Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-8.
Lecture-21.2.
STFT: Time domain and frequency domain formulations.

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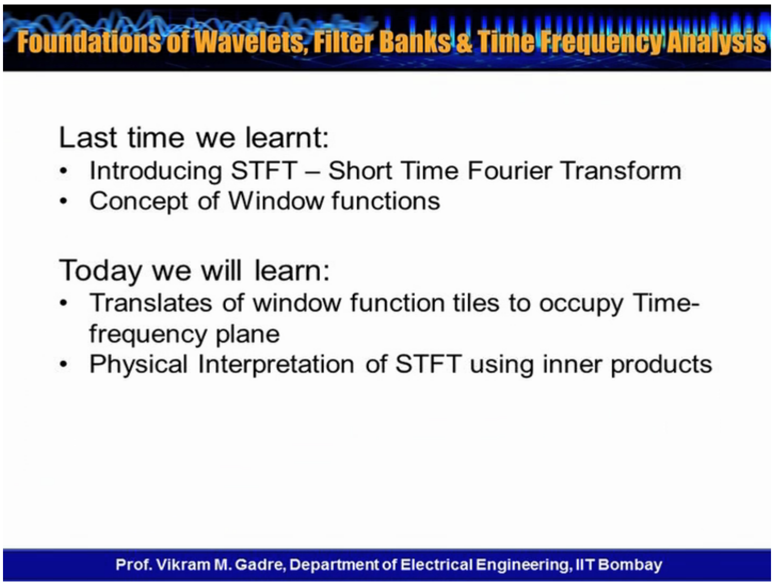

