Principles of Digital Communications
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Lecture – 13
Channel Capacity - IV

We have seen that a band limited signal, which is also constrained to have some mean square value; for a such a signal white Gaussian signal has the largest entropy per second.

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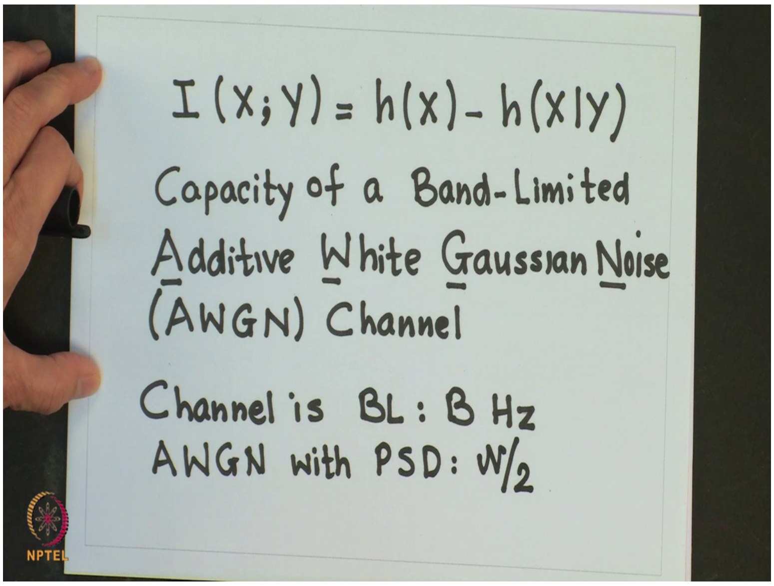

And we have also seen that mutual information for a continuous channel which is I X colon Y is equal to differential entropy of the source minus the conditional differential entropy of the source given that you have observed Y as the output of the channel on which this transmission of X has taken place.

Now, today we will try to find out capacity for a particular specific channel. We have also seen that the capacity is the maximum of this mutual information and we also seen that this for a given channel this capacity turns out to be the function of the input PDF f X ok.

So, now our problem is to calculate the Capacity of a Band-Limited Additive White Gaussian Noise Channel. So, a first assumption is that channel is band limited and let us assume that this is band limited to B Hertz and it is disturbed by Additive white Gaussian noise with power spectral density given by italic N by 2.

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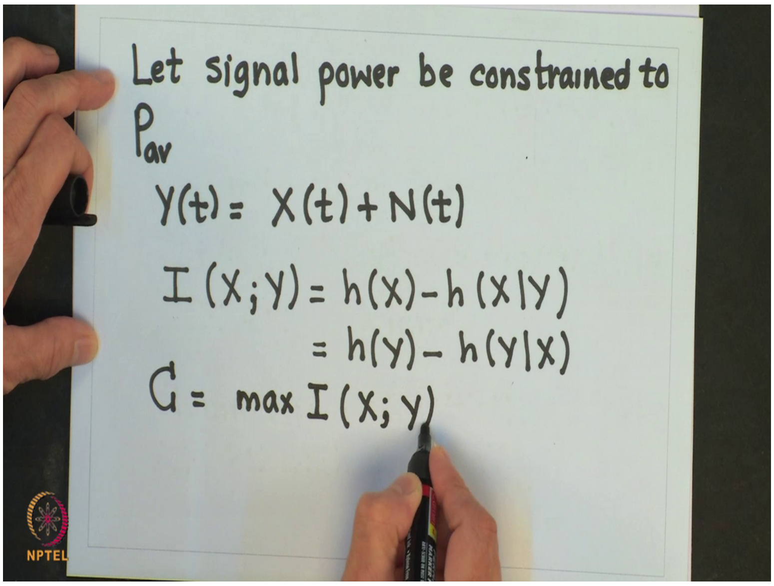

We also assume that the signal power is constrained to a value, which we call it as P average. So, the disturbance is assumed to be additive. So, the received signal which we call it as Y t would be given by X t which is the input signal plus noise which we denote by N t.

Now, all these are random processes correct because our channel is band limited to B Hertz. It implies that your noise process will also get band limited to B Hertz your input process when it is passing through the channel will also get limited to B Hertz and obviously, your output Y t will also get band limited to B Hertz ok.

