Foundations of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-1.
Lecture -3.2.
Ladder of Subspaces.

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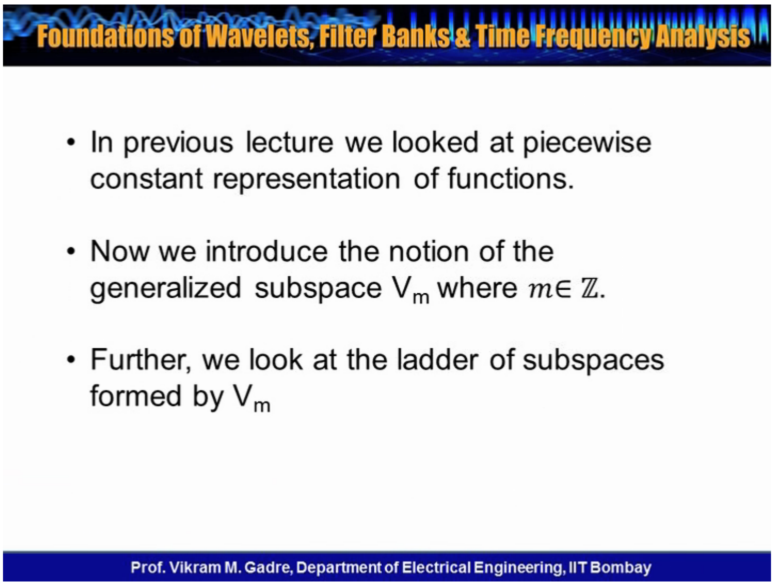

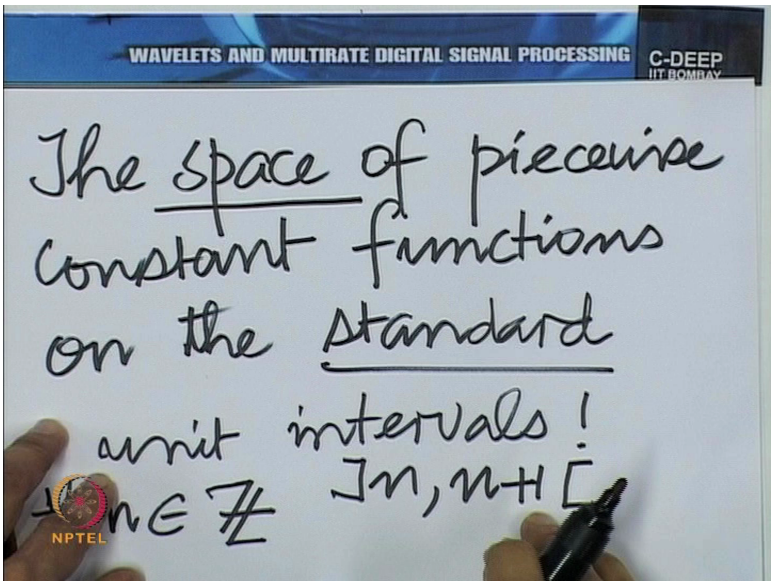

$V_0 : \Big\{ x(t)$, such that

$x(\cdot) \in \mathcal{L}_2(\mathbb{R})$

$\uparrow$ 'belongs to'

and $x(\cdot)$ is piecewise constant on all $]n, n+1[$ $\Big\}$

n integer

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We say V_0, because of piecewise constancy on intervals of size $2^{-0} = 1$

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Similarly

$V_1 : \Big\{ x(t), x \in \mathcal{L}_2(\mathbb{R})$

and $x(\cdot)$ is piecewise constant on standard 2^{-1} intervals $]n \cdot 2^{-1}, (n+1)2^{-1}[$

$\forall n \in \mathbb{Z}$ $\Big\}$

So therefore this set of functions that we talked about a minute ago is indeed a linear space, that is why I have called it a space here. And we will give that space a name. So we will call that space V0. So V0 is a set, now I am going to write mathematical notations. V0 is a set of all x of t, such that, 2 things happen, x of t, now you know x is a function, so when I write like this, what I mean is I am suppressing the explicit value of the independent variable but I recognise there is an independent variable here, then treating the whole thing as an object.

