Principles of Digital Communications
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Lecture – 09
Channel Coding Theorem

In the previous class, we computed channel capacity for a binary symmetric channel and discuss the significance of it from engineer's viewpoint. We have seen that the channel capacity is basically limited by the amount of noise, which is there on a communication channel. For a relatively noisy channel the probability of symbol error could be as high as 10 raised to minus 1, which for all practical purposes is not acceptable; what would be expected in a practical scenario would be 10 raised to minus 6 as the probability of error.

Now, to achieve this kind of performance, we resort to what is known as channel coding. So, channel coding is basically a mapping of incoming data sequences into channel input sequence, and inverse mapping the output channel sequence into an output data sequence. The task of channel coding is performed basically by what is known as channel encoder at the transmitter end and the channel decoder at the receiver end.

So, the block schematic of a digital communication system which includes, channel encoder and channel decoder would be something like this.

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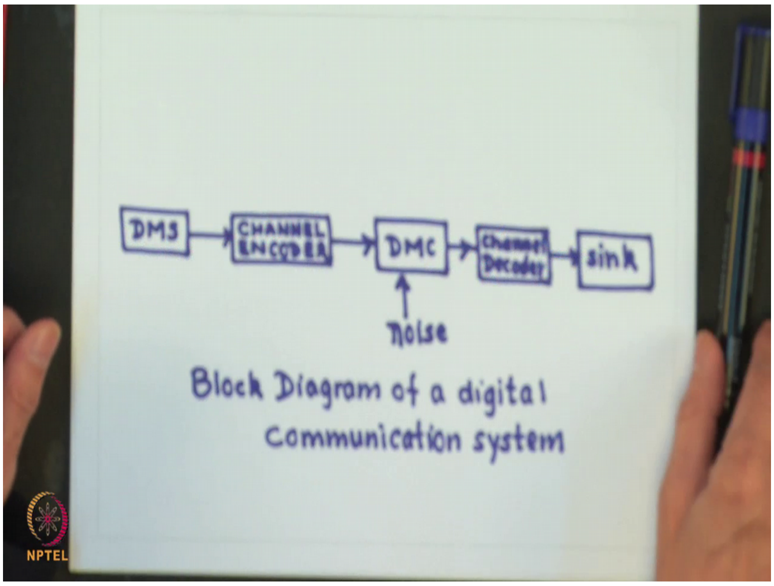

So, we have a discrete memoryless source, in a practical scenario this would be followed by a source encoder, but for the sake of simplicity we have not shown that source encoder and then the output of the source encoder would really go to the what is known as Channel Encoder and then the output of the channel encoder goes to a discrete memoryless channel; this is followed by Channel Decoder which is inverse of what happens in channel encoder and then in a practical scenario this would go to the source decoder, which is again not shown here for the sake of simplicity and then it goes to the sink or destination.

Now, the channel encoder and the channel decoder are both under the designers control and should be designed to optimize the overall reliability of the communication system. Now the approach taken is basically it is to introduce redundancy in the channel encoder in a controlled manner. So as to reconstruct the original source sequence as accurately as possible.

So, you could consider that channel encoding is a duel of source encoding, in source encoding basically we remove the redundancy to improve efficiency of the encoder whereas, in channel encoder the introduced redundancy to improve the reliability of communication.

So, let us see what a channel encoder does and to simplify our discussion let us assume that the input to the channel encoder is consisting of binary sequence that is, binits and what a channel encoder does is that; the input message sequence that is binary sequence which arrives, is divided into a block of k binits correct.

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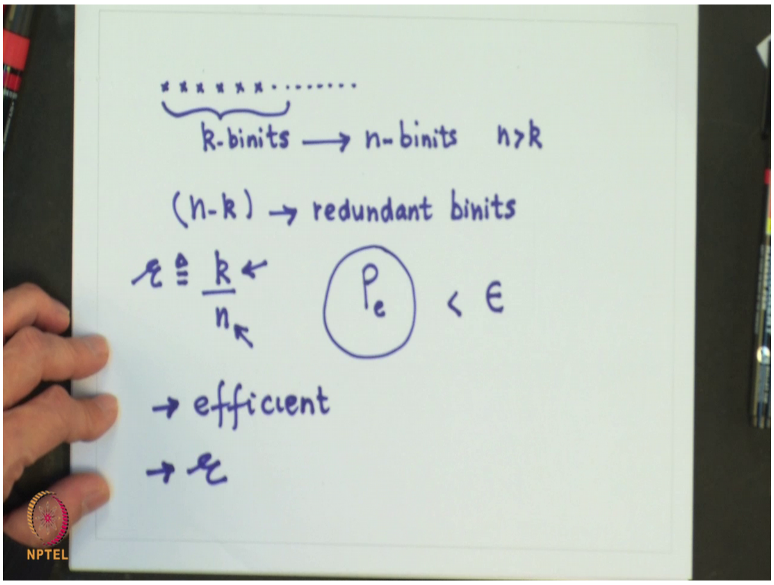

