Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
Professor Vikram M. Gadre.
Department Of Electrical Engineering.
Indian Institute of Technology Bombay.
Week-7.
Lecture-19.3.
Optimal function in the sense of time-bandwidth product.

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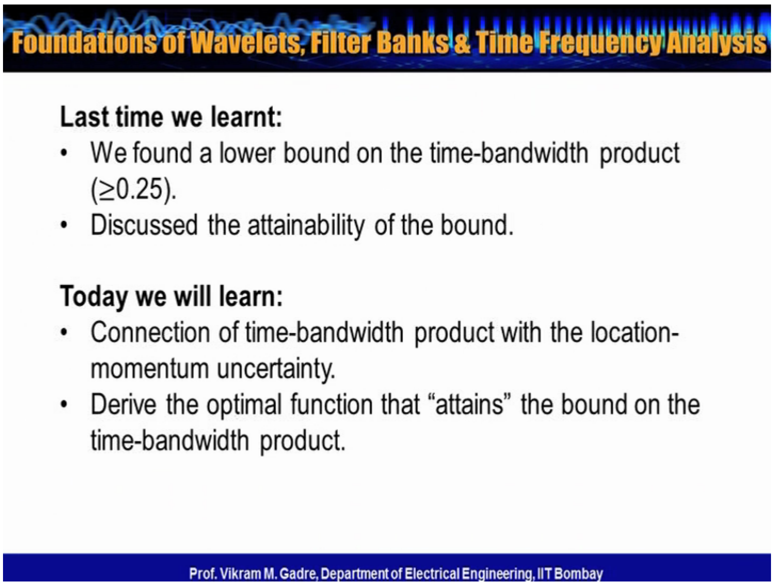

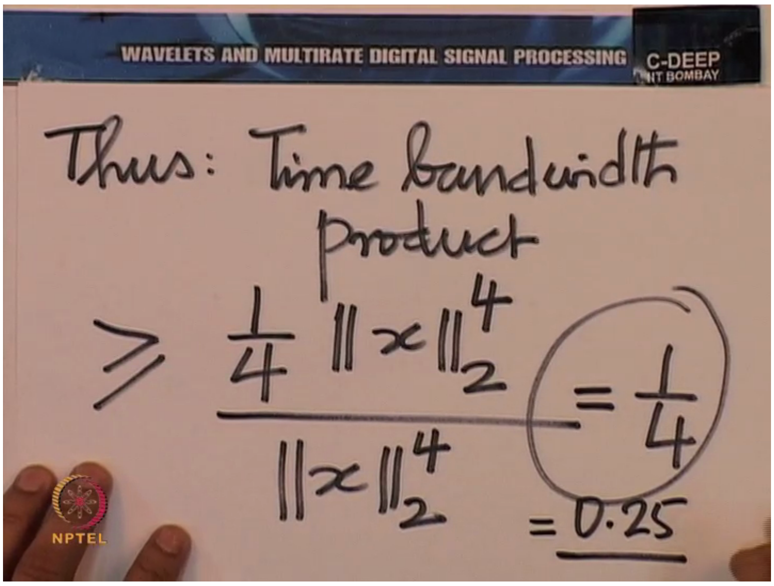

The time bandwidth product can never be less than one fourth, 0.25. So 0.25 is the very lowest value of the time bandwidth product that you can get. And what we have concluded here has nothing to do with the tools available at a particular time, with the technology available the particular time or with the machines and the political situation or whatever. It is fundamental to signal processing. In fact so fundamental is this result that we have derived, that in different manifestations, it is seen in different subjects.

What we call the uncertainty limit in physics is just another version of this. People talk about the inability to locate position and momentum simultaneously, actually it is just another version of this. I shall just give a small hint as to how you might connect this idea of time bandwidth product to the concept of position, momentum uncertainty. And to do that we will actually go back to the expression for the time bandwidth product that we had derived.

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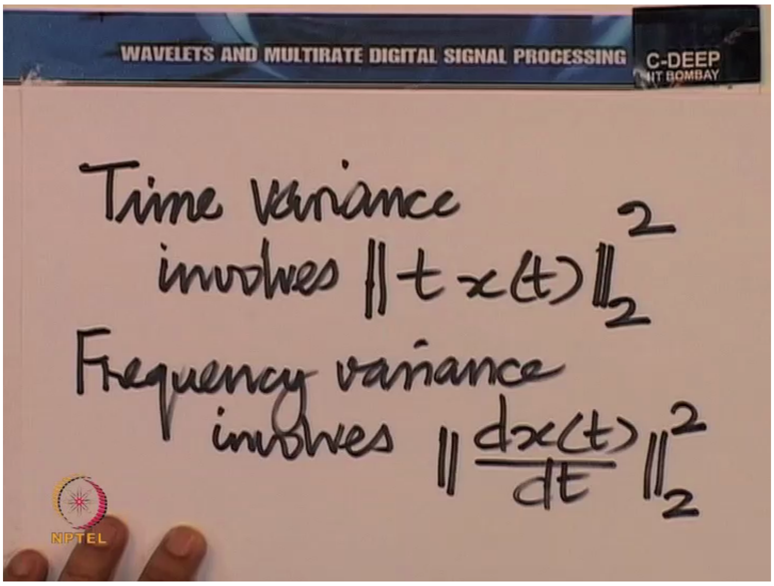

