**Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.**
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**Week-8.**
**Lecture-20.2.**
**Journey from infinite to finite time-bandwidth product of Haar scaling function.**

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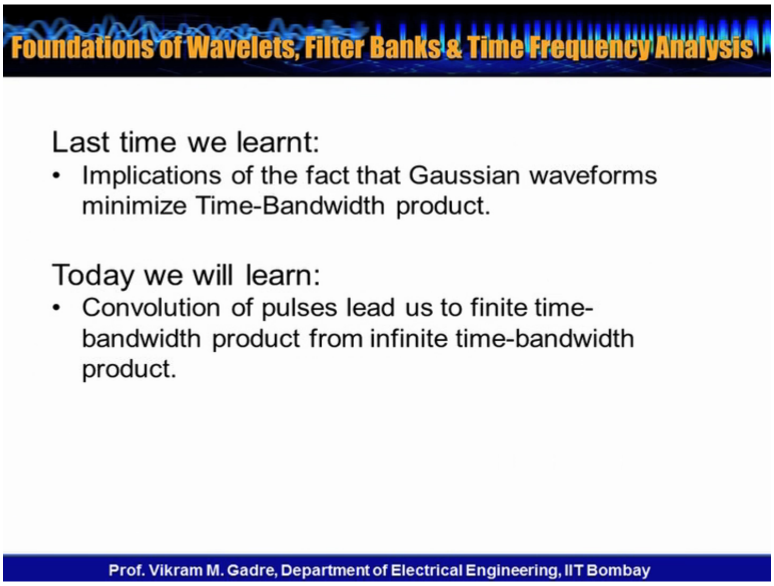