So, again for the sake of simplicity, we will assume the block type of channel coding. So, it takes k binits and then basically it maps to n binits; where n is greater than or equal to k and then so we have n minus k binits has redundant and we also define one parameter what is known as code rate r is equal to by definition k by n.

So, for a prescribed k the code rate r and therefore, the systems coding efficiency will approach 0 as the block length approaches infinity; that means, the number of redundant bits if you keep on adding more and more, the reliability of communication will increase, but simultaneously the code rate will fall down.

Now, the accurate reconstruction of the original source sequence at the destination requires that the average probability of symbol error be as low as possible, in a practical scenario I would like to make it arbitrarily small. So, now the question is that given this scenario does there exist a sophisticated coding scheme such that; the probability that a message symbol will be in error is less than some epsilon; correct this epsilon obviously, is a positive number and this could be as small as we desire. Given this is it possible for me to design a coding scheme which is efficient and the code rate that is r need not be too small. So, I want to satisfy these two conditions. So, I am looking for a coding scheme, which is efficient and simultaneously the code rate is not very small.

Now, is it possible? Fortunately the answer to this is yes and we see basically how is it possible? Ok. So so far in our discussion when we talked about the discrete memoryless source or discrete memoryless channel; time never came into the discussion. As far as the entropy of a source is concerned we always talked in terms of bits per symbol and for discrete memoryless channel we talked in terms of bits per channel use, but transmission of information is obviously, concerned with time also.

So, let us bring in time and we will bring in time as follows. Let us assume that I have a discrete memoryless source and this discrete memoryless source emits symbols from the source alphabet every T s seconds.

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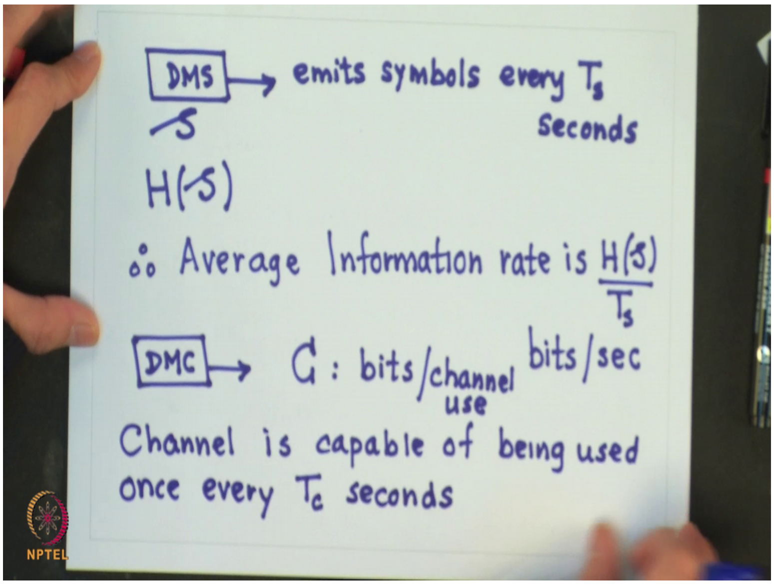

So, with this source we will have an entropy associated with it correct. So, this is the average information which I get per symbol and every symbol is being emitted every T s second therefore, I can say that average information rate is entropy divided by T s and this will give me now bits per second.

Similarly, we have this information being transmitted on a discrete memoryless channel and this has a capacity C which is given in terms of bits per channel use. Now let us assume that this channel is capable of being used once every T c seconds ok. So, this is the condition then I can define my channel capacity in terms of unit time as follows.

