Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
Professor Vikram M. Gadre.
Department Of Electrical Engineering.
Indian Institute of Technology Bombay.
Week-8.
Lecture-21.4.
Continuous Wavelet Transform (CWT).

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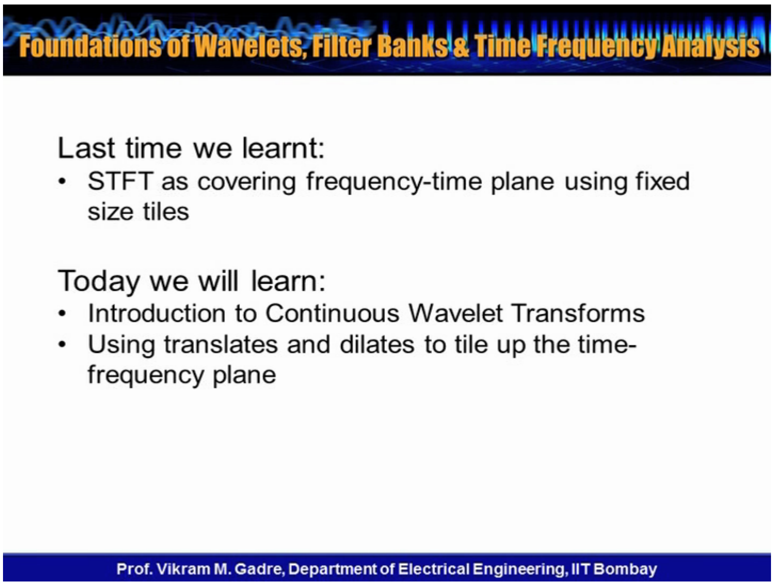

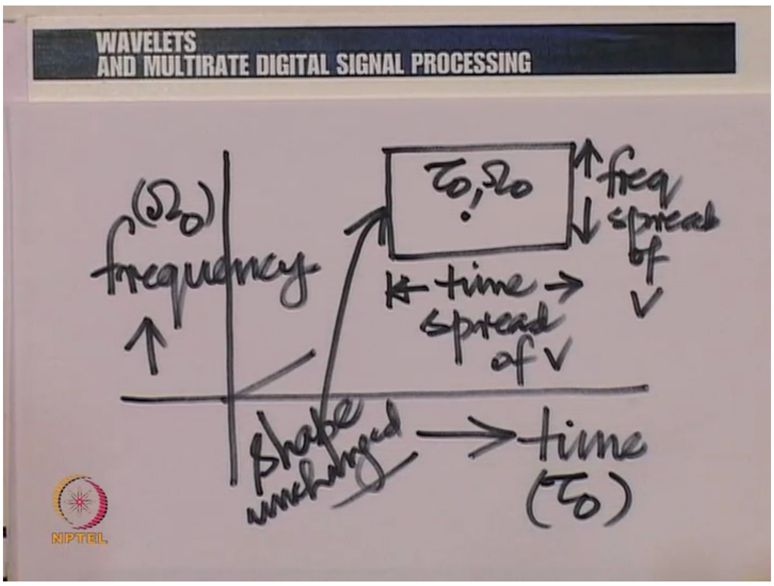

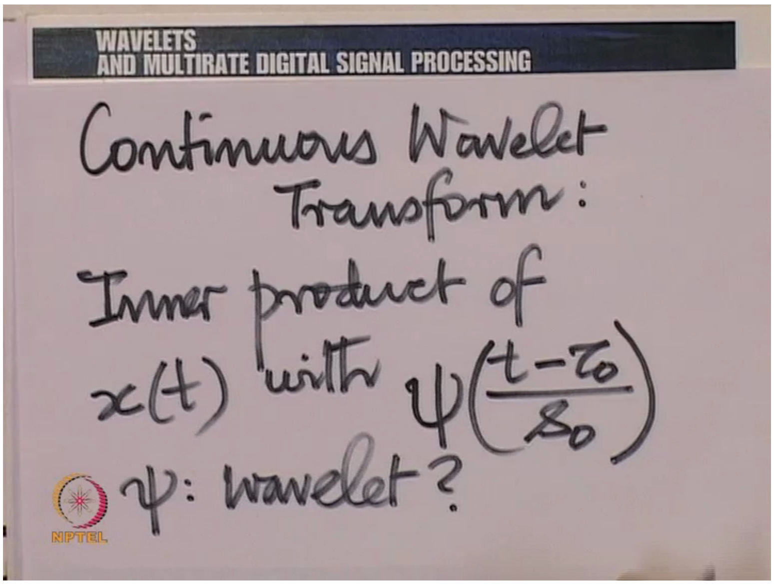

Now in the same spirit, let us look at the continuous wavelet transform. So we are now going to introduce a more general version of the wavelet transform. So far we have been seeing a very specific kind of wavelet transforms, what we call the dyadic discrete wavelet transform. there the scale dparameter is changed in powers of 2, the translation parameter is changed by uniform steps.

the step depends on the scale, so if you look at the Haar multiresolution Analysis, of course the scale is changed dyadically, that means in powers of 2 and the translation is changed in steps of unity when you take the basic or the middle so-called subspace V0. And as you go towards higher subspaces in that ladder, the step size becomes smaller in powers of 2 steps, as you go lower in that ladder, the step size changes again by factors of 2, becomes bigger and bigger.