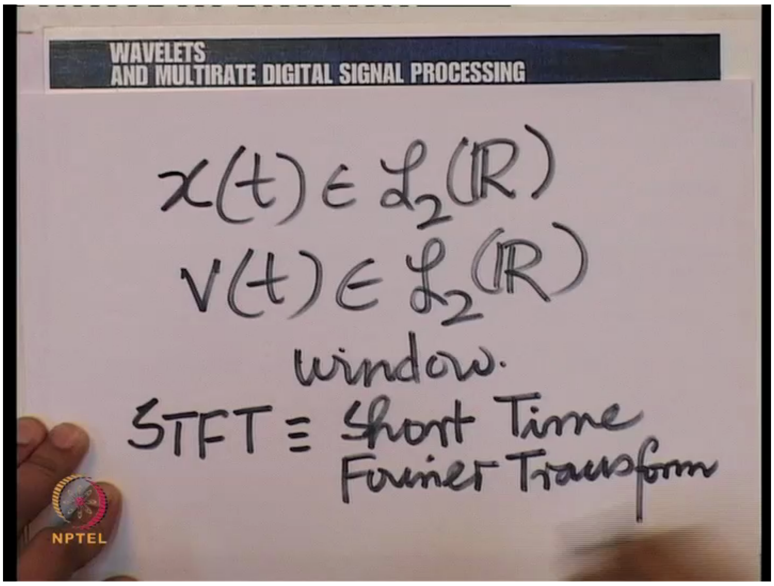

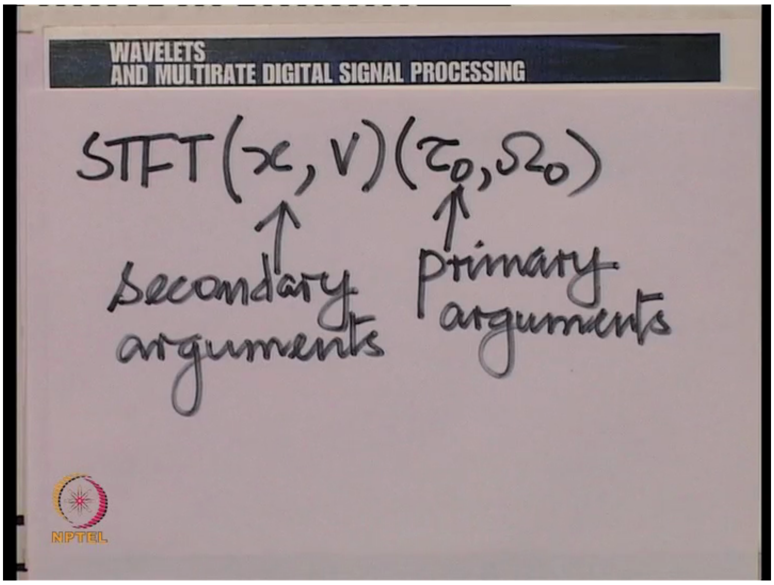

Anyway, with that background let us now use as tiles translates modulates of this window. So what we do is to construct a continuum of the following kinds of dot products. We have a function xt belonging to L2R and we have chosen a window vt, the short time Fourier transform of xt, so the short time various transform is abbreviated often by STFT. So let us use this abbreviation now onwards. The short time Fourier transform or STFT of x with respect to V as we shall call it.

STFT of x with respect to V, so we have 2 arguments, the function whose short time Fourier transform is being constructed and the window with respect to which it is being constructed. Now this is essentially the set of arguments which determine what short time Fourier transform we are constructing but there is also a pair of arguments for this transform which are the translation and the modulation arguments. So these are I would call the so-called primary arguments of the short time Fourier transform.

And these, one should call the secondary arguments. To explain this idea of primary and secondary arguments, let us take for example the Fourier transform. In the Fourier transform, the primary argument is the frequency, the angular frequency capital Omega. The secondary argument is the function whose Fourier transform is being calculated. So when you calculate the Fourier transform of xt, the secondary argument is the function x, the transform is the whole operator.

It takes this function x as a secondary argument and the primary argument is the angular frequency. So with that little explanation let us go back to the short time Fourier transform of x with respect to the window V evaluated, this is how we would read it, the short time Fourier transform of x with respect to the window V evaluated at the translation tao 0 and the modulation Omega 0. Let us write that down. It would be essentially a dot product of the following.

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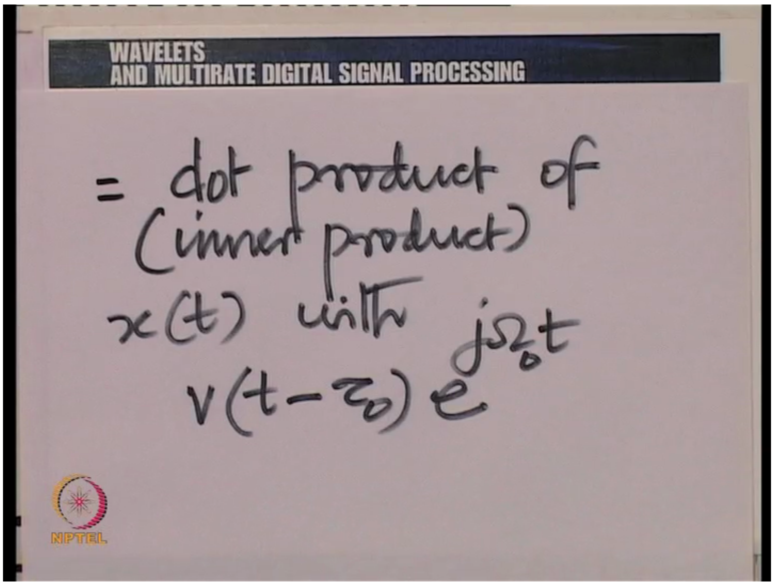