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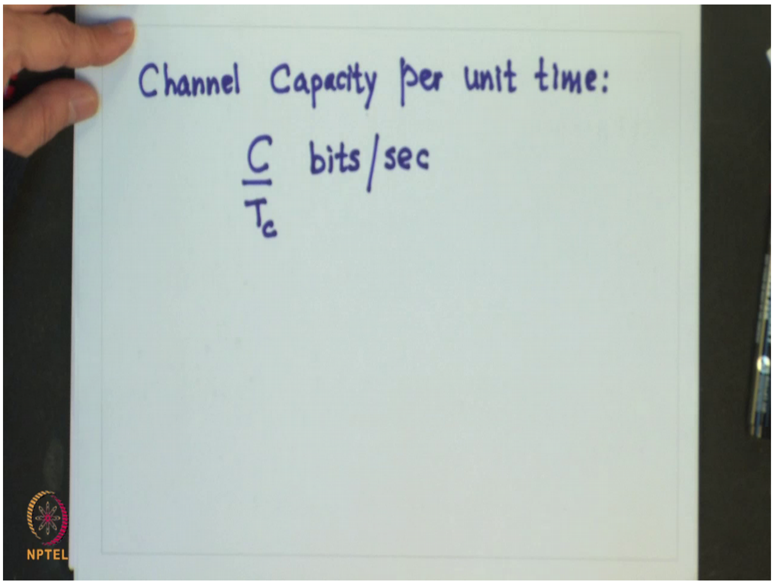

So, channel capacity per unit time, would be given by C by T c s bits per second.

Now, given this we try to answer the question which I had post earlier that is their coding scheme, which is efficient and where the code rate is not very small; when I say efficient means I decide the probability of error epsilon and it could be as small as possible and then I also warned that a code rate should not be too small correct. Is it possible for me to find out that? And the answer to that is given by Shannon second theorem which is also known as Channel Coding Theorem and this channel coding theorem is given in two parts ok.

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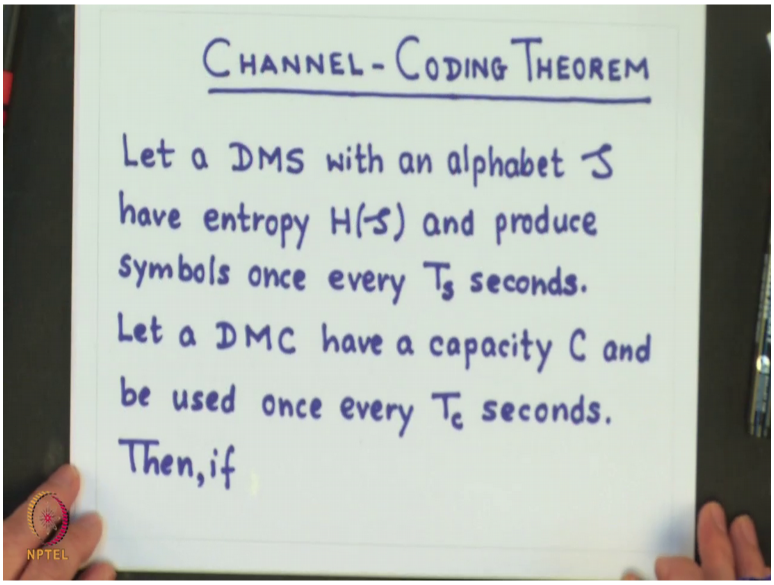

So, what does the theorem say? Let a discrete memoryless source with a source alphabet S have an entropy H S and produce symbols once every T s seconds correct.

Let a discrete memoryless channel have a capacity C and be used once every T c seconds. Then the first part of the theorem says then if this condition is satisfied what it says that, if the information rate which is given by the left hand side of this inequality is lower than the capacity of the channel which is specified in terms of bits per second.