Anyway, we need to understand this from the time frequency plane perspective. So in general what is the continuous version of the wavelet transform, the continuous version of the wavelet transform is essentially, is essentially a dot product again. It is an inner product of xt with, now this time instead of a window, we have a wavelet, a translate and a dilate of a wavelet. So we take a wavelet psi, we translate it by tao 0 and we dilate it by a factor S0.

Psi is a wavelet, now what on earth do we mean by that, what in general is a wavelet? In fact we shall indirectly postpone the answer to that question for a while until we complete this discussion on the continuous wavelet transform. So what qualifies as a wavelet is a question that we need to answer. Well, we know examples of functions that qualify as wavelets. With tongue in cheek, we will say the Haar function qualifies as a wavelet, I said tongue in cheek because it has an infinite frequency variance. So we have trouble there.

But anyway, as I said tongue in cheek. But we know better examples of wavelets, we know the dobash wavelets for example, may be more difficult to construct but nevertheless there for us, and we have a whole family there of dobash wavelets. So we have examples of wavelets, you know which are restricted in time and restricted in frequency. So one thing that we very clearly understand is that the wavelet function needs to be respected in time and restricted in frequency.

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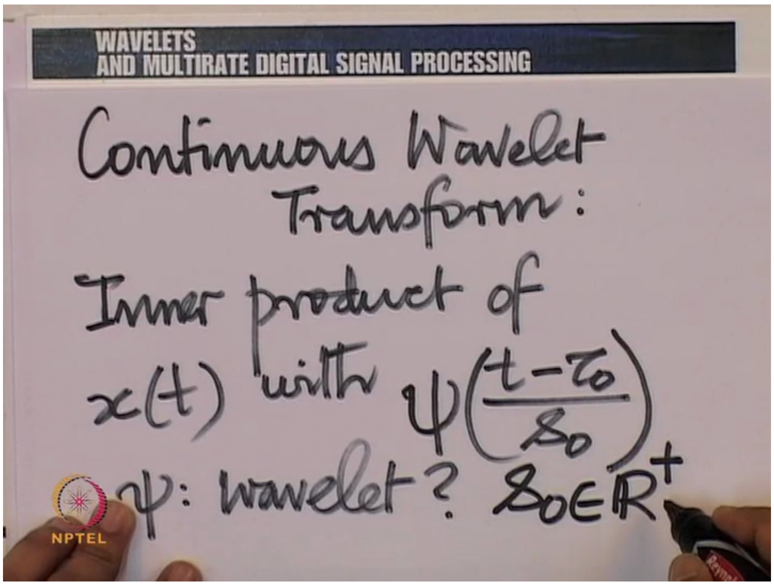

It needs to be a window function in some sense but just any old window function, Ah that way shall take some time to answer. Anyway, so for the moment leaving open the question of what qualifies as a general wavelet, let us come back to this inner product here. We are taking to construct the continuous wavelet transform, an inner product of xt with this wavelet function dilated by S0 and translated by tao 0. Of course, S0 must be nonnegative. So the way to write it is written S0 belongs to R plus, this plus means that S0 is a positive real number excluding 0 and excluding all negative real numbers.

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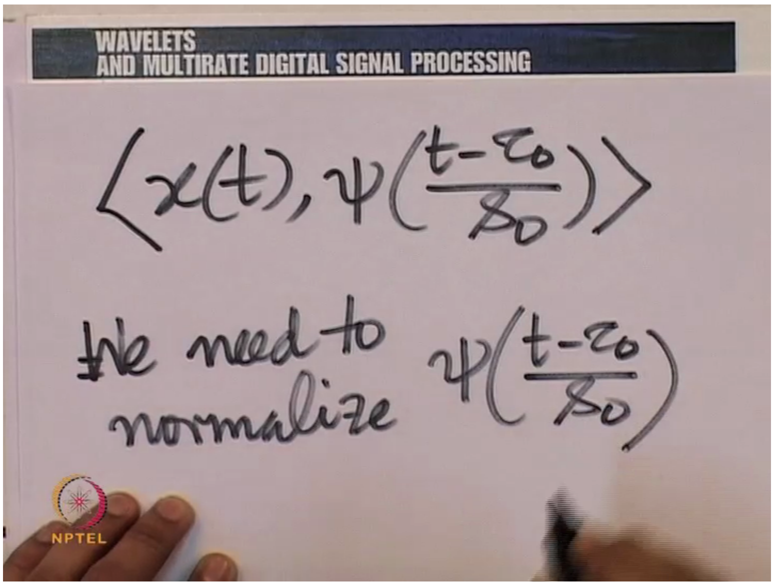