It is a function, I am treating the whole function as an object and this object belongs to L2R, recall that L2R is the space of all functions which are square integrable and this stands for belongs to. So such that x belongs to L2R and x is piecewise constant on all intervals of the form n, n +1, n integer. Now, once we have talked about V0, in fact the reason for giving the subscript 0 here is that we are talking about 2 to the power of 0 as the size of the interval. That is important enough I think to make a noting.

So we say V0 because of piecewise constancy on intervals of size 2 raised to 0 which is one. And similarly therefore, in fact you know you could call it 2 raised to the power of 0,you could call it 2 raised the power of -0. We will prefer to use 2 raised to the power -0 because it will be consistent in future. So we could similarly have V1 then. Similarly V1 is a set of all x, let us define it, the set of all x, x belongs to L2R and x is piecewise constant on standard 2 raised to the power -1 intervals. That is intervals of the form n into 2 raised to -1, n + 1 into 2 raised to -1 for all n integers.

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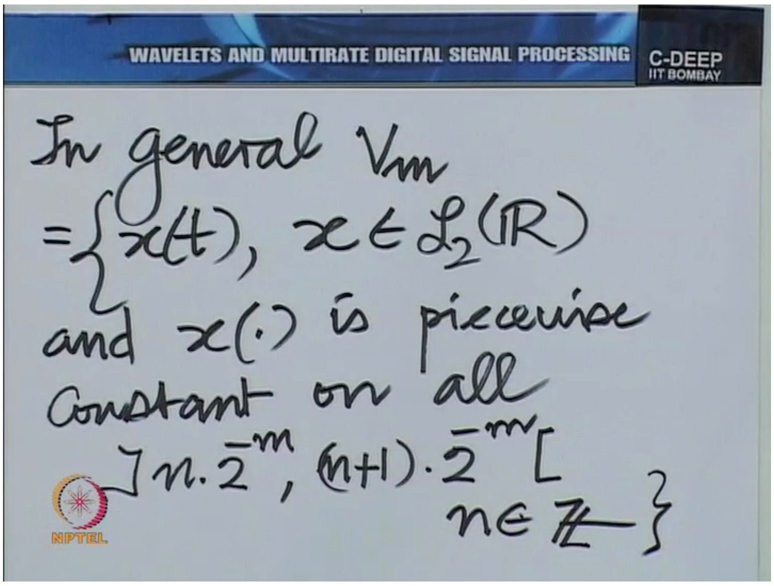
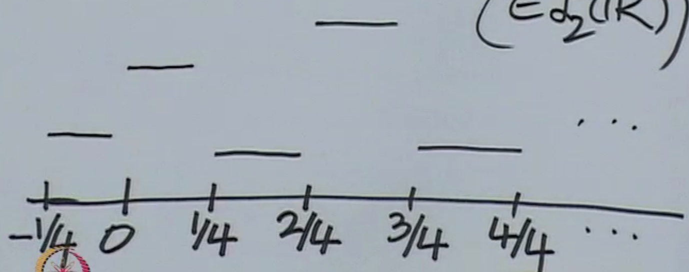


Example of $x(\cdot) \in V_2$

$(\in \mathcal{L}_2(\mathbb{R}))$



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The absolute squared sum of the piecewise constant values, in all V_m, must be convergent $(\in \mathcal{L}_2(\mathbb{R}))$