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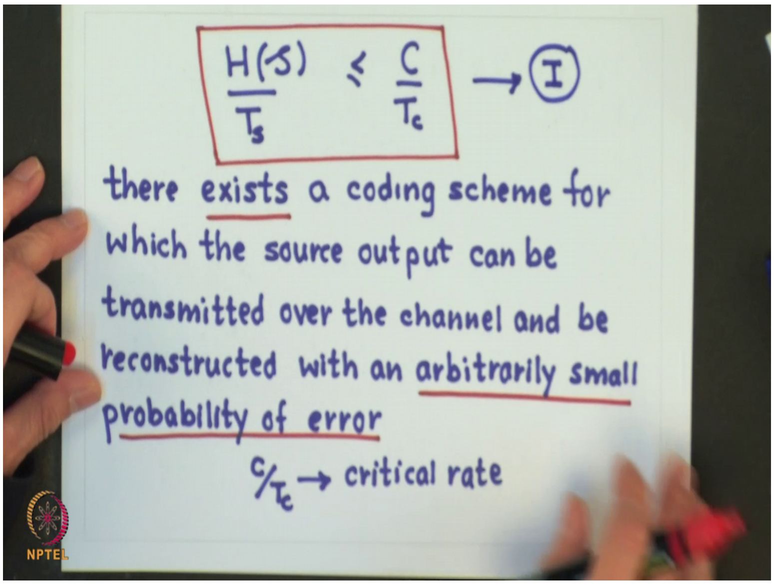

Then there exists a coding scheme for which, the source output can be transmitted over the channel and be reconstructed with an arbitrarily small probability of error correct.

So, and C by T c is known as a critical rate. So, if your information rate transmission is exactly equal to C by T c then what you say that you are signaling at the critical rate. So, this is the first part of the theorem.

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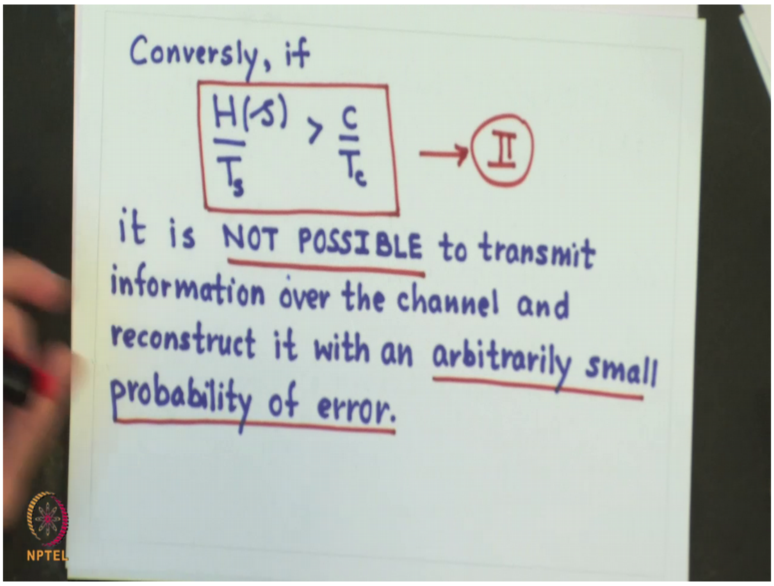

And the second part of the theorem says that conversely if this condition is happens; that means, the information rate is higher than the channel capacity correct. Then it is not possible to transmit information over the channel and reconstruct it with an arbitrary small probability of error correct.

So, these are the thing, but it is important to notice that what these conditions; condition number 1 and condition number 2; this they do not tell us how to construct a good code, it is only an existence proof. It only say exists a coding scheme, but how to construct a good code, satisfying our constraint is not known ok. And the second is that theorem does not have a precise result for the probability of symbol error after decoding the channel output. It only tells that probability of error tends to a 0 as the code length increases again provided the condition 1 is satisfied. So, it is important to note this.

We are not going to the proof of this second theorem by Shannon correct, but let us try to give an application of this channel coding theorem to a binary symmetric channel correct.

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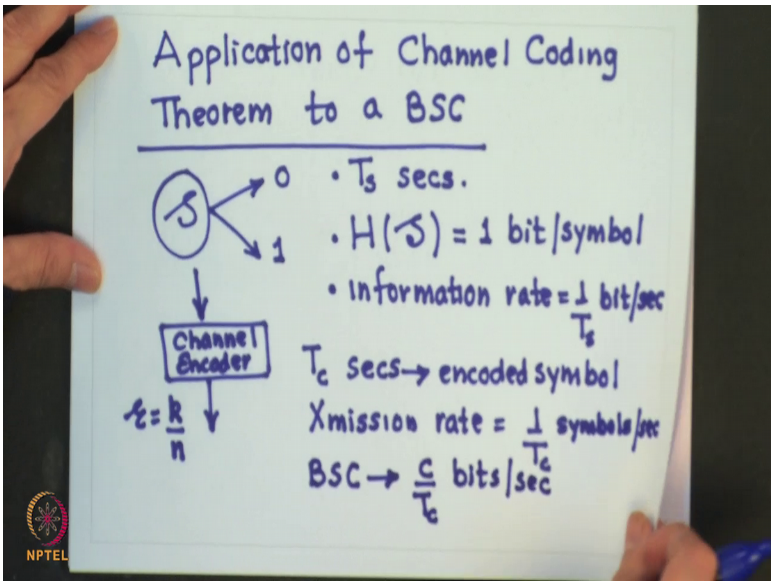