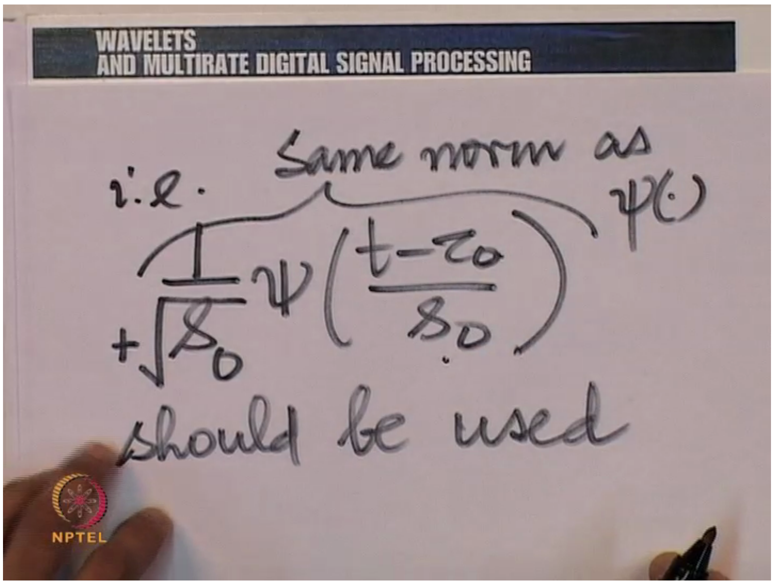

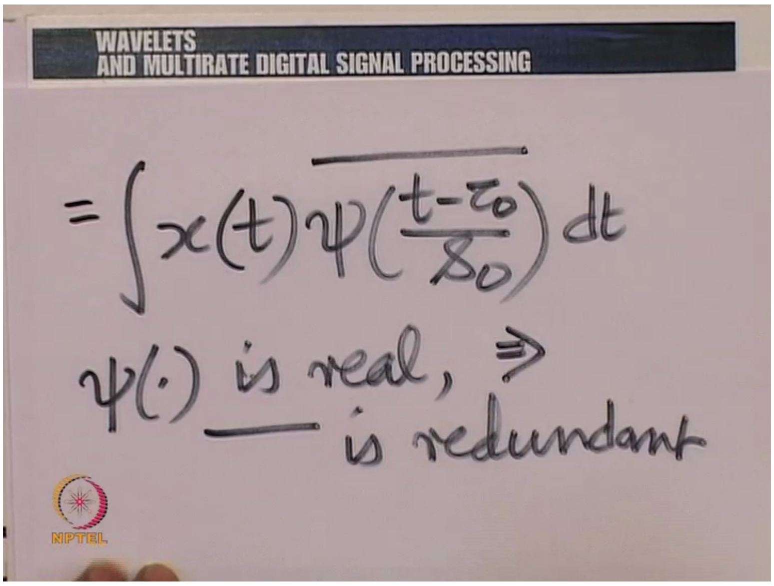

Let us write that inner product down for ourselves. So we are saying this, now again here we need to exercise a bit of caution. You see, if we use this as it is, then we have what is called the problem of normalisation. this problem of normalisation did not come in the short time Fourier transform because when we modulated and when we translated in time, the norm of the function was unchanged. But here when we dilate, than the norm changes and we need to a care of that, so we need to normalise.

And that I leave to you to prove, can be done by multiplying by a factor of 1 by square root of A0. So if we take one by S0 Positive square root times psi t minus tao 0 by S0, then it is normalised, all right. It has a norm equal to the norm of psi t. So this has the same norm as psi t without the translation and dilation. Anyway, so let us construct this dot product. the dot product becomes integral xt psi t minus tao 0 by S0 dt with a complex conjugate on this.

Now of course if psi t is real, then the complex conjugate is redundant. And that is what we had been doing in all the real wavelets that we had been using, ignoring that complex conjugate. Now, let us interpret this also in the frequency domain sense, what are we doing there.