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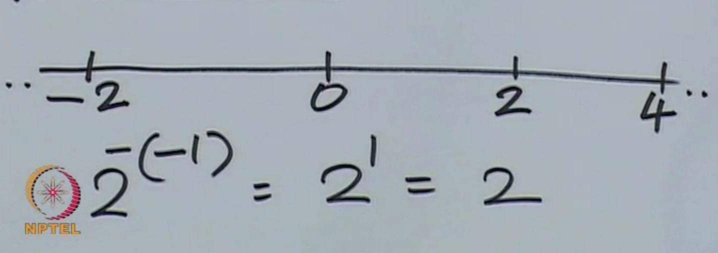
Example of $x(\cdot) \in V_{-1}$



$$2^{-(-1)} = 2^1 = 2$$

In general we have Vm, the set of all xt, for completeness we should write down the definition properly, and x is piecewise constant on all open intervals of the form, simple enough. To fix our ideas, let us sketch a couple of examples. So let us take an example of x belonging to V2. It would look something like this, you would have intervals of one fourth here. And in fact to be complete, we should also include intervals before 0 and so on.

And we have piecewise constancy on these and so on then. And please remember x is also in L2R, so when you say it is in V2, it is automatically of course in L2R and that means that if I take the sums squared, of all these constants that sum square is going to be finite, that is an important observation. The constants that we assign here must be such that when we sum the square of all of them, magnitude square of all of them, that sum must converge. This observation is so important that I think we should make a note of it.

So we are saying the sums squared, the absolute squared sum of the piecewise constant values come in all these Vm must be convergent and this follows from belonging to L2R. Let us also take an example of a function belonging to V -1. So -1 means intervals of size of 2, 2 raised to the power - of -1, so intervals of size 2 and so on there and so on there and we are piecewise constants there and so on here and so on there. So now we get our ideas fixed, what we mean by the spaces VM.

Now the moment we put down these spaces with these examples so clearly, we see a containment relationship, so there is a relation between these spaces, they are not arbitrary, they are not just totally disjoint and unrelated. In fact, you can notice that if a function belongs to V0 for example, which means that it is piecewise constant on the standard unit intervals, it is also going to be piecewise constant on the standard half intervals. And for that matter, if a function belongs to V1, which means that it is piecewise constant on the standard half intervals, it is automatically going to be piecewise constant on the standard one 4^{th} intervals.

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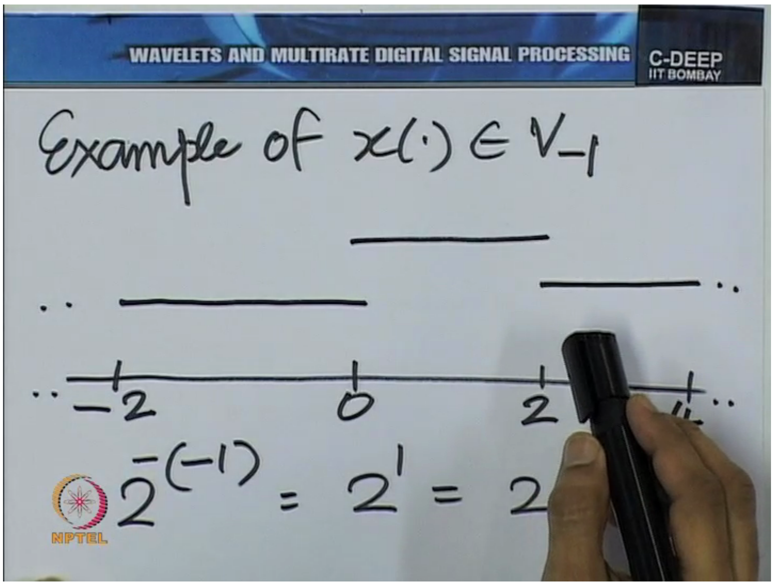