So, we will consider the Application of Channel Coding to a binary symmetric channel. Though we are not going to the formal proof of this Shannon's second theorem, that is channel coding theorem; earlier when we were trying to discuss the physical significance of the capacity, we did discuss about a binary symmetric channel and there we provided some kind of an argument to show that error free transmission is possible; provided by transmission we do not exceed the capacity of a binary symmetric channel correct fine.

So, let us take a source, which is a binary source and it is a discrete memoryless source. So, the two symbols let us consider them to be 0 and 1 and let us assume that the symbols are being emitted at the rate of T s seconds correct because, a discrete memoryless source; the entropy of this would be equal to 1 bit per symbol and the information rate would be equal to 1 by T s bit per second correct. Let us take this output of the source and feed it to the Channel Encoder, which has a rate r equal to k by n. So, we are going to take the input to this channel encoder, in terms of blocks of k binits and the output of that would be n binits where n is larger than k and let us assume that each symbol is basically transmitted every T c seconds correct. So, the transmission rate would be equal to 1 by T c s symbols per second.

So, if you assume that we have a binary symmetric channel, then the transmission rate would be given by C T T c bits per second.

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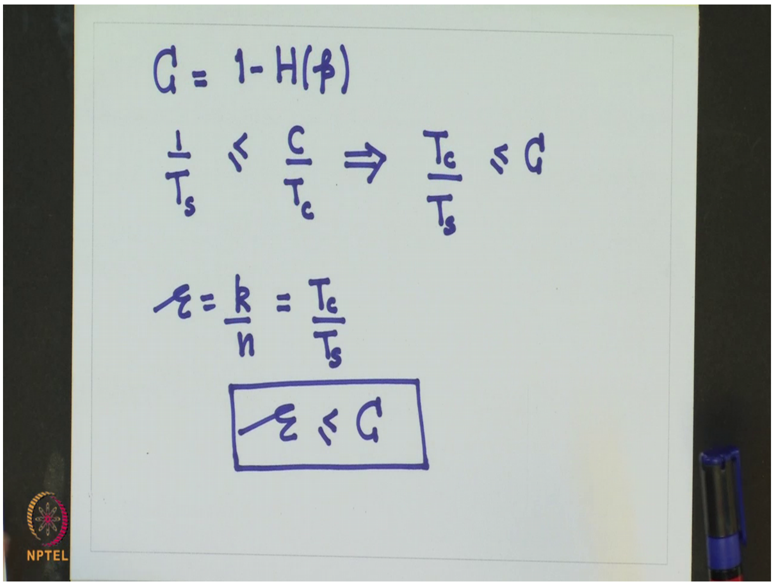

Where your channel capacity for a binary symmetric channel is equal to 1 minus H p; where your p is the crossover probability or transition probability.

Now, what does the channel coding theorem say is that, the information rate which is 1 by T s should be less than equal to channel capacity, which is equal to C by T c. So, this implies that T c by T s should be less than equal to channel capacity C. Now it is easy to see that your code rate, which is equal to k by n. So, k input symbols get mapped to n output symbols, the duration of each symbol here is T s the duration of each symbol here is T c. So, this ratio is nothing, but equal to T c by T s. So, what it implies from this that your rate should be less than or equal to C correct. So, your rate that is given by r correct it should be less than or equal to the channel capacity C fine.

So, now let us take illustrative example to highlight the significance of this channel coding theorem. So, let us say we have a binary symmetric channel with transition probability given as 10 raised to minus 2.