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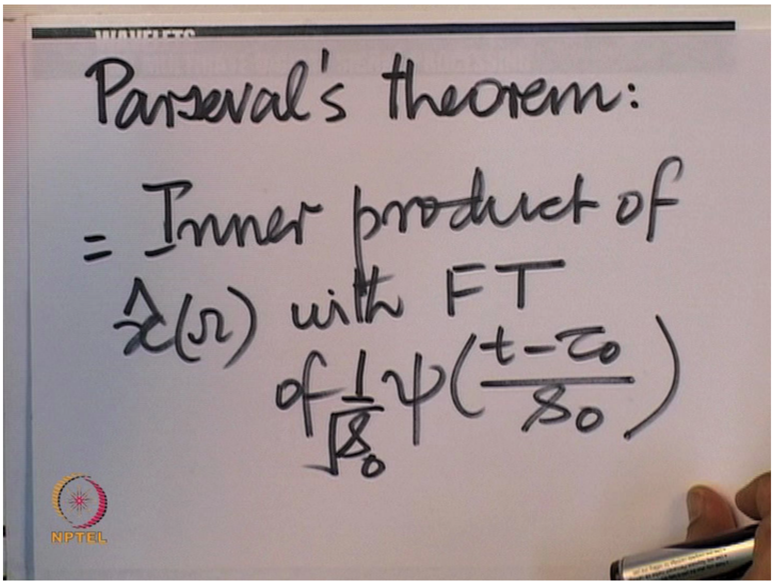

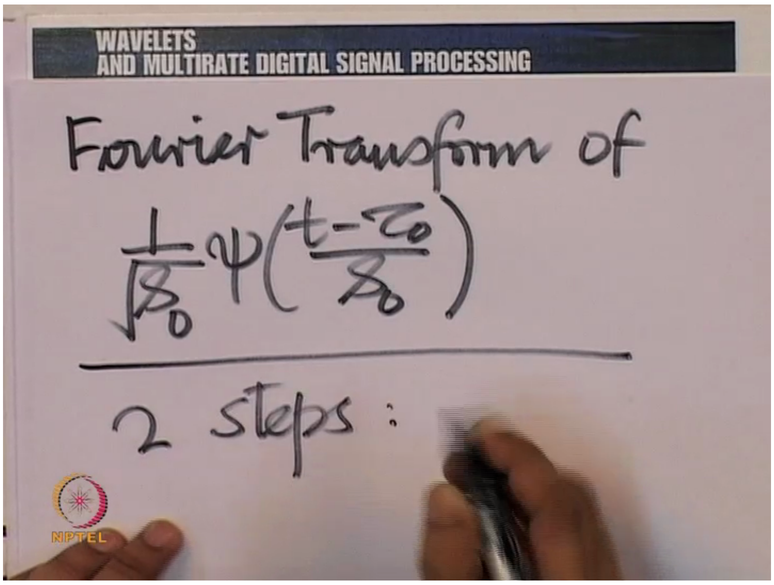

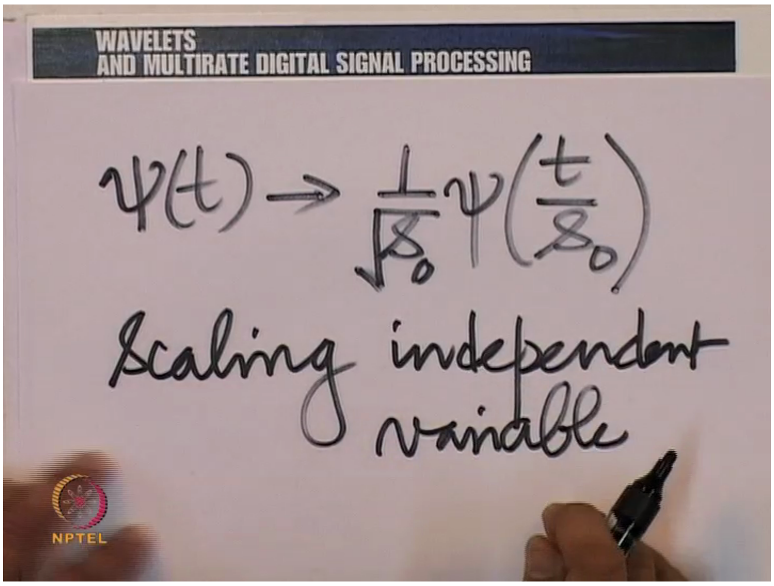

Let us use Parseval's theorem again, this is also equal then to the inner product of x cap omega with the Fourier transform of psi t minus tao 0 by S 0. And let us evaluate the Fourier transform of t minus tao 0 by S0, of course normalised with this. And here again, although I did not need it inside the integral sign, I should keep the one by S0 to the power of half outside the integral sign. So I must bring in the factor here for completeness. So let us evaluate this. Now, the Fourier transform of 1 by S 0 square root psi t minus tao 0 by S0 can be calculated as follows.