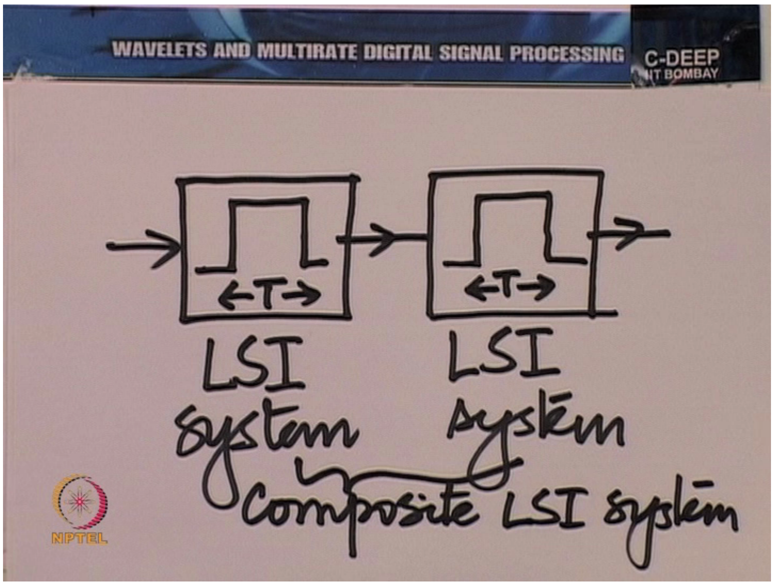

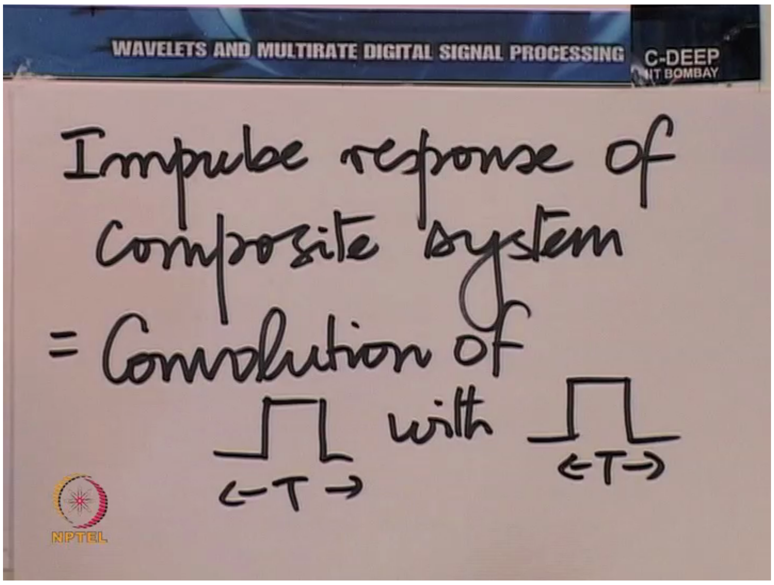


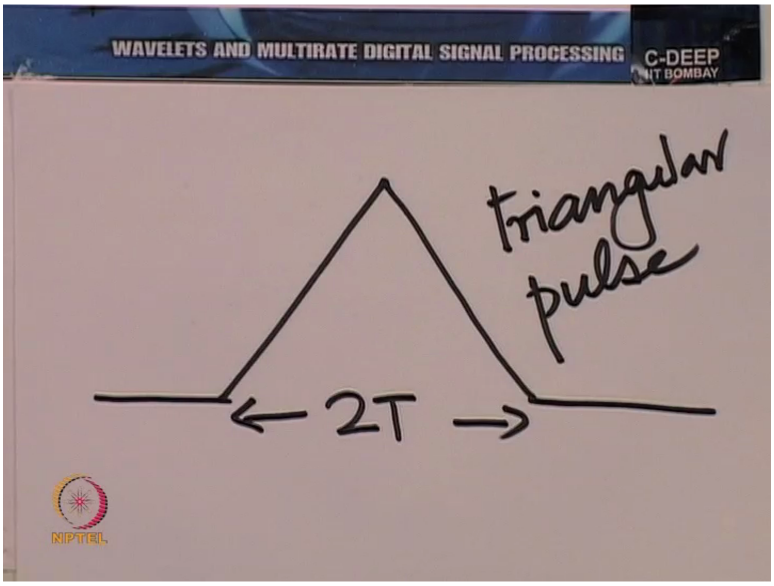


Suppose we took a cascade of 2 systems whose impulse response is essentially a pulse. What I mean by that is instead of taking just one pulse, take a cascade of them. Suppose we have 2 systems, each of whose impulse response is essentially a pulse, say of the same width. This is a linear shift invariant system, this is another linear shift invariant system. The impulse response here is essentially a pulse in the impulse response your too is a pulse. Both pulses of the same width, let us say T.

We cascade them and we note of course that together this also forms a compose LSI system in the impulse response of that composite LSI system is essentially the convolution. And we know that convolution very well. That convolution looks something like this, I hardly need to work it out, this is something very similar to us from any basic course on signals or systems.

The convolution looks like this, essentially what is called a triangular pulse. Now we can give this a physical interpretation.

You know, when you have this LSI system with an impulse response equal to a pulse, what you are essentially doing is a sample and hold process. So if an impulse results in a pulse, you are essentially sampling a pulse at a point in holding it for the duration of that pulse. That is what the physical meaning of that impulse response given by a pulse is. So if you have to such sample and hold, then you are effectively talking about a triangular impulse response.