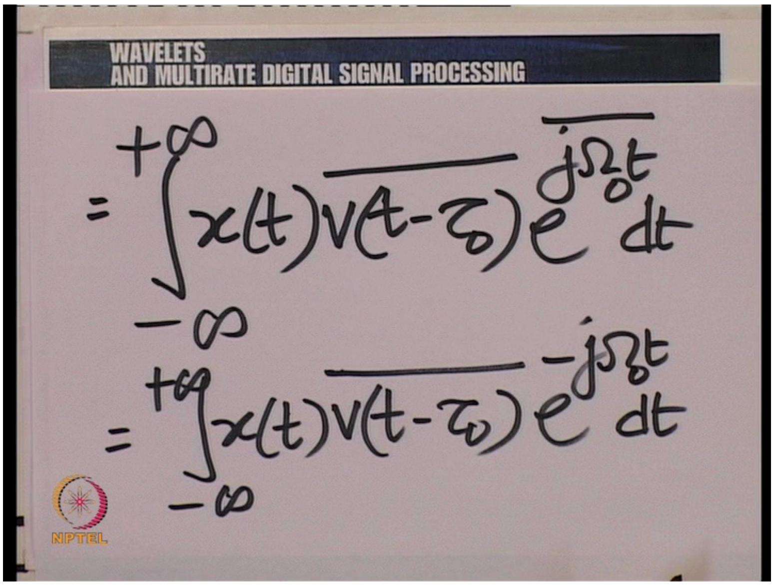

Dot product or the inner product of xt with V translated by tao 0 and modulated with e raised to the power J Omega 0T. And how would dot product look, let us write it down. xt multiplied by vt - 0 bar e raised to the power J Omega 0T bar dt. And if you please, we could simplify this. Now a couple of words here. If vt is the real function, as we have been considering all this while, whether it is a Gaussian or whether it is a raised cosine of the triangular function or if you like the extreme case of the rectangular window, this complex conjugation is redundant, it is not required.

If you look at it, what we are doing here in alternate sense is to 1st multiply xt by a window appropriately translated, so tao 0 is the location of the window. Essentially what we are trying is to extract information of xt around the point t equal to tao 0. And then we are taking a Fourier transform, that is another way to interpret it. The short time Fourier transform is essentially a process of piecing or breaking the function into pieces followed by a Fourier transformation.

Here we are assuming a continuous tao 0 and a continuous capital Omega 0. And therefore we are talking about the continuous short time Fourier transform. Now, let us invoke Parseval's theorem to get different perspective on the same expression here. So let us go back to the expression here. We have taken a dot product of xt with a translated version of the window and then a Fourier transformation on this product. So for example if vt were to be a triangular window, essentially it would extract the information of xt around the point tao 0 with some weighting done by the rectangular, by the triangular function or the rectangular function.

In the rectangular function, there is no weighting, in the raised cosine or the triangular or the Gaussian function there is a weighting, different weights given to different points around tao 0, followed by a Fourier transformation, so looking at the frequency content in a certain region. Now if we invoke Parseval's theorem on this, what does it give us?

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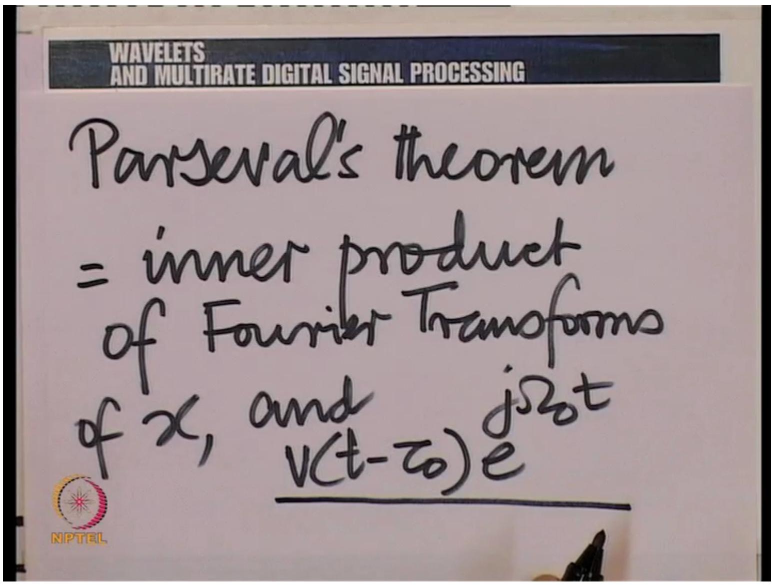