We will do it in 2 steps. We will 1st go from psi t to 1 by square root S or square root S0 psi t by 0 and we will make use of the property of the Fourier transform pertaining to scaling being independent variable. So we will make use of that property 1st. So it is very easy to see that the Fourier transform of 1 by S0 to the power half psi t by S0 becomes S0 to the power of half psi cap S0 omega. So because of the normalisation, the one by square root of 0 here and one by S0 become square root of S0 here and S0 there.

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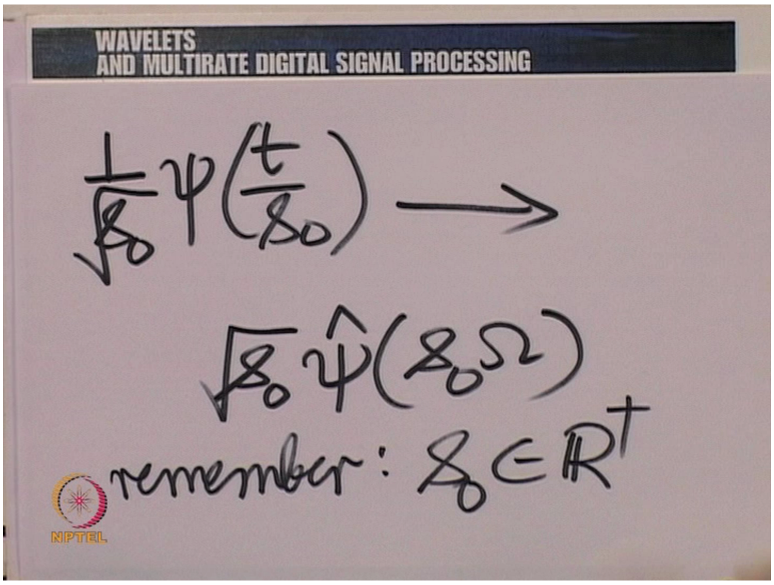