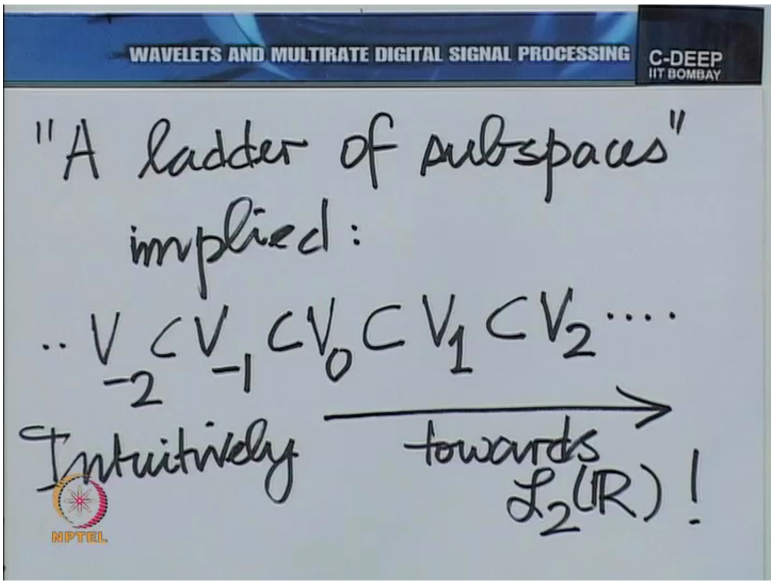

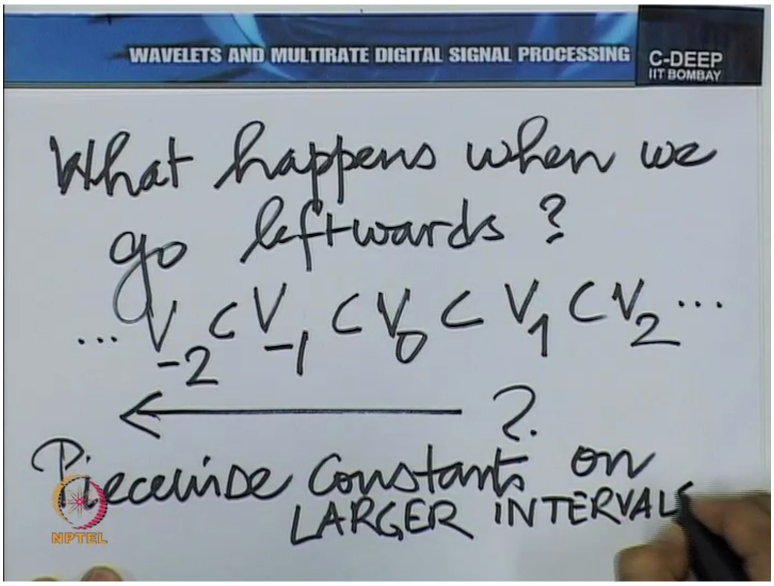

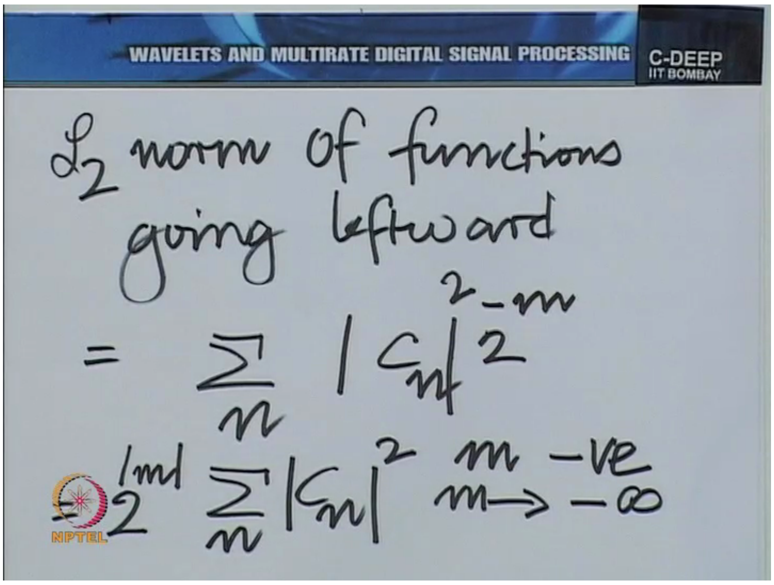

To exemplify this, let me go back to this example of x belong into V -1 that I have here. Notice that this function is piecewise constant on the standard intervals of size 2. So obviously should take the standard intervals of size 1, for example 0 to 1, 1 to 2, 2 to 3, 3 to 4, -2 to -1, -1 to 0 and so on, the function is still piecewise constant. So therefore a function that belongs to V -1 automatically belongs to V0, a function that belongs to V0 automatically belongs to V1.