Now, we the channel capacity is by definition, the maximum rate of information transmission over a channel and the mutual information we have seen turns out to be this expression. This we have justified this in a previous class and we have also shown that your mutual information is symmetric with respect to X and Y. So, I can also write this as h Y minus conditional entropy of Y given X; whenever I say entropy basically it is presumed that it is a differential entropy ok.

So, the channel capacity C is the maximum value of this mutual information.

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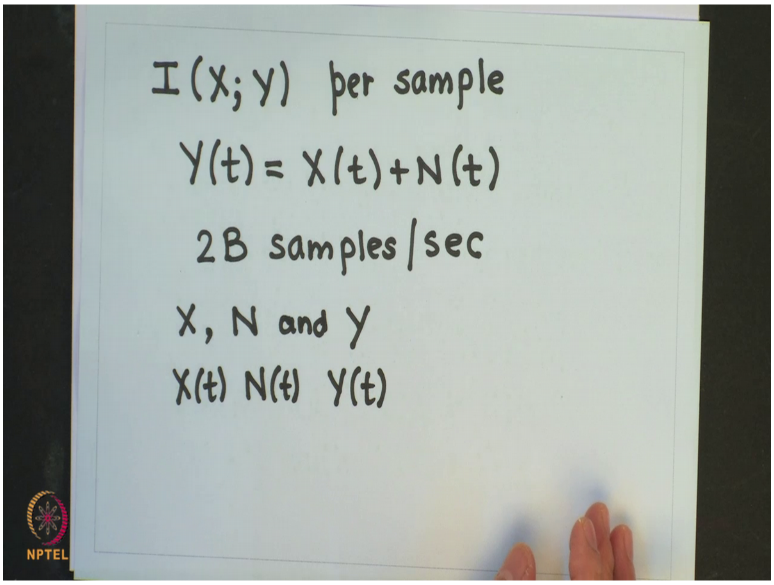

Now let us first find the maximum value of mutual information per sample correct ok. So, we will try to do this assuming that the channel is band limited to B Hertz and disturbed by a white Gaussian noise or power spectral density italic by 2 correct.

Now, this Y t is equal to X t plus N t, all these signals are band limited to B Hertz. So, all these signals can therefore, be completely specified by samples taken at the uniform rate of 2 B samples per second and the problem is to find out the maximum information, that can be transmitted per sample. So, let us use X, N and Y as sample values of X t N t and Y t respectively correct fine.

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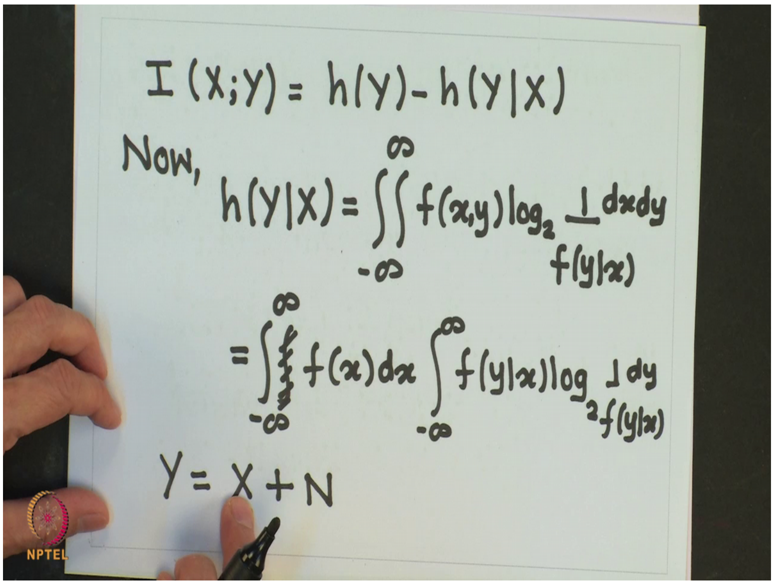

So, the information that is your mutual information, transmitted per sample is equal to we will use this expression for computation, because it is easier to do that ok. Now your conditional differential entropy is equal to joint PDF log to the base 2 of 1 by conditional PDF and integrate it over x and y. This can be written as follows by Bayes' rule sorry this integral we have breaking it correct fine ok.