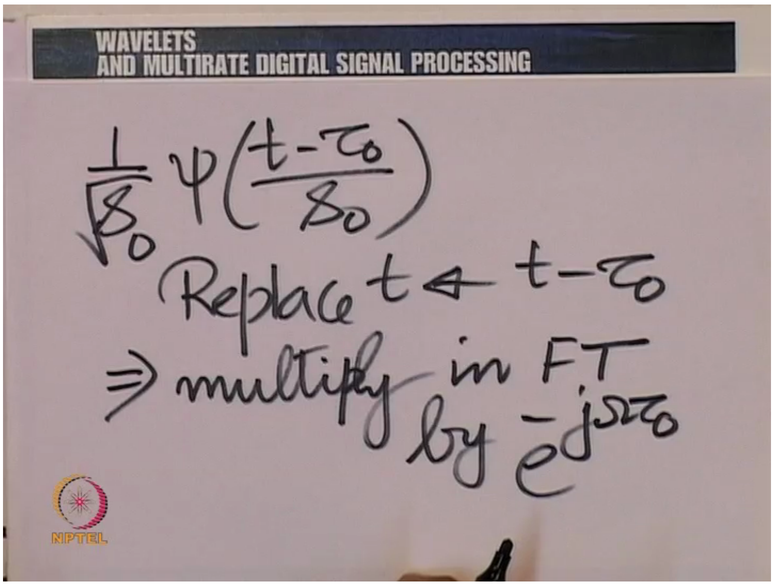

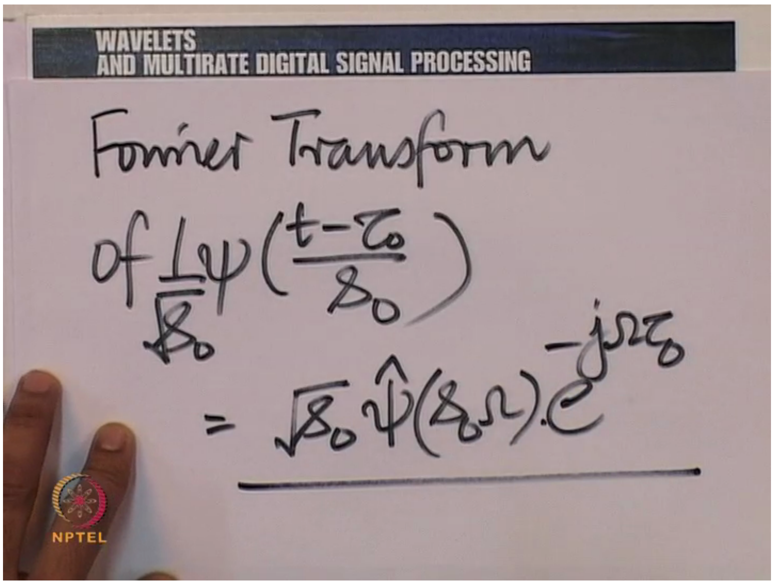

And of course remember S0 is positive and real. So we do not need to worry about the sign, modulus is not required. Now if we wish to find the Fourier terms of 1 by S0 square root psi t minus tao 0 by 0, all that we are doing is to replace t by t minus tao 0 and that amounts to multiplying in the Fourier domain by e raised to the power minus J Omega tao 0, so we will do exactly that.

Therefore we have the Fourier transforms is of psi t minus tau 0 by is 0 into 1 by S0 square root is square root of S0 psi cap S0 omega multiplied by e raised to the power minus J Omega tao 0. And we put this back in the expression that we had for the continuous wavelet transform. Remember the continuous wavelet transform is a continuous function of the translation tao 0 and the scaling S0. S0 is only positive real, tao 0 is both negative and positive real, let us emphasise this.

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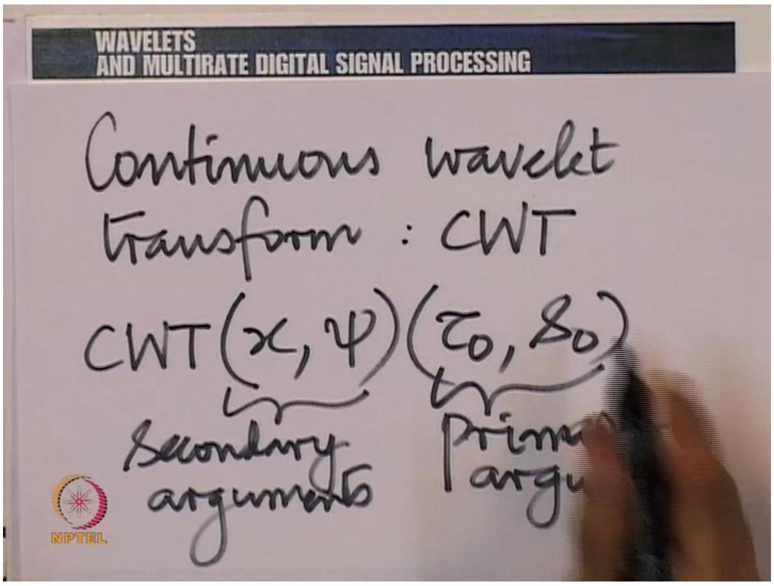

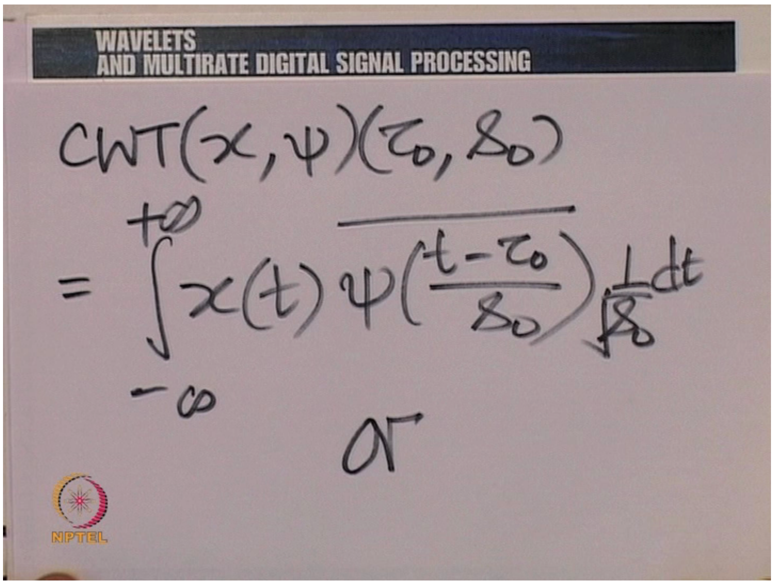

So let us write both of these down and let us introduce notation here. The continuous wavelet transform which we shall abbreviate by CWT, now here again it has primary and secondary argument. So CWT will have the secondary argument x and psi and the primary arguments tao 0 and S0. And this reads as the continuous wavelet transform of the function x which presumably belongs to L2R with respect to the wavelets psi evaluated at the translation tao 0 and the scale S0.

