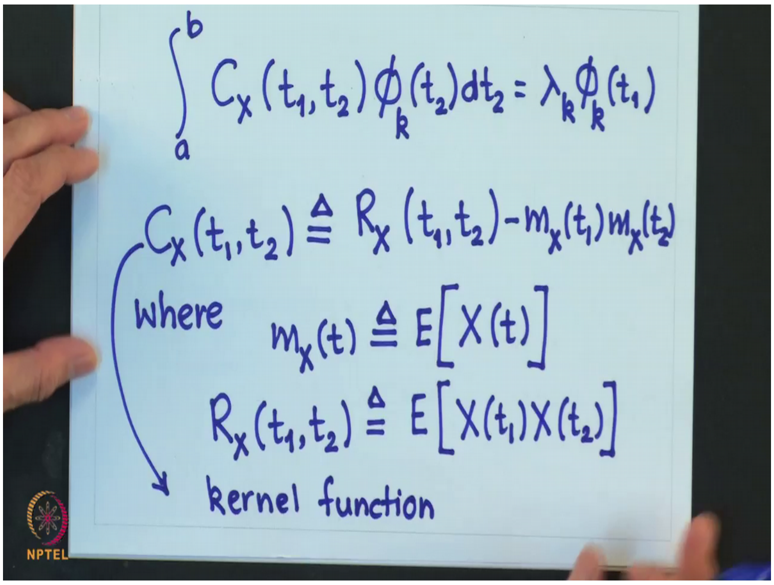

So, in this your auto covariance function C X t 1 comma t 2 is by definition, auto correlation function minus the means. So, where your means are defined as follows, this is the expectation of the random process X t correct, this is the auto correlation of the random process correct. And this is a kernel function, in this equation correct.

So, solving this integral equation results in the orthonormal basis phi k t and projecting the random process on this basis results in the sequence of uncorrelated random variables X k ok. So, you are here this function once you get you have to normalize it properly.

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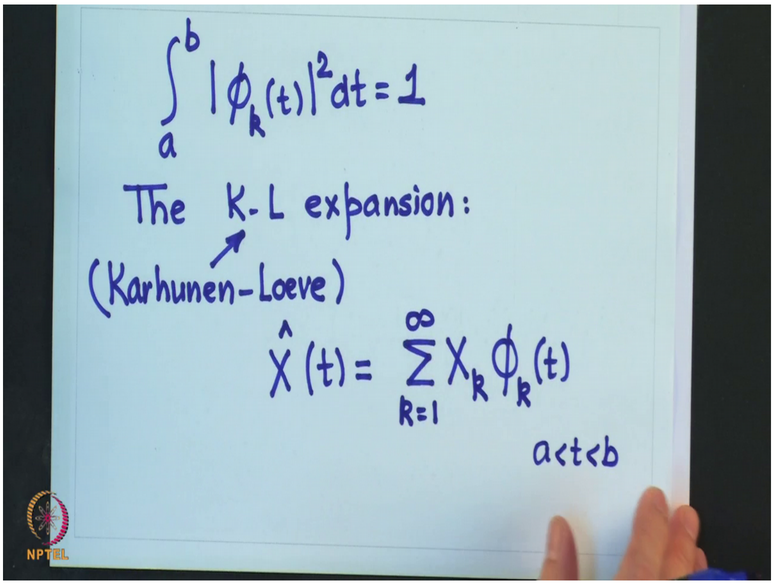

So, this condition is satisfied and then the K L expansion, this is known as Karhunen-Loeve or K L expansion and, it is given as follows correct. So, this equality is in the mean square sense it is important to note that please ok.

So, now if your random process has 0 mean correct, then you could substitute it though auto covariance function by autocorrelation function and, then it would be something like this correct.

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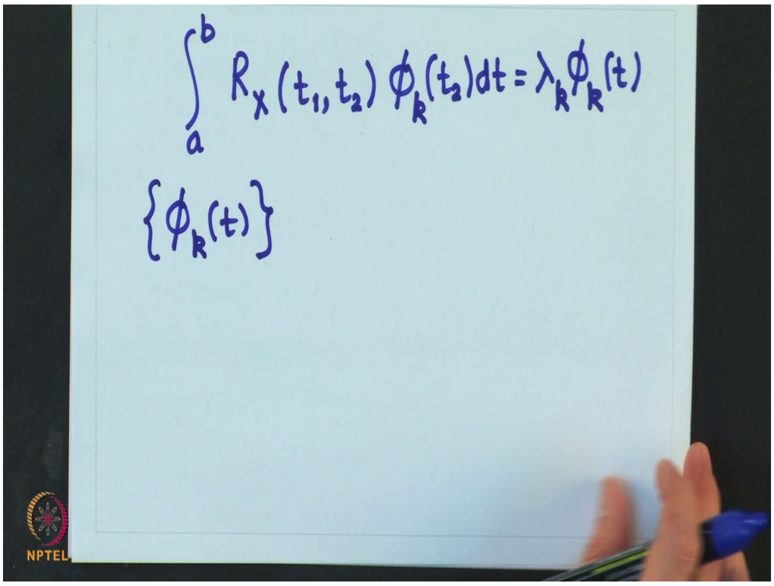

