Principles of Digital Communications
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Lecture – 12
Channel Capacity - III

We have defined the average information measured for a continuous information source in terms of what we called as differential or relative entropy. The practical utility of this definition and the significance of this definition will become very clear, when we derive Shannon's capacity formula; which says that capacity of a communication channel is equal to the bandwidth multiplied by log to the base 2 of the term 1 plus signal to noise ratio. So, the channel capacity will be related to engineering parameters like bandwidth and signal to noise ratio.

Now, let us address the issue of transmission of the output of a continuous source onto a communication channel.

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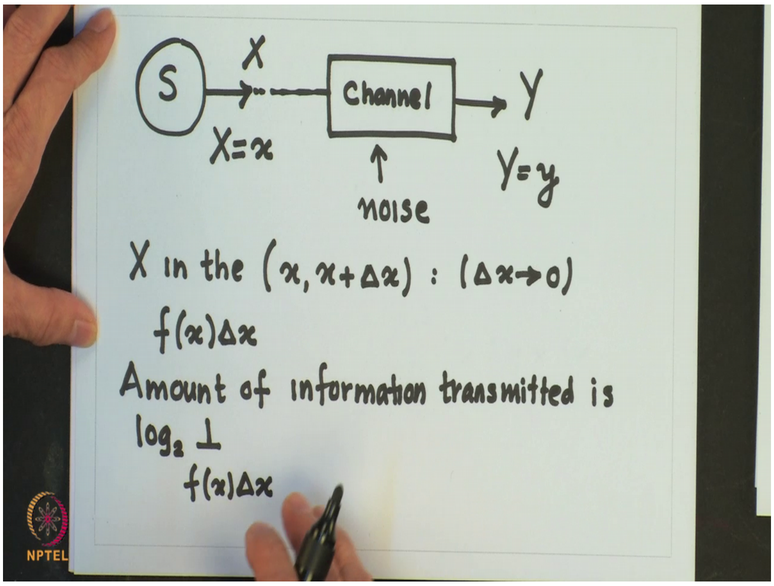

So, a problem is we have a continuous source and we model this output as a random variable X and this is transmitted over a communication channel. So, each value of X in a continuous range is now a message and this message can be transmitted say as a pulse of height X ok. The message recovered by the receiver after it has passed through the channel, let us call it as a random variable Y.

So, if the channel were noise free, the received value Y equal to say small y, would uniquely determine the transmitted value; say X is equal to small x ok. But channel noise introduces a certain uncertainty about the true value of x correct ok. Now consider the event that at the transmitter a value of X in the interval x, x plus delta x has been transmitted there we assume that delta x approaches 0.

So, the probability of this event is f x d x in the limit as delta x tends to 0 ok. So, the amount of information transmitted is log to the base 2 1 by the probability fine ok. Now let the Y value of Y at the receiver be Y is equal to small y.

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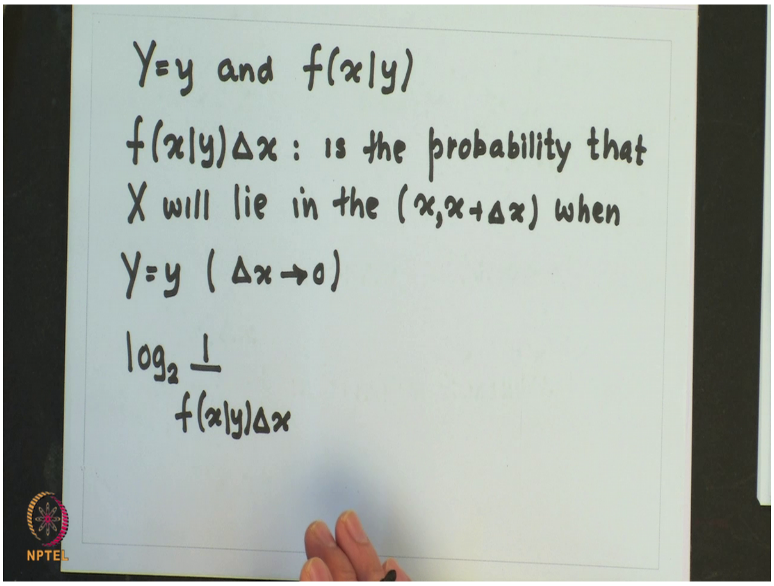

And let conditional probability density function f x given y be specified ok. Then this conditional probability density function multiplied by delta x is the probability that, x will lie in the interval x, x plus delta x; when Y is equal to small y provided delta x tends to 0.