So, now you are we have also said that the sample values of these random signals we are going to denote as Y X and N. So, we will get Y is equal to X plus N hence the PDF of the random variable Y, when X is given is obtained as follows.

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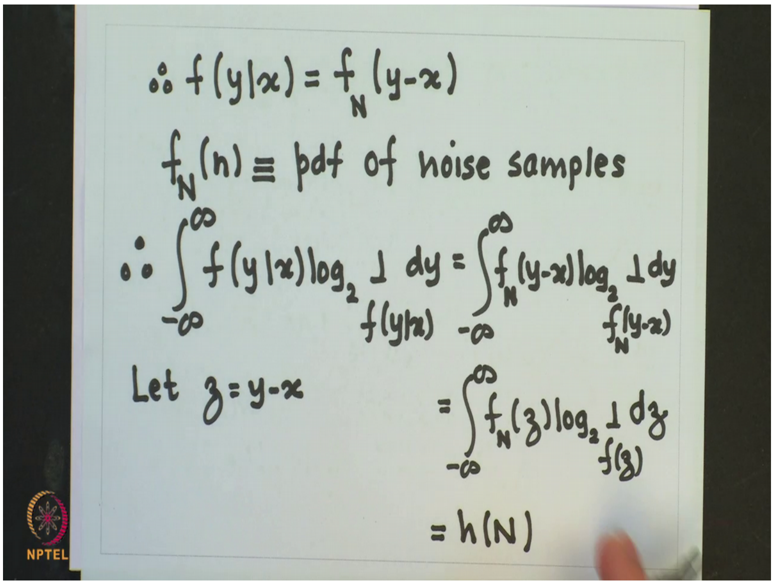

Therefore, PDF of conditional PDF of y given x will be the same as the PDF of the noise except for the translation by x.

So, this will be equal to this ok because y is equal to x plus n and x has been given. So, the PDF of y given x will be equal to this quantity, which is the PDF of the noise except for the translation by x; where your f N n is PDF of the noise sample. Therefore, let us evaluate this integral, this we can rewrite it as; let us substitute Z equal to so this is some Z equal to y minus x correct. So, if I do this and substitute into this I will get it as this integral.

Now, this by definition is the differential entropy of the noise samples.

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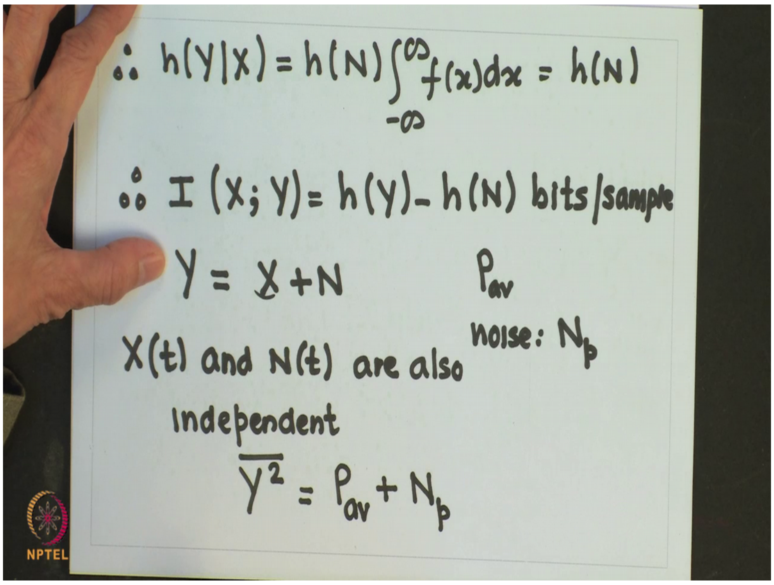

We get the conditional differential entropy equal to h N integral of f x d x this by property of the PDF is equal to 1. So, this is equal to h N.

Now, this relationship is applicable to all types of additive noise, no assumption is made about the noise except it is additive ok. Therefore, your mutual information is equal to h Y minus h N bits per sample correct. Now our Y is equal to X plus N, we have assumed that the mean square value of the signal x t is constrained to have a value P average and the mean square value of the noise is let us assume to be N p and let us assume that this signal X t and N t are additive that we have done are also independent correct.