And therefore there is a ladder of subspaces that is implied here. What is that ladder, the space V0 is contained in V1, the space V1 is contained in V2 and so on this way and of course the space V -1 is contained in V0, the space V -2 in V -1 and so on. And we expect intuitively as we move in this direction, we should be going towards L2R. Of course it is an important question, what happens when we go in this direction. That is interesting; we will spend a minute now and reflect on that.

So you see, what happens when we go leftwards? What I mean by that is, of course you have V0 contained in V1 contained in V2 and then V -1 contained in, I mean contained in V0, yes and so on here and so on there. What happens when we go this way? What do we think should happen, what are we doing, we are taking piecewise constant function on larger and larger intervals, let us write that down, so piecewise constant functions on larger intervals.

Now, you see what is the L2 norm of functions as you go leftwards? What kind of a form will it have? It is going to have a form like this, summation on n, now we see, remember, the L 2 norm is the integral of the absolute square of the function. And please remember the function is piecewise constant, so you have one constant, let us call it Cn on the nth interval and the interval is of size 2 raised to the power - m. So this is essentially, you know you are talking about integrating mod Cn square, it is a constant over an interval of -2 raised to the power - m, and please remember m is negative and m goes towards - infinity as a go leftwards.

That is the same thing as, 2 raised, now you see 2 raised to the power - m is 2 raised to the power mod m in the context of negative m and summation on n mod Cn square. Now you see the subtle point is that if this needs to be finite irrespective of how large m is, we have no control on this, except that this part must be finite. But then when we say finite, if it is nonzero and if we allow m to grow without bound, this is going to diverge.

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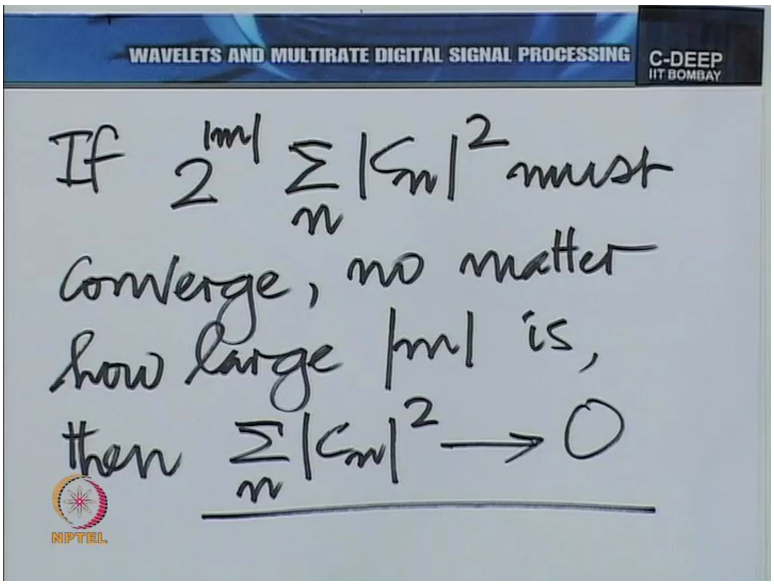

So the only way in which this can converge, no matter how large, I mean, large in the sense, large in magnitude, how large in magnitude m is, no matter how large in magnitude m as if this is to converge, then this must be 0. A very important conclusion. So we are saying that if 2 raised to the power mod m summation n mod Cn square must converge no matter how large or how negative, then we must have summation over n Cn square tending to 0. So essentially what we are seeing is as we move leftwards, we are going towards the 0 function.