So, these are known as Eigen values and this is known as Eigen function. So, in this case basically this autocorrelation function becomes your kernel function correct. Now, what it says basically that given a random process and, if you desire the coefficients of expansions, which are going to be random variables to be uncorrelated, then this basis function basically are obtained by solution of this integral equation; that means this basis function cannot be selected arbitrarily correct. And depending on the random process, obtaining this basis function orthonormal basis function may not be very trivial task ok.

So, let us consider one special case of this random process and we will consider for that matter to be Stationary White Noise.

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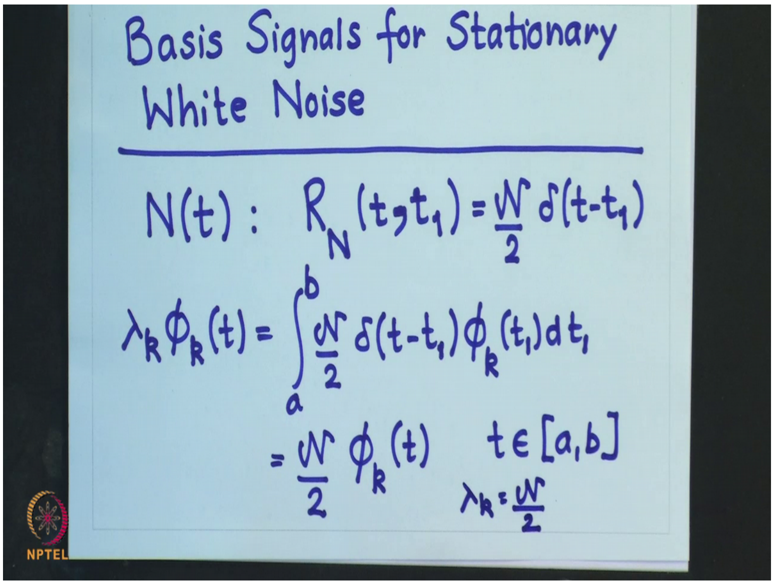

So, we will determinant the basis signals, for stationary white noise process ok. So, we know that for a stationary white noise process let me assume the white noise process to be N t I am considering noise. So, I am writing is N t and the autocorrelation for this, because it is a white noise and the stationary what it implies is that. Since the noise is white its power spectral density will be flagged. So, autocorrelation will be an impulse ok.

So, for this spatial kernel the integral equation, this will reduce to the simple form as follows correct. Now, using the property of an impulse this is easily shown to be equal to N by 2 phi k t belonging to the interval a b correct.

So, what this implies that any complete orthonormal set of basis signals, can be used to represent the stationary white noise process correct. And additionally all the Eigen values are identical and lambda k is equal to italic N by 2 a very important result ok.

Now, let us consider this white noise process to be having 0 mean correct. So, if that is the condition then, let us look at the expansion of this white noise process in terms of the basis signals. So, as we have said we could use any complete orthonormal basis signal set.

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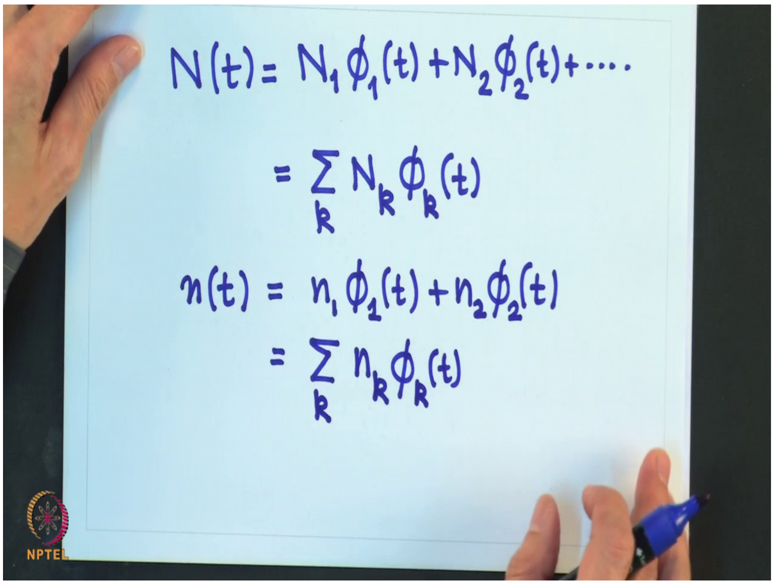