So, what will it imply that? This sample X and N random variables will also be independent. So, if that is true then we can find out the mean square value of this random variable Y will be equal to the sum of the mean square value of this plus the mean square value of this, because they are independent random variables ok. So, this will be equal to this value. So, this is your noise power and this is your signal power fine.

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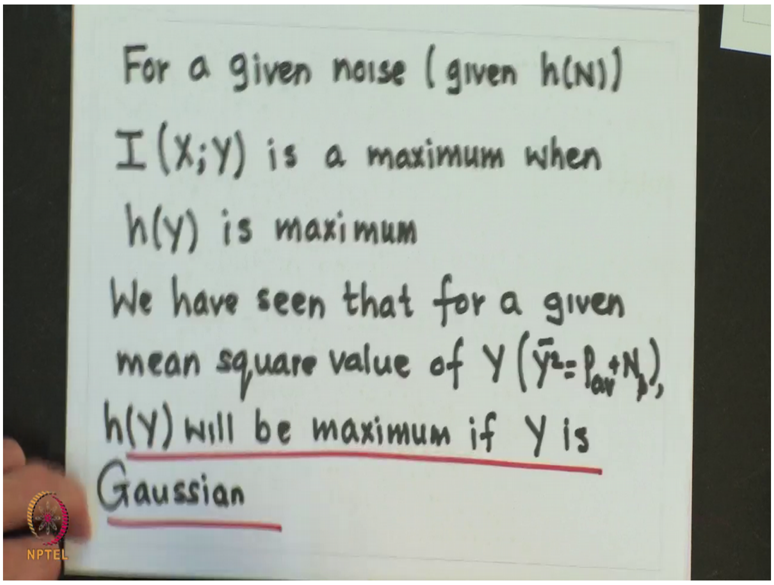

So, for a given noise; obviously, your differential entropy is given. So, in that case your mutual information is a maximum; when h Y is maximum. It comes from this is given so this will be maximum when this is maximum.

Now, we have seen that for a given for a given mean square value of Y, which is equal to P average plus noise power h Y will be maximum, if Y is Gaussian this we have studied earlier right. So, it is important to do this right; so h Y h Y will be maximum if Y is Gaussian correct.

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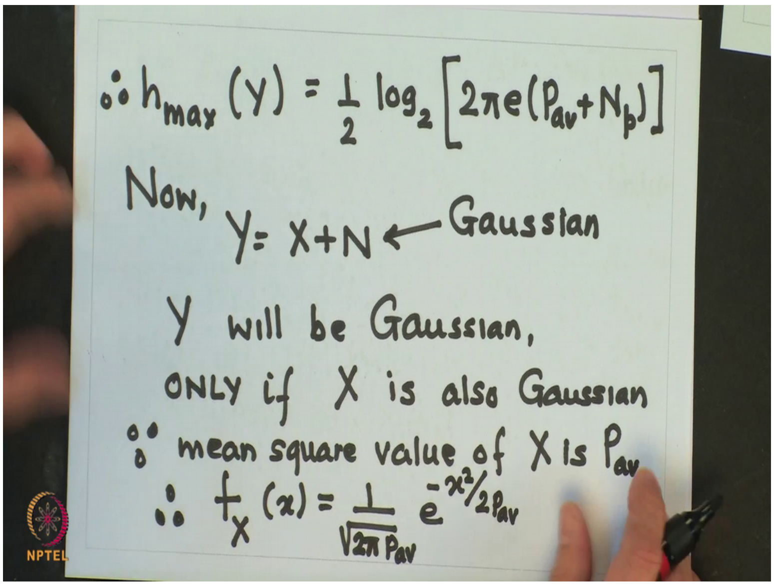

So, what this implies that your maximum value of that if Y is Gaussian will be equal to half log to the base 2 2 pi e power signal power plus noise power ok. Now Y is equal to X plus N, this we are assuming it to be Gaussian and if we want Y to be Gaussian, only if X is also Gaussian ok.

Now, because mean square value of X is P average which we have assume therefore, it implies that the PDF of my X should be given by this Gaussian PDF which will be equal to this expression fine.