Now obviously, there is a uncertainty about the event that X lies in this interval, this uncertainty which is given by log to the base 2 of conditional PDF multiplied by delta x; arises because of channel noise and therefore, this represents a loss of information correct.

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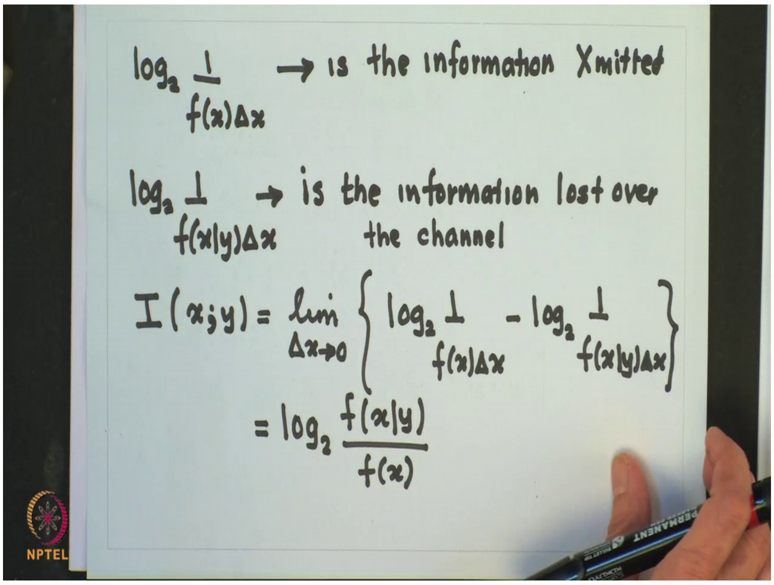

Now log to the base 2 1 by f x delta x is the information transmitted log to the base 2 1 by this conditional PDF multiplied by delta x, this represent the probability is the information lost over the channel or we could also say that this is the uncertainty about the event that x lies in the interval x plus, x and x plus delta x correct after I observe y.

So, this also can be interpreted that way. So, this is the information lost over the channel. So, the net information received which I call it as i x; i x colon y is equal to limit delta x tending to 0 of this quantity, this is the information which I have transmitted and this is the information which I have lost. So, difference gives me basically the information, which I have received or gained. So, this turns out to be log to the base 2 of conditional PDF by the this PDF ok.

So, note this relation is true in the limit as delta x tends to 0. Therefore, this quantity represents the information transmitted over a channel if we receive; capital Y is equal to small y when capital X that is a random variable is equal to small x is transmitted. So, now, what we are interested is in finding the average information transmitted over a channel, when some x is transmitted and certain y is received ok.

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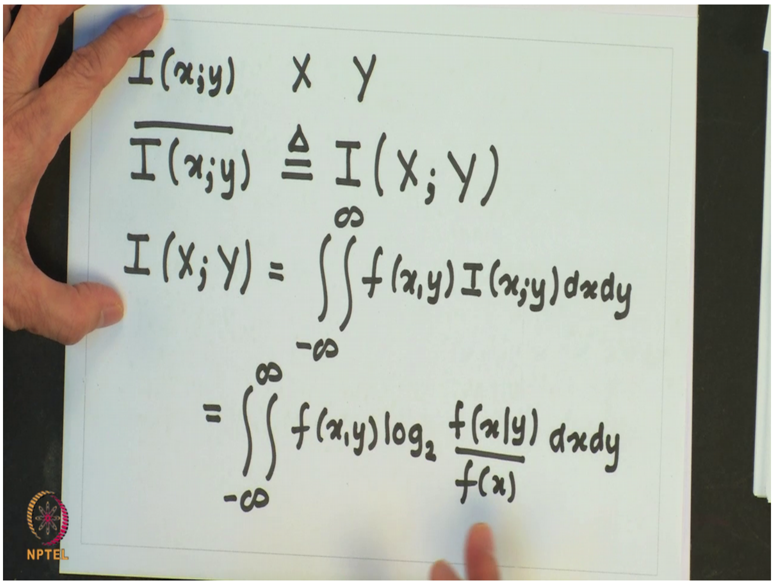

So, we must therefore, average this quantity correct over all values of the random variable X and Y. The average information transmitted will be denoted by then. So, this is the average value and this will be denoted as we had done earlier for the discrete case, mutual information between 2 random variables X and Y but in this case both are continuous ok.

And so, this I can write it as, I have to take the average of this. So, this will be double integral joint PDF multiplied by I mutual information I x y ok. So, this I can rewrite it as we know I x y here; so here is this quantity. So, I am just going to substitute it out there fine. So, this will give me the average information; mutual information ok.