And I will be able to represent my noise process N t as N 1 phi 1 t plus N 2 phi 2 t plus correct. So, this is summation of it is important to note that this N 1, N 2 are random variables because this is a random process which consists of ensembles. And we are taking the projection of that onto this phi 1 2 phi 1 t phi 2 t and all that.

So, depending on a given sample function from this random process, you will get different coefficients of expansions. So, what it means basically each coefficient of expansion is going to be a random variable correct. So, it is important to note that when you write this as a random process, then this is a random variable, but if I write it as a sample function of this noise process, then this I should write it as n 1 phi 1 t plus n 2 phi 2 t and this will be the summations like this correct right. So, please note this difference between a random process and the sample function of the random process.

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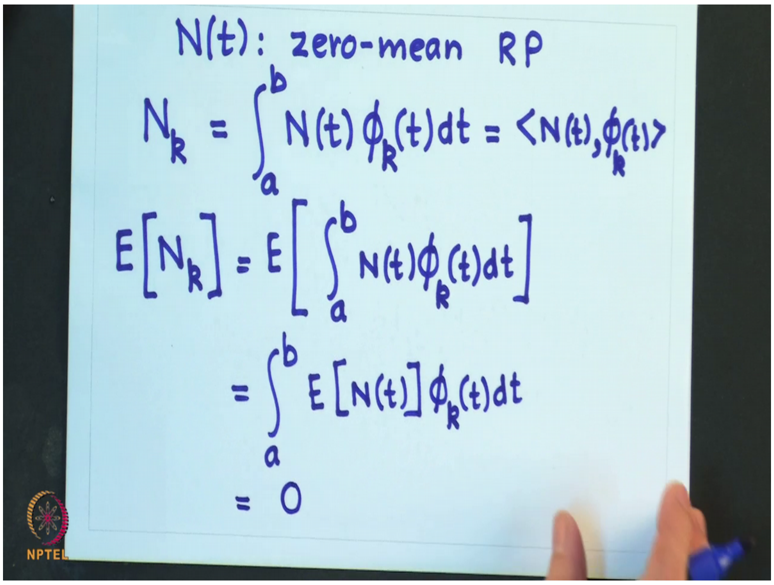

So, now if we assume that our N t is a zero mean random process. Let us look at the properties of this random variables N k which is nothing, but the projection of the noise process onto the basis signals correct right. So, we take the average of this random variable, this will be equal to average of this quantity, I can change the order. Now, this quantity is equal to 0 because, we assumed it to be 0 mean random process. So, it means basically this is equal to 0 fine ok.

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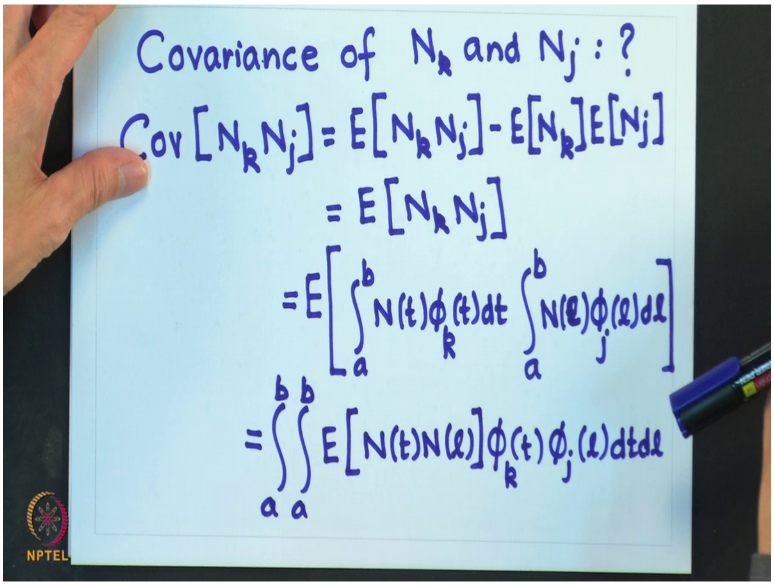

Now, let us look at the covariance of two random variables N k and N j. So, I am interested in finding out the covariance of N k and N j correct. So, we will write this as this is by definition equal to this. Now, we are assuming 0 mean process and also stationary process correct. So, this will reduce to just this quantity, this will go to 0 and this I can write it as follows.