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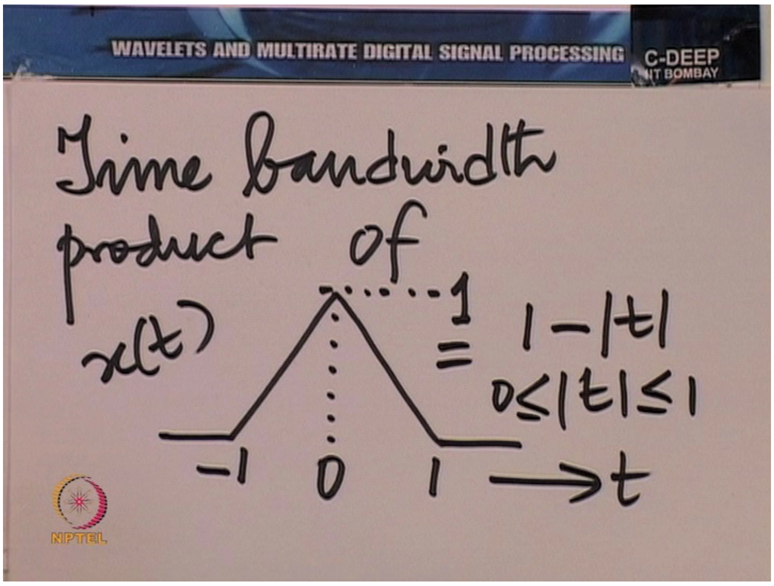


So there is some underlying physical meaning. Now the natural question to ask is what can we say about the time bandwidth product of this angular pulse, how bad or good as it compared to the Gaussian, that is the next question that we shall answer. So the time bandwidth product of this triangular pulse. And you know all you remember that we do not need to worry where this wrangle a pulse lies so we can as well centre it in put it at 0. We do not need to worry how wide this angular pulse is as long as we have kept it symmetric.

So we can put this from minus1 to1 and we do not need to worry what the height is and we can as well therefore make the height equal to1, good. All this is because of the invariance properties of the time bandwidth product. It is invariant to scaling on the dependent variable, it is invariant to scaling on the independent variable and it is invariant to translation. So we shall find out the time bandwidth product of this. In fact we can describe this function, let us call this function x of t as a function of t and describe it.

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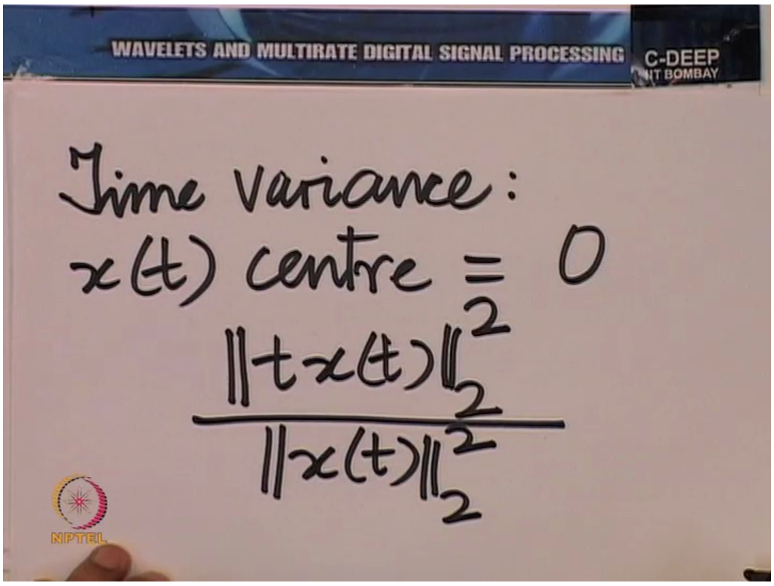