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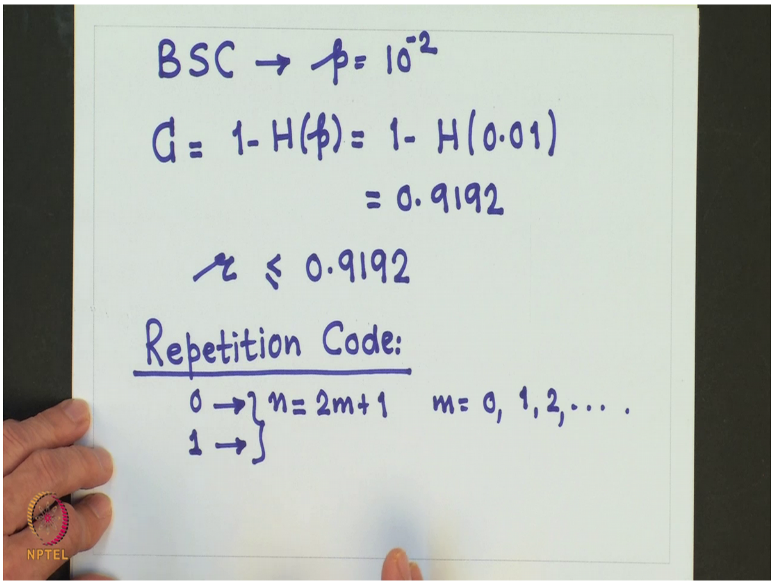

Now for this channel we can calculate the channel capacity as equal to 1 minus entropy function, this is equal to 1 minus H of 0.01 and this comes out to be 0.9192.

So, in the light of the channel coding theorem what it says that, now you should were to find a code with a code rate r which has to be less than or equal to 0.9192 to get almost error free transmission; that means, make the probability of error as small as possible.

Now, let us take a simple example of channel coding in the form of what is known as Repetition Code. Now in this coding what we do is basically any binary digit say 0 or 1 is transmitted n times correct. So, n is equal to 2 m plus 1 where m can be 0, 1, 2 and all that. So, basically n is odd integer. So, at the receiver basically we deploy what is known as; majority logic decoding procedure and the idea is that given n binits, we decide in the favor of 0 if the majority of binits in this sequence of length m is 0 otherwise we decide in favor of 1.

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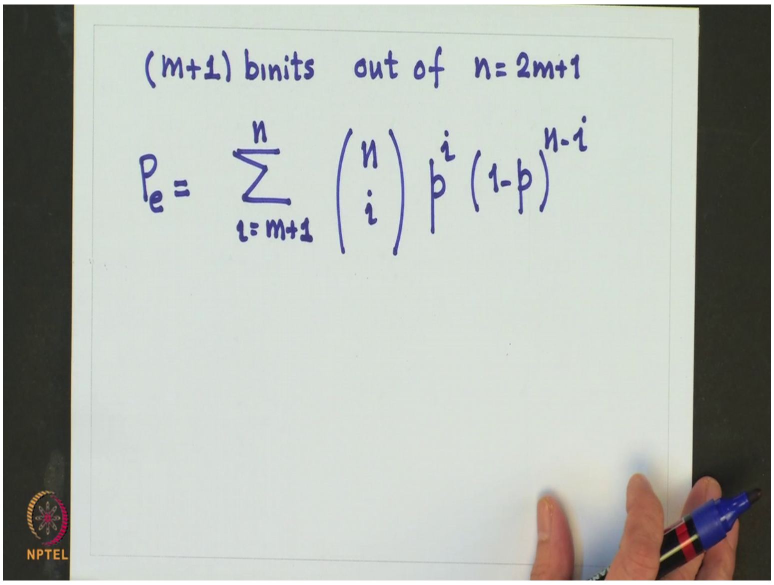

So, in such a case for error to occur, it is required that n plus 1 or more binits out of n binits, which is equal to 2 m plus 1 should go wrong correct. So because the channel is symmetric so in this case irrespective of the prior probabilities of binary digit 0 and 1, it is easy to see that the probability of error will be equal to summation of I is equal to m plus 1 up to n this is n c i combinations, p is a transition probability of a binary symmetric channel, 1 minus p n minus i correct. So, in the sequence of length n this denotes i bits going on and this denotes number of binits received correctly ok.

So, for this basically if we try to now for different value of m or indirectly m, if we try to find out the probability of error we will get their result as follows.