So this quantity, CWT of x with respect to psi evaluated at tao 0 and S0 is either, if you wish to look at it that way, the dot product of xt with psi t minus tao 0 by S0 normalised with 1by S0, either this or, as we have just seen using Parseval's theorem the following.

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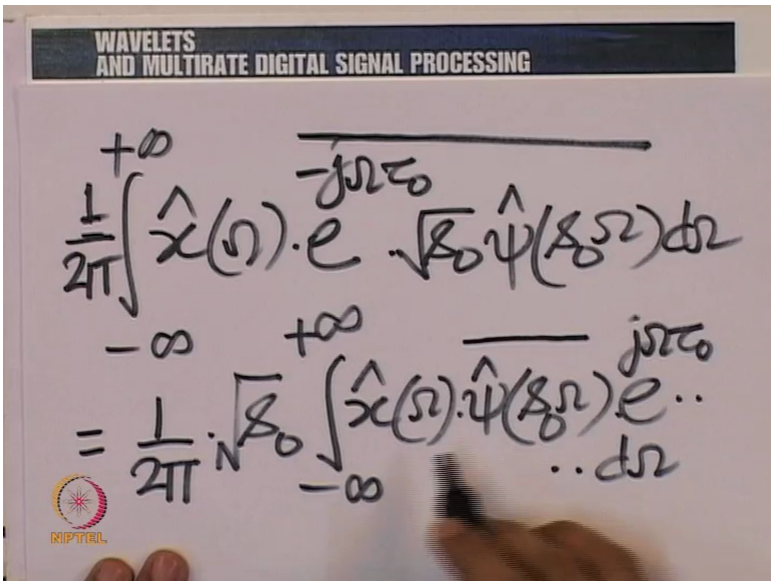

Of course with the factor of 1 by 2 pie, so let me keep the factor of 1 by 2 pie here. This is of course complex conjugated, so I need to rewrite this little bit. There is a complex conjugate here in general, I have taken care of the complex conjugate here by replacing the minus by plus. Now this has a very interesting interpretation, provided we recall the nature of the Fourier transform of psi from the examples that we have seen so far.

You will recall that if we considered the Haar wavelet for example, psi t was a band pass filter, in fact just to recall, let me put down the magnitude pattern of the Fourier transform of society in the Haar case. Recall the Haar case, it had a magnitude Fourier transform which looked something like this, which had the 1st null at 4 pie and subsequent else of side lobes at all multiples of 4 pie beyond and the main lobe essentially was a band between 0 and 4 pie.

So it was a bandpass function, by a bandpass function I mean it did not have a non-null Fourier transform at 0, as omega tends to infinity, again the Fourier transform decays towards 0. So the Fourier transform magnitude is 0 at 0, 0 at infinity and maximum at somewhere in between, it emphasises a band of frequencies. We thought that especially in the case of a Haar function. I encourage all of you to effortlessly calculate the Fourier transform of the dobash 4 wavelet for example, it will be interesting to do, it has to be done numerically.

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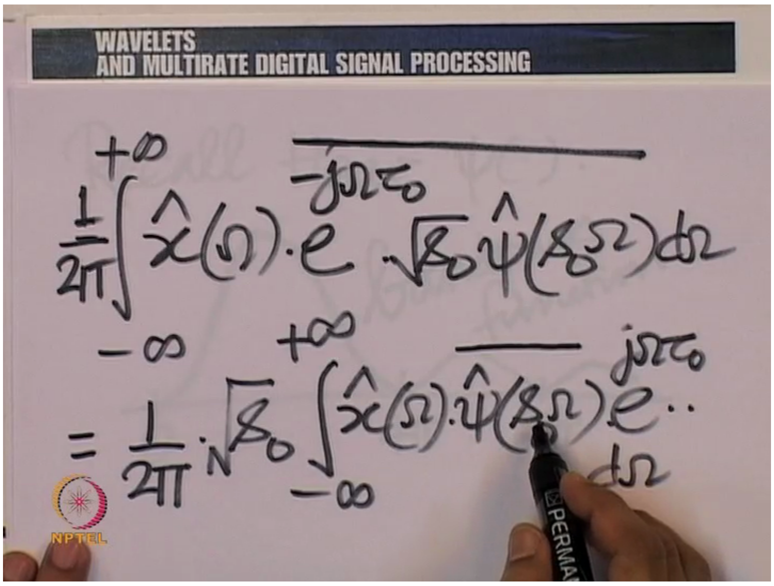

And verify that that would also have this bandpass character. So, we see the trend in these so-called wavelet functions. They have a bandpass character and if we allow the interpretation, then what we have written here has a beautiful meaning. It means that we are multiplying the Fourier transform of x with a bandpass function, scaled in the Fourier domain by the factor S0 and we are calculating the inverse Fourier transform of the same. Of course this factor square root of S0 is here to normalise.