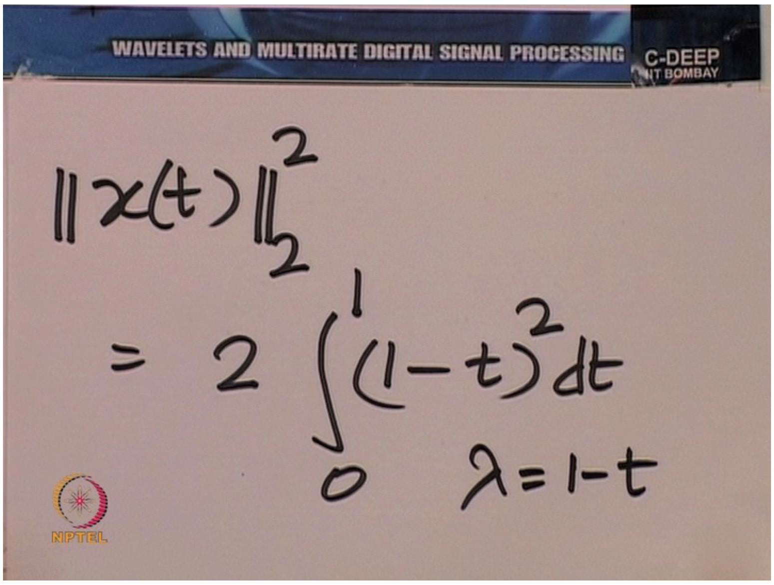


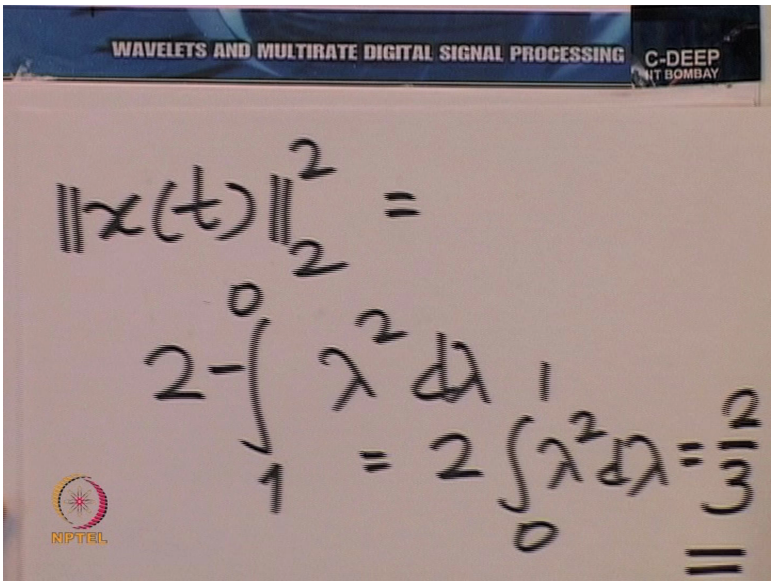

It is essentially 1 minus mod t for mod t it mean 0 and 1. So, how would we find the time bandwidth product? We shall 1st obtain the time variance. And you will recall that since the function is centred, that means the centre of xt is at 0, the time variance is going to be described by the norm of t xt the whole squared in L2R divided by the norm of x in L2R the whole squared. Now we shall be requiring this norm of x in L2R the whole square again and again.

So let us begin by calculating this norm 1st. The norm of xt in L2R is easily seen to be 2 times integral from 0 to 1, 1 - t the whole square dt. This 2 times comes because of the symmetry around t equal to 0. So essentially the area on the negative in the positive side is the same. Now this is an easy integral to evaluate, we can easily make this substitution lambda is 1- t and evaluate this integral and that gives us the norm of xt in L2R squared is 2 again, this is lambda squared, now d lambda is minus dt, so we could write minus d lambda here but the limits also change from 1 to 0 in that case and therefore this is the same as 2 integral 0 to 1 lambda squared d lambda.

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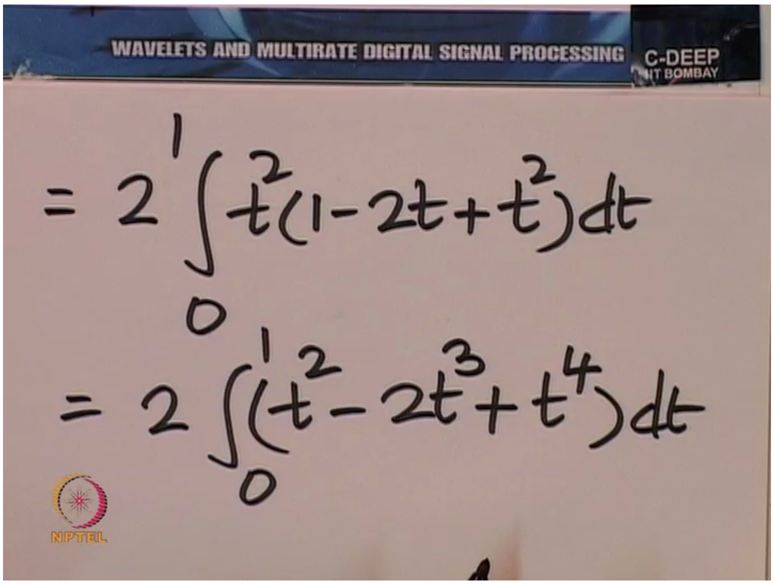