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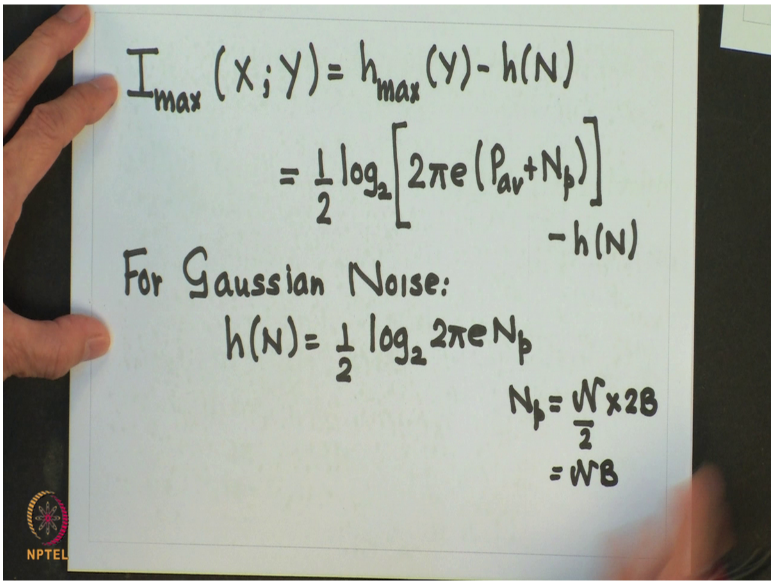

I max which is equal to h max Y minus h N is equal to now half log to the base 2 2 pi e power average plus noise power minus differential entropy of noise. So, for Gaussian noise, we know that differential entropy is equal to half log to the base to 2, 2 pi e N p ok.

Now, remember the noise power correct will be equal to the power spectral density of the noise, multiplied by the twice into the bandwidth of the channel it is a band limited. So, this turns out to be this value.

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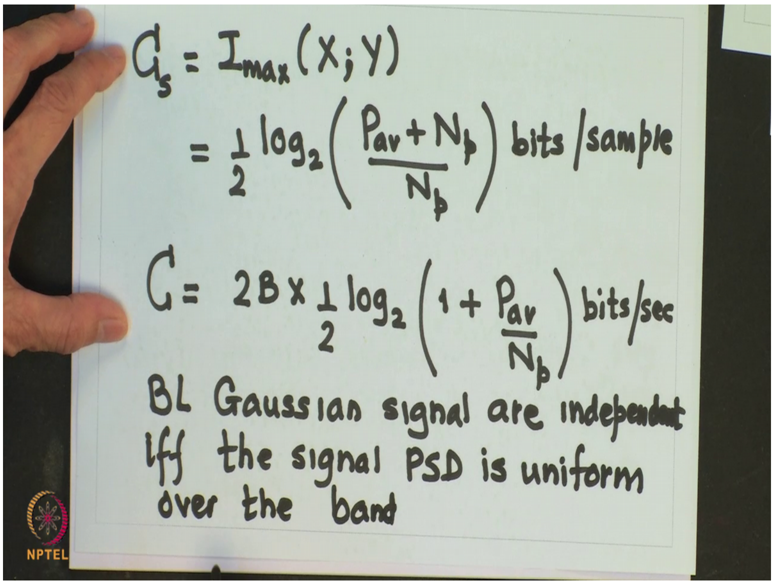

So, what happens is that, from this we get the channel capacity per sample as follows bits per sample.

Now, if all the samples are statistically independent the total information transmitted per second will be 2 b times this value C s correct. So now channel capacity C represents the maximum possible information transmitted per second correct; so then I can write that C is equal to twice B multiplied by this value which is the capacity per sample.

So, remember that samples of a band limited Gaussian signal are independent if and only if the signal power spectral density is uniform is uniform over the band. Therefore, the information can be transmitted at the maximum rate, when the power spectral density of signal y t is also uniform ok fine.