So, this is the projection I will get N k N j I will get the projection of N t sorry, we will use this phi j l dl correct. So, this I can write by changing the order as follows right. Now, assuming that your noise is white, then I can substitute this by impulse.

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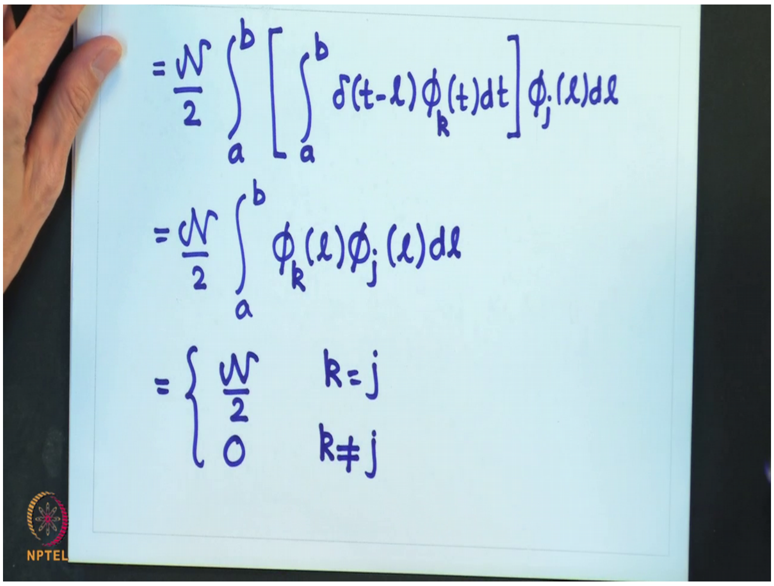

So, this I can write it as it is a white stationary noise, this can be simplified as, this will be equal to phi k and t equal to l only and this value will be existing. So, what this implies is this will be equal to Italic N by 2, when k is equal to j is equal to 0, when k is not equal to j by the property of the orthonormal set correct ok.

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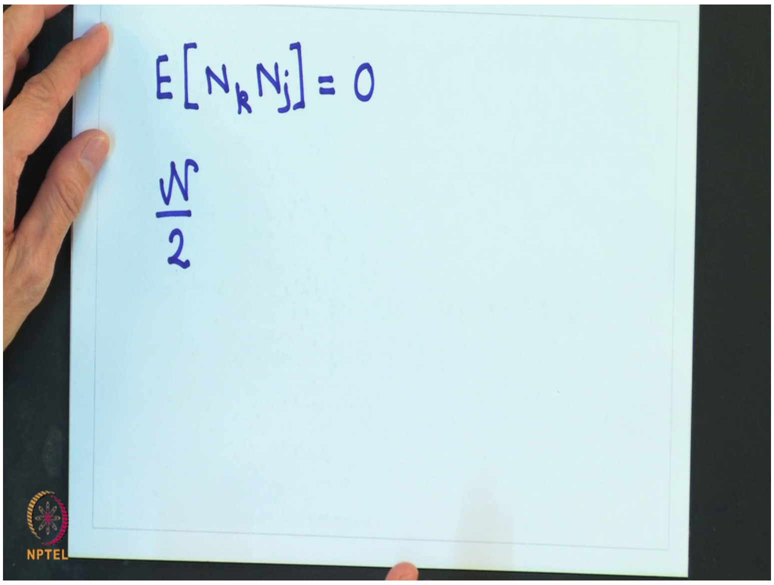

So, what this shows is basically that I get my random variables N k N j to be uncorrelated correct. So, and the variance of each of this random variable turns out to be Italic N by 2 correct.

So, now if the this noise process was a Gaussian, then my N k N j would also be Gaussian random variables, in that case when Gaussian random variables are uncorrelated, it will also imply that they are independent correct. So, for a white stationary Gaussian noise it will imply independence of N k and N j random variables ok.

Now, having studied this we will see basically how to incorporate, this idea of vector representation of a noise process, into a design of an optimum receiver for M-ary communication in additive white Gaussian noise scenario. And we will continue the study in the next class.

Thank you.