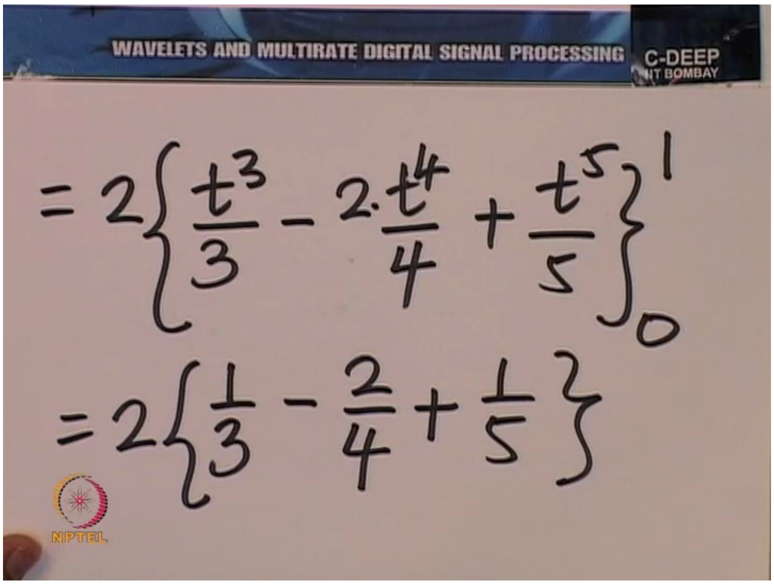


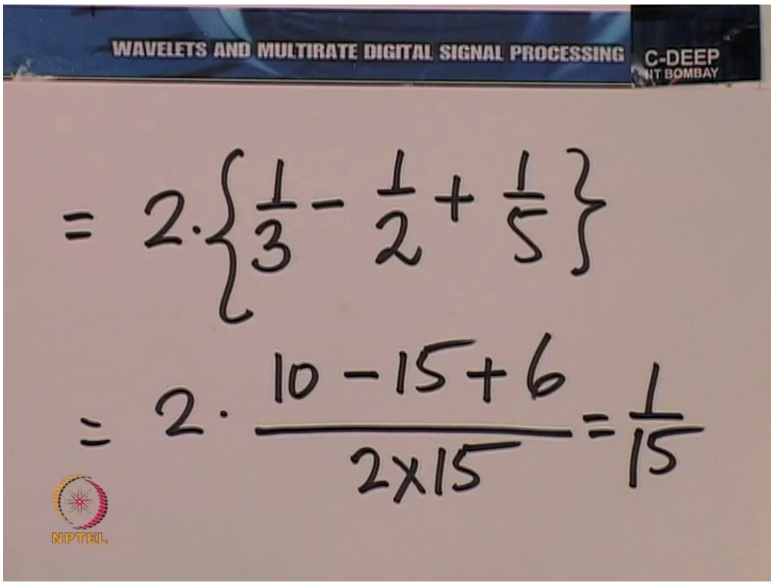


This is easy to evaluate, this essentially evaluates to 2 by 3. That is easy to evaluate, lambda cube by 3 from 0 to 1. Anyway, so, so much so for the L2R norm. Now let us take the norm of t xt, let us evaluate norm t xt the whole squared. Now here again we use symmetry. That is 2 times the integral from 0 to 1, t times 1 minus t dt, using symmetry, the whole square of course. Now it is not going to help very much to make a substitution of variable because you know if you substitute lambda is 1 minus t, we will get a 1 minus lambda here which is not so convenient.