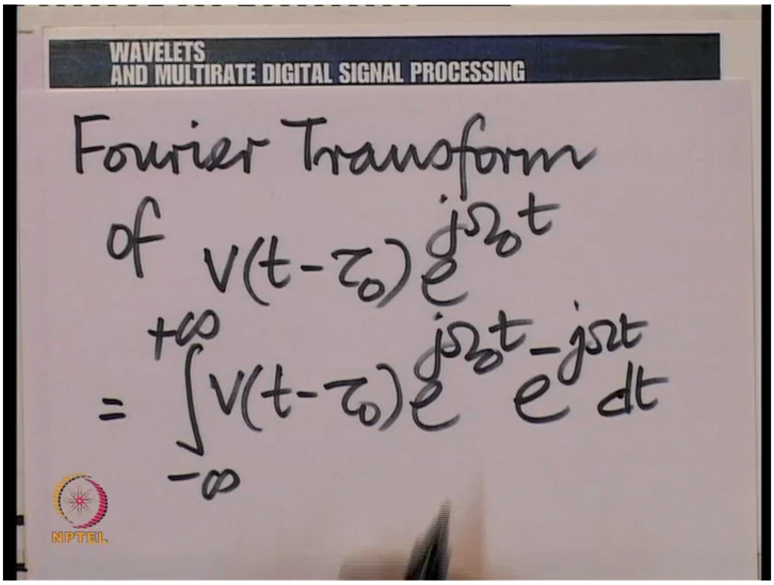

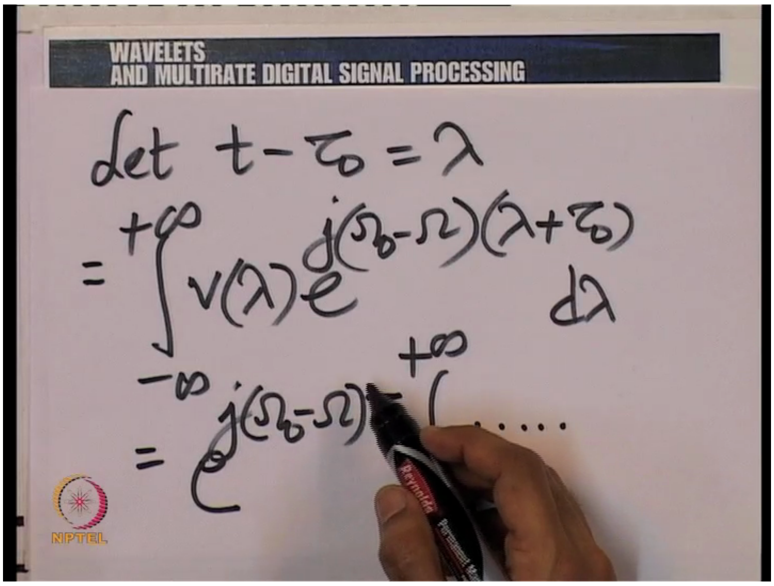

Parseval's theorem says, this is also the inner product of the Fourier transforms of x and V translated by tao 0 and modulated by capital Omega 0. Therefore we need to calculate the Fourier transform of this quantity, that is a task that we need to do. The Fourier transform of vt minus tao 0 e raised to the power J Omega 0T can easily be seem to be the following. So it is minus to plus infinity vt - 0 is a so the power minus, well J Omega 0T multiplied by e raised the power minus J Omega t integrated with respect to t.

And let us simplify this, so of course we employ the standard process of replacement of argument. So let us replace the argument t minus tao 0. And noting that for a fixed tao 0, when t runs from minus to plus infinity, Lambda also runs from minus to plus infinity, this would become integral from minus to plus infinity V Lambda e raised to the power J, now of course collecting terms, this is Omega 0 minus Omega times t which is lambda plus tao 0 d Lambda.

And now of course we have a simple answer that. We can rewrite this little bit, taking out common terms. So you will notice that when we expand this product, we have that e raised to the power J Omega 0 minus, rather Omega 0 minus Omega times tao 0. And e raised to the power J Omega 0 minus Omega times tao 0 is independent of Lambda, so I could bring it outside the integral. So I am saying this becomes e raised to the power J Omega 0 minus Omega times tao 0 and rest of it inside.