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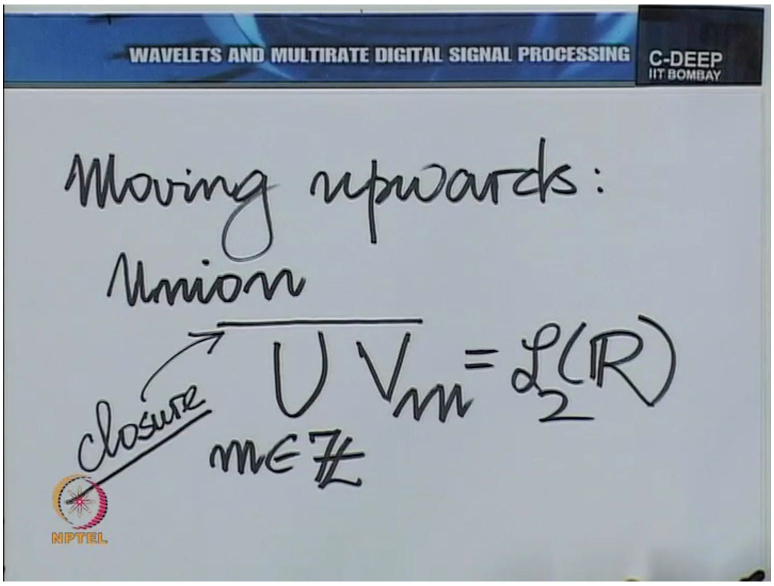

A point that takes a minute to understand but is not so difficult as you can see. So now we have very clearly an idea of our destination, as you move up this ladder towards + infinity and as we move down the ladder towards - infinity, and we can formalise that. What we are saying is moving upwards, now you know one has to use proper notations. We would have been tempted to say something like limit as m tends to infinity or + infinity or something like that but you see it is not really correct to talk about limits of sets.

So we need to use notation that is appropriate in the context of sets, namely union. So when we take a union of 2 sets and if one set is contained in the other, we are automatically taking the larger set. So moving upwards is attained by using union, in other words we are saying the union of Vm, m over all the integer should almost be L2R, now that is where the little catch is.

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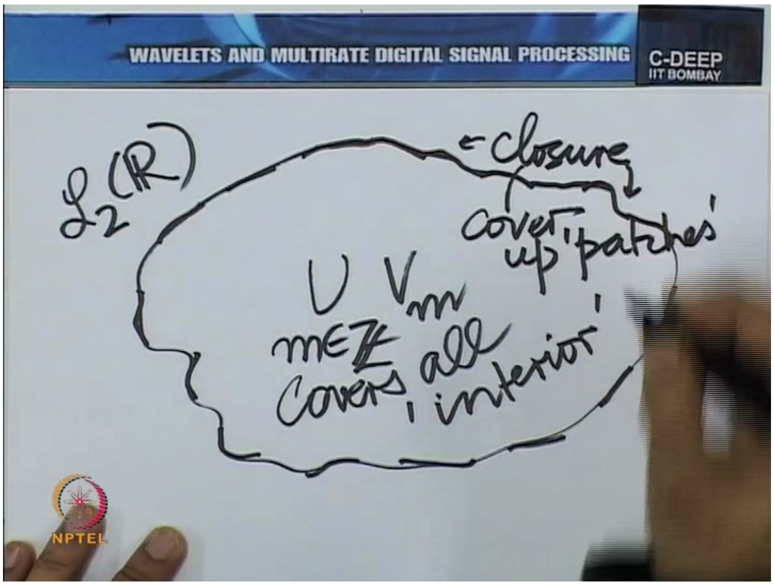

I mean, we would have been happy to write is equal to L2R, but, you know we need to make a little detail here, we need to put something called a closure. I will explain what I mean by a closure. You know, suppose you were to visualize L2R to be like an object with a boundary. So suppose this were L2R, just notionally, and this is the boundary of L2R so to speak, so it is a space you know. Now what we are saying is as we go in union, that is union m Overall integers of Vm, it would cover all the all the inside, cover all the interior.