So let us keep it in integral in t and let us evaluate the integral bravely so to speak. So that is 2 integral from 0 to 1, t squared 1- 2t plus t square dt which is 2 integral 0 to 1 t square minus 2 t cube plus t to the power 4 dt. Easy integrals to evaluate and we do that, t cube by 3 there, t to the power 4 by 4 there and t to the power 5 by 5 here, evaluated from 0 to 1. And that is 2 1 by 3 - 2 by 4 + 1 by 5, simple enough. Let us simplify a little bit. So, 1 by 15 is what we have. 16 - 15, that is 1, 2 and 2 cancel and there you are. Now, this is the L2 norm of t xt squared. I mean the L2 norm of t xt the whole squared in L2R, that is what I mean.

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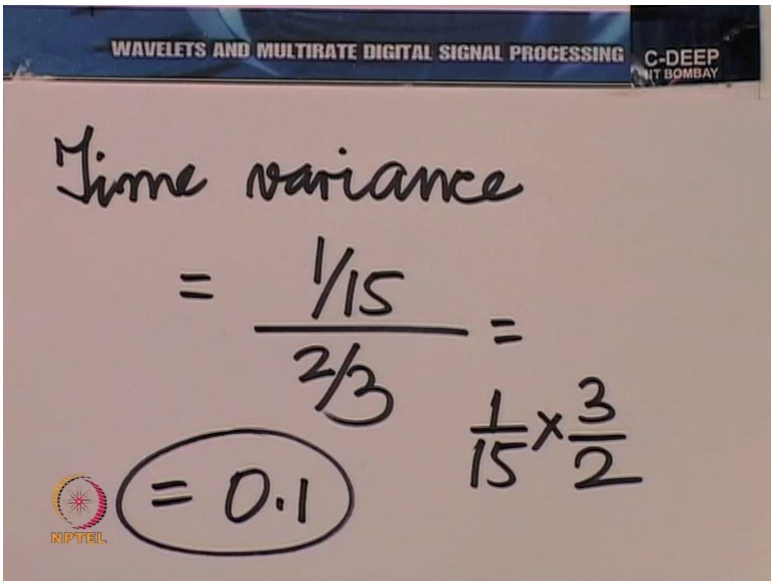