Now if you accept that psi is a bandpass function, then what you are doing here is essentially to extract a region of frequencies in the Fourier transform of x which lies around the capropriate dilates of that Fourier transform of psi and you are calculating the inverse Fourier transform. The inverse Fourier transform, this integral after multiplication with e raised to the power J Omega 0, omega tao 0 with respect to omega essentially means the output after doing this work in the frequency domain. So what we are saying in effect is the following.

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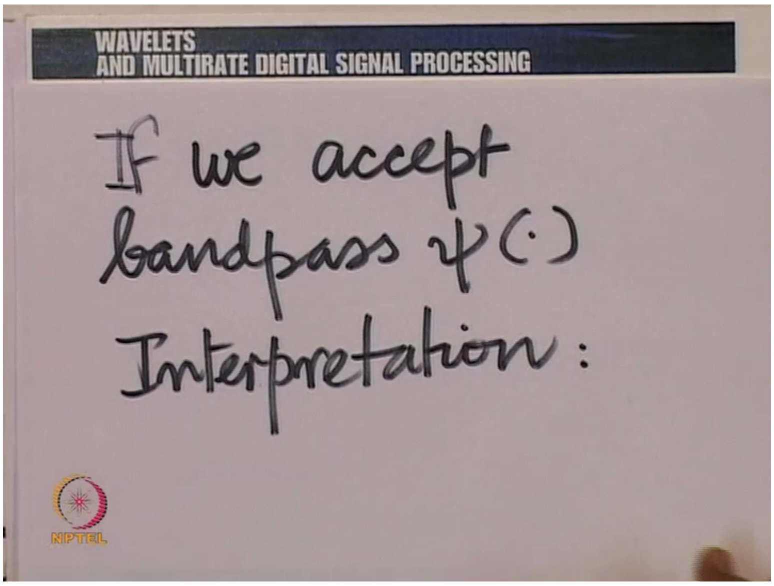

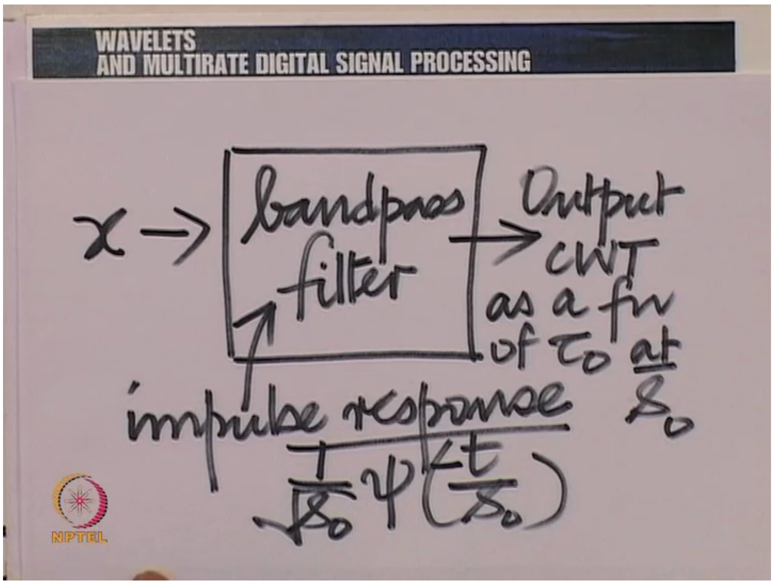

We are saying that in effect if we accept that psi is a bandpass function, then the interpretation is as follows. In the continuous wavelet transform, we are taking x, we are passing it through a bandpass filter whose impulse response is essentially 1 by S0 square root psi t by S0. Well, if you like, one should complex conjugate this, because you are doing a complex conjugation there as well. So you complex conjugate this and strictly you should also put a - sign here.

Because this is, this would be the inverse Fourier transform when you complex conjugate in frequency and then scaled by S0. The output is the CWT as a function of tao 0 at the scale S0. So at every scale there is a different filter. You have a continuum of filters indexed by S0. For every scale S0 there is a different filter. It extracts information in x cap around the Centre frequency capropriately scaled by S0.

And the band is also scaled, remember, when you scale the Centre frequency, also scale the band, recall all this discussion in the Haar, now we are doing it for a general bandpass function. And that inverse Fourier transform is operated by tao 0, so you are calculating the output that each point tao 0, this is the interpretation of the continuous wavelet transform. So with that then we come to the end of this lecture where what we have seen is essentially the definition and the interpretation of the short time Fourier transform and the continuous wavelet transform. Thank you.