But then it might leave out some peripheral things on the boundary, so it may also cover some part of the boundary. Now of course, do not ask me at this stage what we mean by boundary and interiors, save that, you know you are talking about situations, you know boundary now, informally, when you say boundary, you are talking about functions where moving in a certain direction does not remain in L2R, moving in the other one does.

So you know this boundary and the interior at the moment needs to be understood only informally. But what we are saying is as far as this union goes, it can take you almost all over L2R, it covers all the interior, it may also cover quite a sizeable part of the boundary but it might leave some patches of the boundary untouched. And therefore when we do a closure, we are covering up those patches. What we just did was covering up those patches, so closure means cover-up boundary patches.

Now this is a small detail and we need not spend too much of time in reflecting about this idea of closure and so on. But to be mathematically accurate, we do need to note that it is after closure that the union overall m integers of Vm becomes L2R. Otherwise, it is almost

L2R, which means that when you take this union, that is when you make piecewise constant approximation is on smaller and smaller and smaller intervals. You can go as close as you desire to a function in L2R, so you can reduce the L2 norm of the function to 0.

So if you look at it, that is what we mean by, that is what mean, that is what is implied by boundary. You know, you can you can go as close as you like to a certain function, you can make the L2 norm 0 but still it would not quite reach there. So you know you could just visualize and you might just be a teeny-weeny bit inside that boundary but not quite on the boundary. And how teeny-weeny, as small as you like, they touch where the union takes you, that is the subtle idea of closure.

Anyway, as I said we do not need to spend too much of time in talking about this closure but we should be aware of this idea because when we need literature on wavelets or for that matter when we really wish to put down the axioms of multiresolution analysis properly, we must be aware that this closure is required. So much so, anyway. Now let us take the 2nd of our inferences here, moving downwards so to speak. So how do we move downwards?

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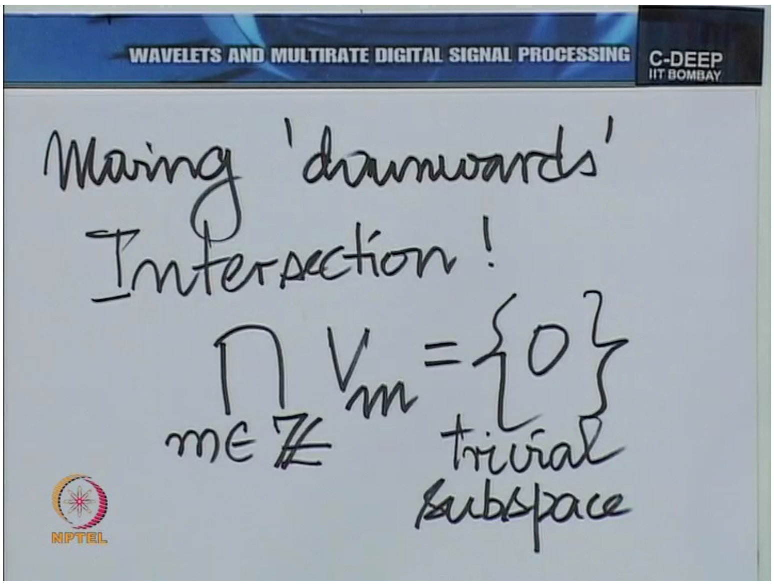

Just as union takes you upwards, Intersection takes you downwards. So if you take an Intersection on all m belonging to Z on VM, there I do not need to worry about closure and anything of that kind. I can simply put down, this is a trivial subspace, essentially the subspace of L2R with only the 0 function included. This is called the trivial subspace. Now again I must make an observation here to clarify. The trivial subspace is not the same as the null space. The trivial subspace has only the trivial 0 element in it, the null subspace does not have any element. So that is a subtle distinction and we must bear in mind that we are talking about the trivial subspace.