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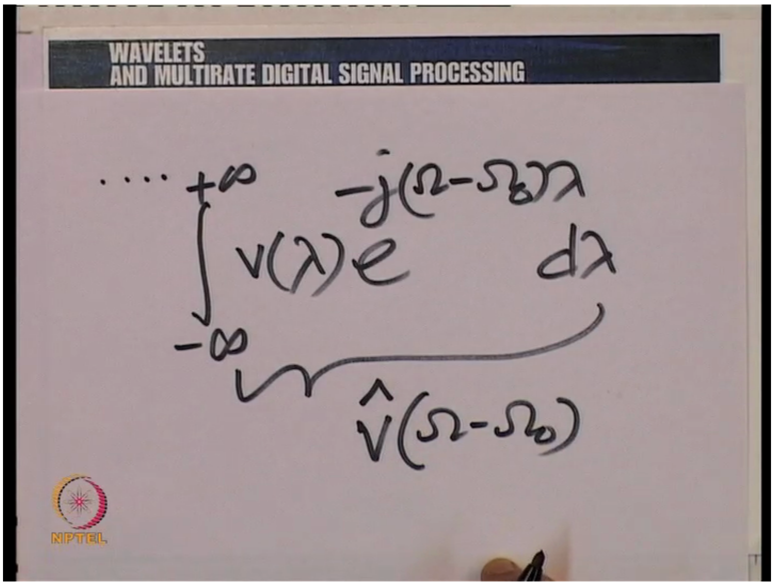