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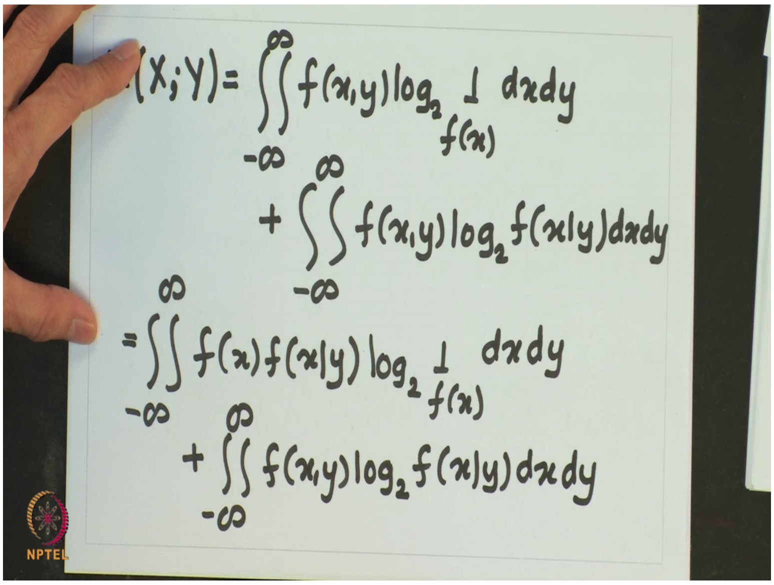

So, now let us try to reorganize this integral as follows. So, I rewrite it as right I have just broken up in 2 terms fine ok. So, this we can again rewrite it as follows. This I can rewrite it as this term this term I can rewrite by bayes' theorem, this is equal to fine. Now again I can rewrite this term. So, the first term in this as follows is can separate it out correct.

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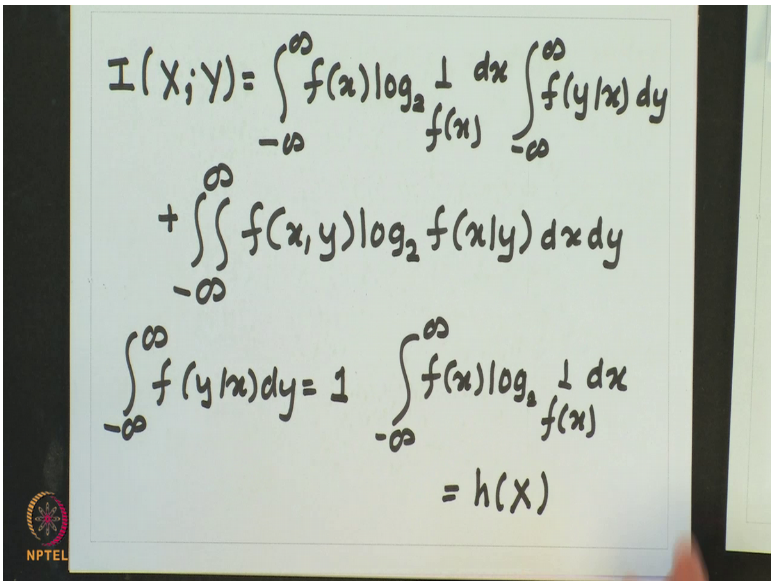

Now, note that this integral is equal to 1 and this integral is equal to the differential entropy of the random variable X. So, if I can do this basically then I can rewrite my mutual information as now this I can again rewrite it just by small manipulation as follows.

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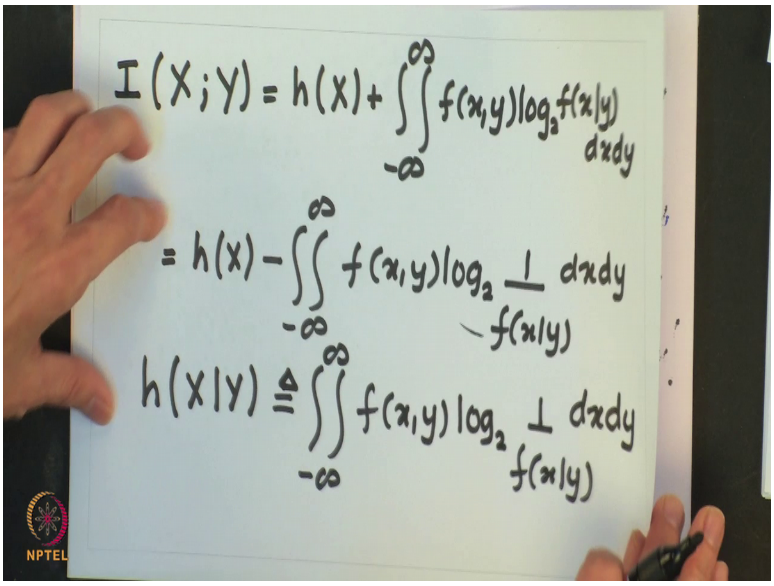

Now note the second term out here correct. So, this is the average over the random variable x and y of this term, log to the base 2 of 1 by f x y correct, but this term basically represents the uncertainty about x when y is received.