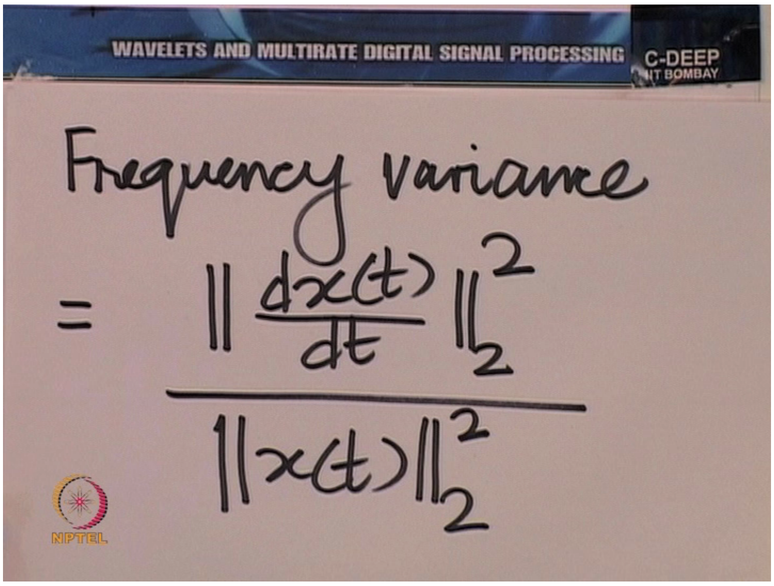

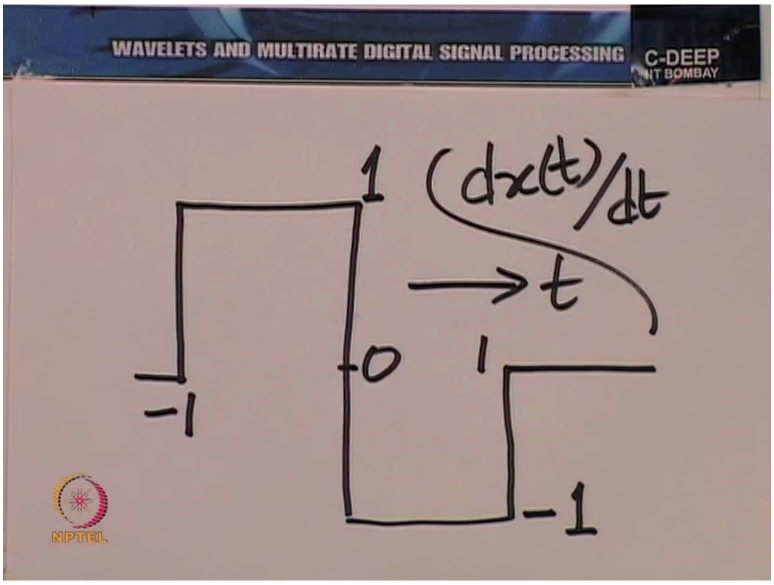


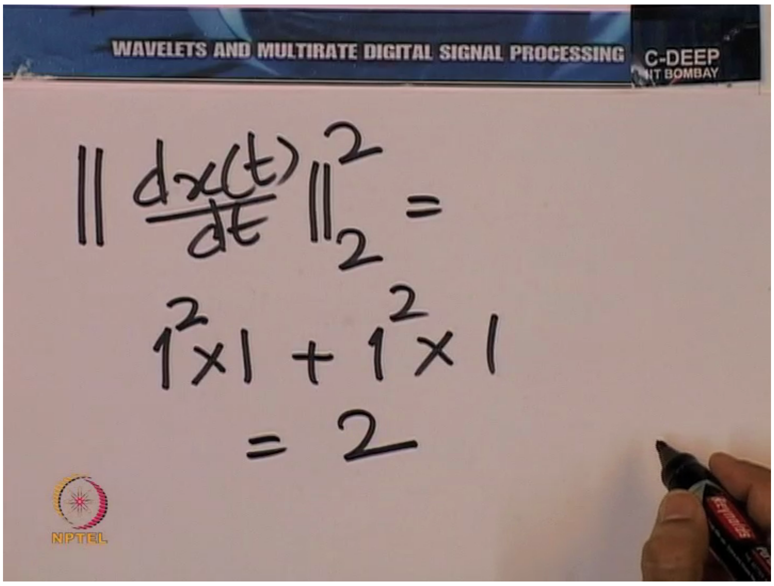


So we have the time variance ready for us. The time variance is therefore 1 by 15 divided by 2 by 3, which is 1 by 15 into 3 by 2 or that is 1 by 10. Now let us look at the frequency domain, in fact the frequency domain will be a little easier, because we are going to make use of the principle of bringing the frequency in time domain in calculating variance. That is a very easy integral to evaluate. The frequency variance is going to be given by the L2 norm of d xt dt the whole squared divided by the L2 norm of xt the whole squared.

And d xt dt is a very simple function to evaluate. In fact d xt dt has the following appearance. It is interesting, d xt dt here, it is interesting. d xt dt here has the appearance of Haar wavelet. It is very easy to calculate the energy in this. The L2 norm of d xt dt is simply, L2 norm square I mean, is simply 1 squared into 1 class 1 square into 1, looking at the areas of the rectangle, so that is 2. And we already know the L2 norm of the function squared too.