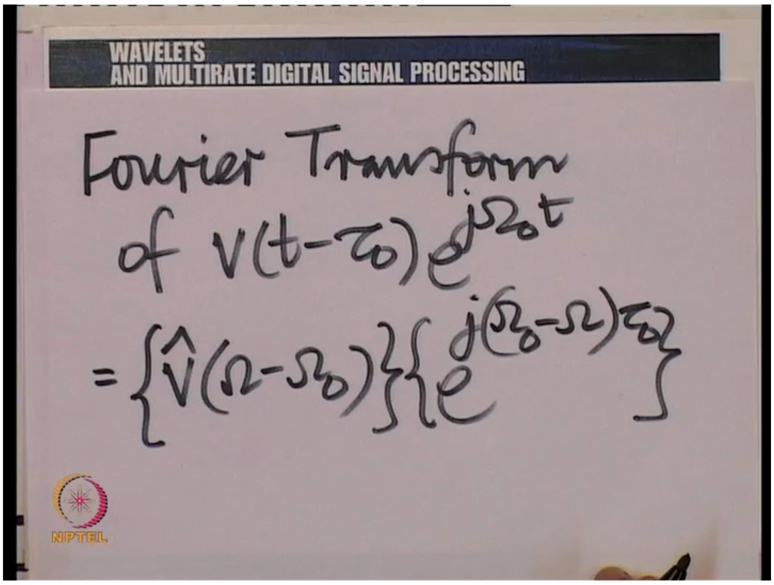

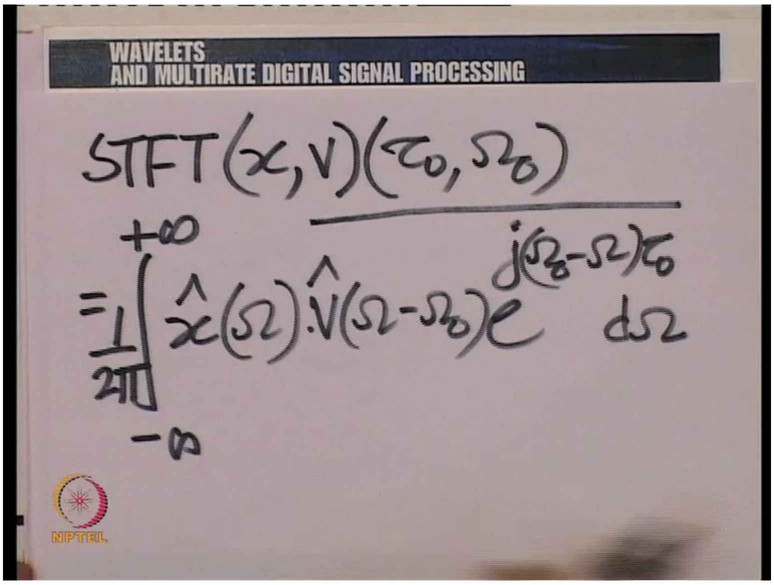

And now let me see what remains inside, what remains inside is V Lambda E0 raised to the power J Omega 0 minus Omega times Lambda d Lambda. So let me write that part down. So now here I should write only the integral, I have left out the other terms. The integral is essentially V lambda e raised to the power, I will rewrite it, minus J Omega minus Omega 0 times Lambda d Lambda which is easily seen to be the Fourier transform of V but evaluated at Omega minus Omega 0 instead of at Omega.

So all in all we have the following. We have the Fourier transform of vt minus tao 0, e raised to the power J Omega 0 t is the Fourier transform of V evaluated at Omega minus Omega 0 multiplied by e raised to the power J Omega 0 minus Omega times tao 0. And we shall substitute this in the Parseval's theorem expression. So we have the short time Fourier transform, secondary argument of x with respect to the window v evaluated at tao 0 and capital Omega 0 is essentially the following inner product with a factor of 1 by 2 pie remember.

So it is x cap Omega v cap Omega minus Omega not e raised to the power J Omega 0 minus Omega times tao 0 d Omega with this whole complex conjugate. And let us simplify that and once again let us note that when we expand this, we get 2 terms e raised to the power J Omega 0 tao 0 and e raised to the power minus J Omega tao 0, out of which only the 2nd depends on capital Omega, the 1st does not, the 1st can be brought outside the integral.

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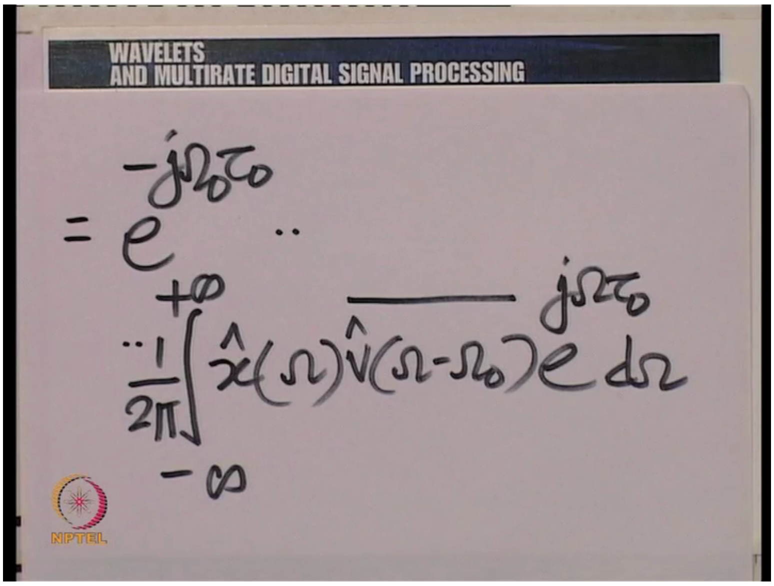

So what we have in effect is e raised to the power minus J Omega 0 tao 0 times the following. 1 by 2 pie, just the constant x cap Omega we cap Omega minus Omega 0 e raised to the power J Omega 0J Omega tao 0 D Omega. This is complex conjugated here, here the complex conjugate has been taken care of as it is. And now we have a very beautiful interpretation for this, this looks very much like an inverse Fourier transform if you think about it. It is the inverse Fourier transform evaluated at the point tao 0.

Inverse Fourier transform of what, of the Fourier transform of x multiplied by the Fourier transform of the window shifted to lie around Omega 0. So now this makes a lot of sense. The short time Fourier transform as we expected has an interpretation, a very similar interpretation both in time and frequency.