Thus we can see that, this is the average information lost over the channel because this you are taking the average of this quantity with a join PDF f x y correct and so, we define this thing as a conditional differential entropy or equivocation of random variable X with respect to Y correct? I by definition this is equal to this. This is similar to what we did in the discrete case. So, this represents the average information lost over the channel.

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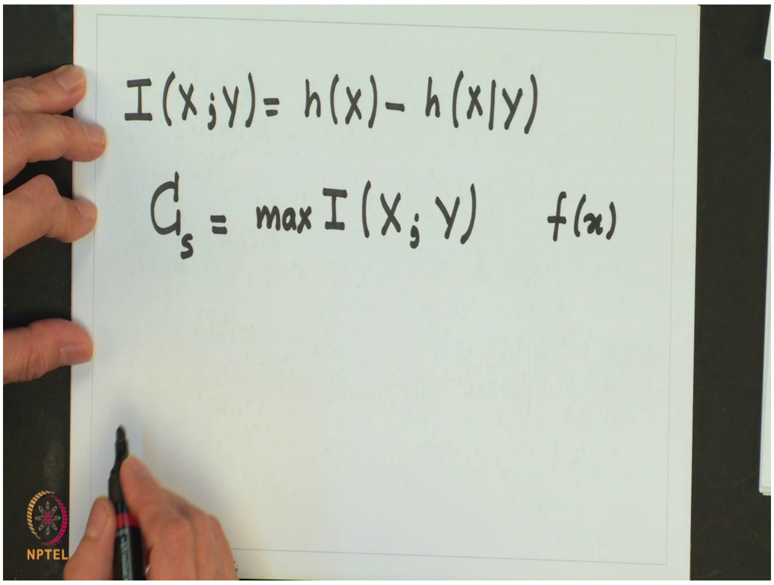

So, finally, what I get? My mutual information to be as equal to differential entropy of the source minus the equivocation of X with respect to Y; where X is the input to the channel and Y is the output of the channel ok. Now so, so this basically represents the average information transmitted over the channel correct? Now we can define the channel capacity C has the maximum amount of information that can be transmitted on the average per sample or per value transmitted.

So, this basically this channel capacity I will define it as maximum of this mutual information correct. Now for a given channel this quantity mutual information is a function of the input probability density function, that is basically f x alone correct and this can be clearly seen as follows; we have f x y is equal to f x f y given x, which I can write it as correct this I can rewrite it as.

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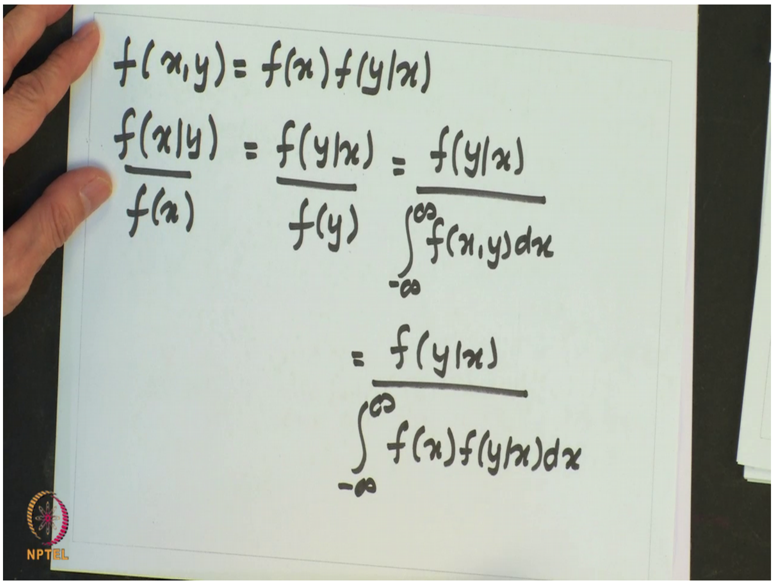

This I can obtain, this marginal PDF I can obtain from the joint PDF by integration and this also I can rewrite it by using the bayes' rule as correct.

So, if I do this basically correct if I use this relationship for this quantity on the left hand side here, what I get my mutual information to be as follows right.