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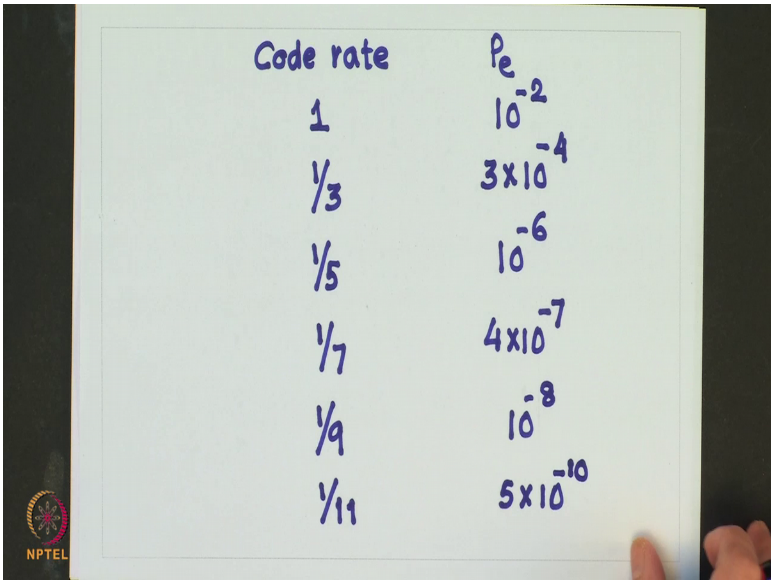

So, the choice of n will decide the code rate. So, when this is equal to 1, this is obvious 10 raised to minus 2 and when n is equal to 3, that mean the code rate is one third you can compute the probability of error turns out to be 3 multiplied by 10 raised to minus 5, code rate 1 by 5. This is 10 raised to minus 6, code rate 1 by 7, this turns out to be 4 multiplied by 10 raised to minus 7, 1 by 9 gives 10 by 8, 1 by 11 gives 5 multiplied by 10 raised to minus 10 correct.

So, if we plot this probability of error versus the code rate, this is the curve which I will get. So, what this curve shows that, as I keep on decreasing my code rate the probability of error also decreases right.

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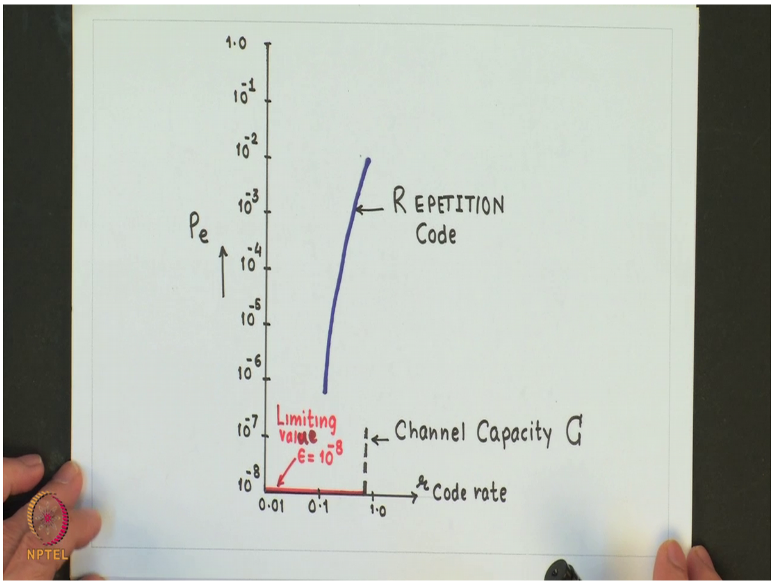

So, but if you look from the channel coding theorem; what we get is this red line. What it says that I have just taken a specific value of this prop this error to be 10 raised to minus 8, is a just without loss of generality correct.

So, this is the Limiting value correct. So, what it says basically that you can design a code or there exists a code, for the code rate less than 0.9192; which is the capacity of the binary symmetric channel, you should be able to find the code, such that the probability of error would be as small as possible that is 10 raised to minus 8 in this case correct, but what I get from the repetition code is this curve correct.

So, only it says this this channel coding theorem helps me to know what is the maximum code rate; which I can achieve and that is equal to 0.9192 in this case. So, far in our study we have limited our discussion to discrete sources and discrete channel. In the next class we will extend this to a continuous case.

Thank you.