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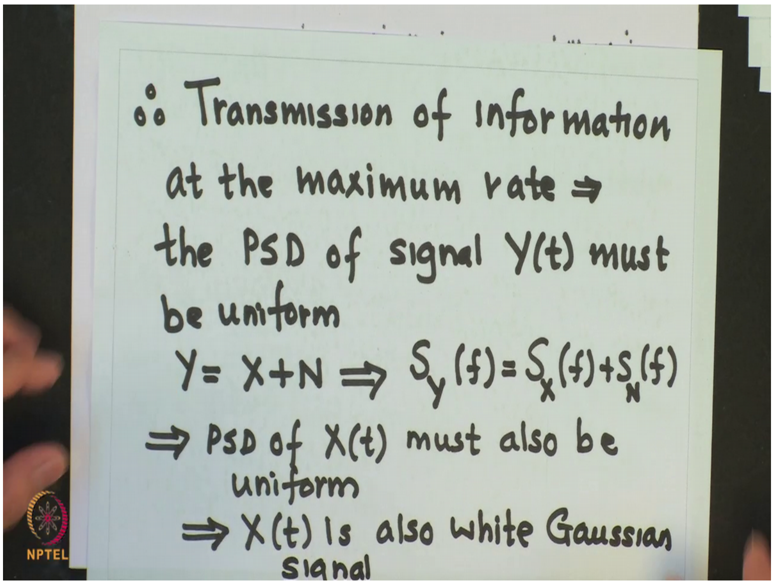

So, therefore so maxi transmission of information at the maximum rate implies that, the power spectral density of signal Y t is also uniform. Remember because the power spectral density of the noise is flat is white; correct and we want this relation to hold good. So, it implies that your Y being Gaussian should also be flat the power spectral density of that correct fine.

So, this implies that a power spectral density of X t must be also uniform or it must be flat ok. So, quickly recapitulate Y is equal to X plus N implies that, power spectral density this here f denotes the frequency, not to be confused with the PDF please would be equal to this correct; because we are assuming to be additive process correct and we want this to be white this is already given to be white. So, it implies that power spectral density of X t must also be uniform and we are also seen that X t has to be Gaussian. So, it implies that X t is also white Gaussian signal.

So, to recapitulate when the channel noise is Additive white and Gaussian with mean square value say n p correct and that that is the power, which is equal to power spectral density of the noise multiplied by twice the bandwidth of the channel because it is limited to B. So, when the signal noise is Additive white and Gaussian with mean square value, the channel capacity C of a band limited channel under the constraint of a given signal power P average is given by this falling expression bits per second; where B is the channel bandwidth in Hertz.

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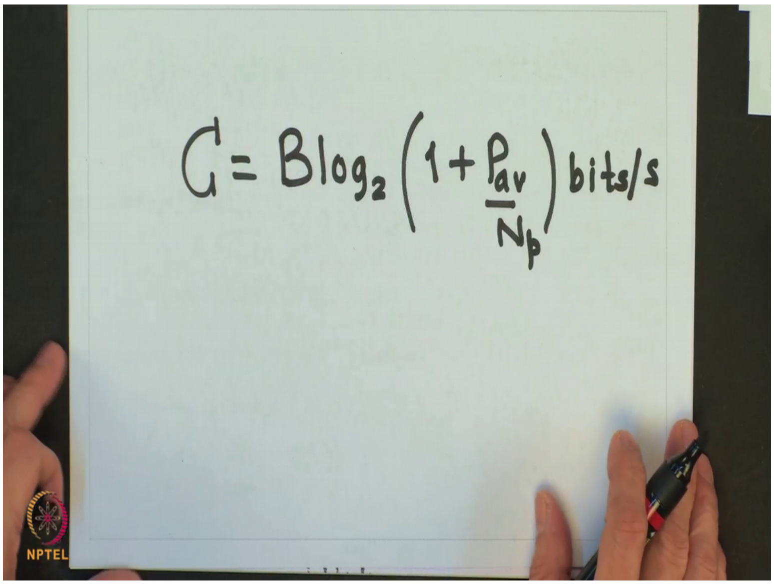

So, this is a maximum rate of transmission which can be realized and this will happen when the input signal is a white Gaussian signal ok.

Now, let us see what happens to this capacity, when I change the input parameters to the channel. For example, if I increase P average what happens? From here it appears that capacity will increase and so that is fine that will happen, but what will happen if bandwidth of the channel increases? If the bandwidth of the B that is B increases to infinity will I get channel capacity also to be infinite? This is the question which I have and we will find out the answer to this in the next class.

Thank you.