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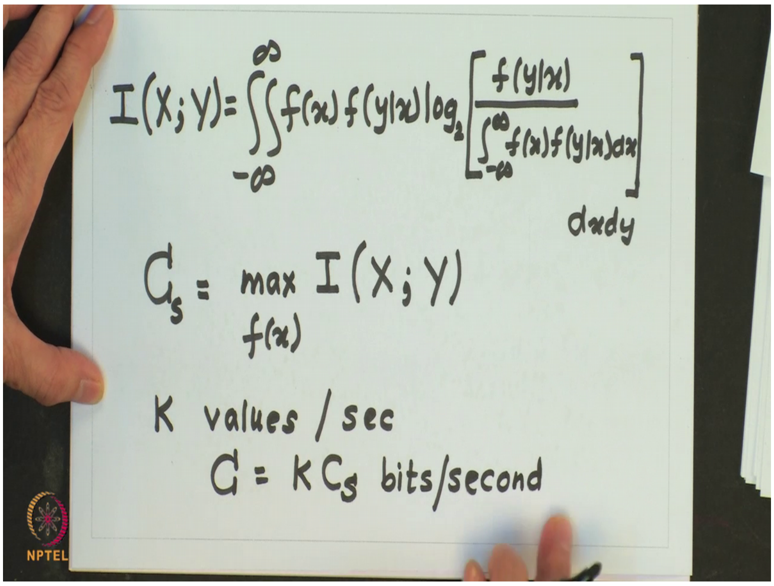

So, this whole double integral is there. Now if you notice this basically out here correct here mutual inform, this basically is a function of only PDF f x for a given channel. Given channel means the conditional PDF has been specified. So, what it means basically that now your channel capacity maximization problem for the given channel reduces to finding out maximum of mutual information over your input PDF f x ok. So, this is what we get from here ok.

So, now let us say that the channel is capable of transmitting K values per second correct. Then the channel capacity per second will turn out to be C is equal to K time C S this is basically per sample. So, this will become bits per second correct. So, this is the channel capacity we I get fine and let us take one more thing see basically look at the mutual information which we had written as follows right.

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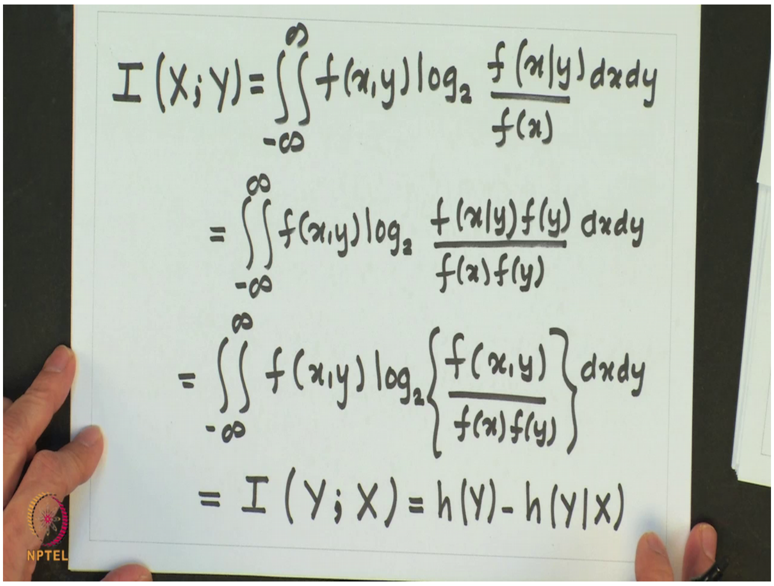

This is what I have for the definition of my mutual information, we justify it mathematically also how this we arrive at ok. So, this I can rewrite it in a different form as follows. I just multiply the numerator and denominator by the PDF of y this is what I get right and this I can rewrite it as, joint PDF log to the base 2.

This equation shows that mutual information is symmetrical with respect to the random variable x and y. So, I can write this then as also mutual information between Y and X and this by definition I can show that this is nothing but differential entropy of random variable Y minus conditional differential entropy of Y given X. So, this result is again similar to what we got for the discrete case.

So, we have seen basically how to define the average information for a continuous source in form of differential relative entropy and then we basically have also seen the concept or the definition of the mutual information when we transmit, the output from a continuous source onto a communication channel. Then we define the channel capacity for this continuous channel and the result we got was that, it turns out that it is maximum over the input PDF of the mutual information ok. So, in the next class basically we will try to evaluate the capacity of a band limited Additive white Gaussian noise channel, which is of importance in most of the practical communication scenarios.

Thank you.