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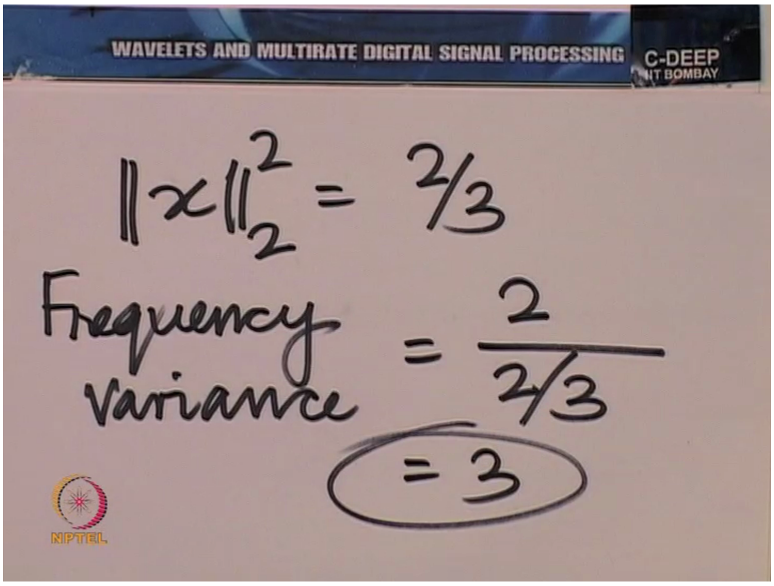


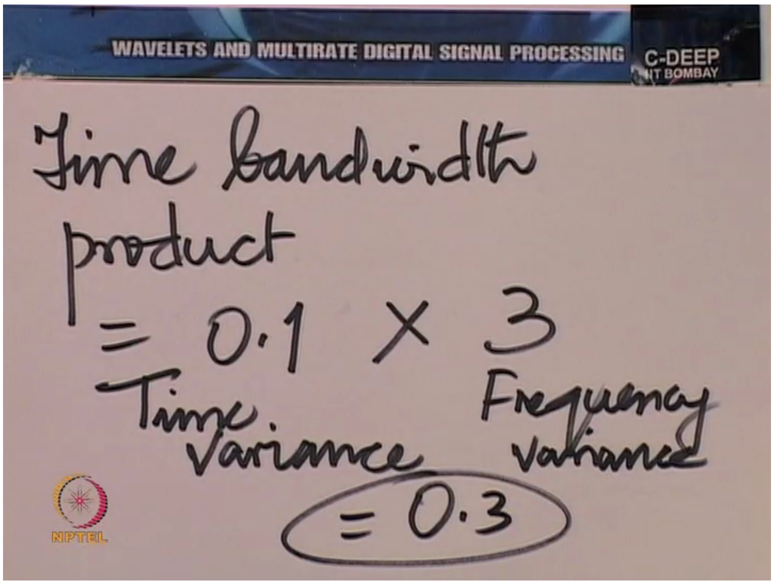


So we know this, we know the L2 norm of x, it is 2 by 3 and therefore the frequency variance turns out to be 2 divided by 2 by 3 which is 3. Now, we can calculate the time bandwidth product. So the time bandwidth product is 0.1, the time variance multiplied by 3, the frequency variance. Lo and behold, it is 0.3. So that is very good news actually if you think about it. We know the minimum we can go to is 0.25, we have come all the way down to 0.3, pretty good.

From infinity we have come all the way to 0.3 just by cascading the system with itself once again, not bad at all. So although there was bad news in the uncertainty principle that you cannot reduce the simultaneous localisation in time and frequency below 0.25 in the sense of the time bandwidth product, there was good news in that you knew what the optimal function was, namely the Gaussian. Then there was bad news again that the Gaussian was physically unrealisable as a function but now we have some good news, namely that we can go all the way down to 0.3 by a very meaningful function.