You see, you notice that we have shown that essentially the time bandwidth product relates to a time variance, you know the time variance relating to the norm of t xt the whole squared. And you are assuming the object is centred, so now think of xt as descriptive, descriptive of an object, remember that we have talked about the one-dimensional mass. Now here we are talking about xt then as a kind of mass distribution of the object or rather, if not mod, if not xt, it is really mod xt squared which is like a mass distribution of that object.

So division, the denominator part is essentially a normalisation, make the object unit mass if you like. But here in the time variance the numerator is indicative of the uncertainty in position. What is the variance of the object as far as its spread around the Centre is concerned, that is what the time variance tells you. Now, if you look at the other term there, the frequency variance, so time variance involves this, frequency variance involves this.

And you know, involving this, the derivative of xt with respect to t, now xt or mod xt square as I said is indicative of in some sense the presence of that mass or that body on t. So d xt dt is indicative of the movement of that body, the change with respect to time. So here in some sense a change, it is indicative of a change. So in a broad sense you can see the connection between position and momentum, uncertainty in where it is and uncertainty in how it changes.

Let me not dwell too much further on this. To interpret precisely how this is the uncertainty principle in physics, I think we should leave it to a physicist, a person who specialises in that but this is what I have just given you is an intuitive indication, that is all. However, it is just to bring out the various meanings that this time bandwidth product or this uncertainty limit has. It has meanings in different subjects.

Anyway, coming back to its object with which we are dealing, this is the whole basis of the subject of wavelets or for that matter time frequency methods. What it tells us is that no matter what we do, we are not going to get any function which has finite energy and which can be confined beyond a certain point in the 2 domains, time and frequency simultaneously, that is bad news. What was worse is that if you look at the Haar case, it is not confined at all, the time bandwidth product was infinity.

So now we naturally ask the next question. Is there a function which gives us this 0.25 or is it something which we should never seek? That is not very difficult to answer. In fact that can be answered again by using a vectorial interpretation. When does the numerator become equal in the Cauchy Schwarz inequality to the expression that we derived?

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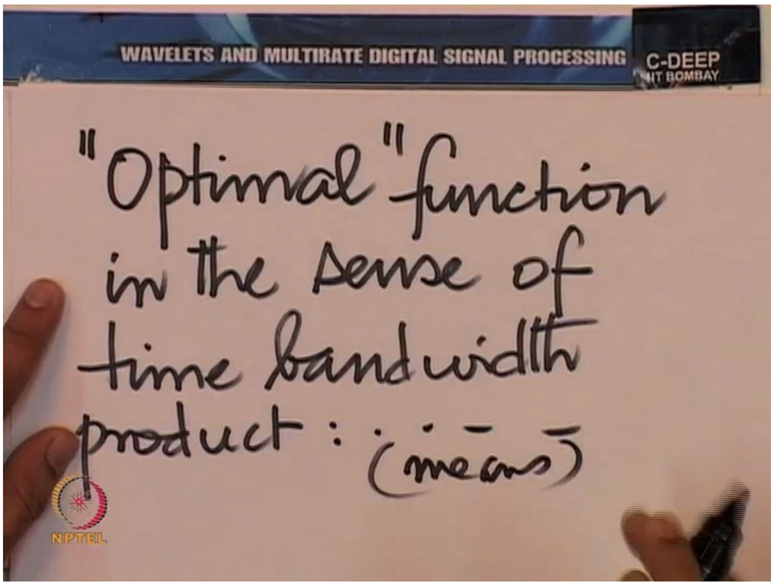

The Cauchy Schwarz inequality becomes an <u>equality</u>.

That is <u>"$\cos^2\theta$" = 1</u>

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We need "vectors" $t\,x(t)$ and $\frac{dx(t)}{dt}$ to be "COLLINEAR" (linearly <u>dependent</u>)

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Thus:

$$\frac{dx(t)}{dt} = \gamma_0 t\, x(t)$$

<u>Solve</u>

So the optimal function in the sense of time bandwidth product means the Cauchy Schwarz inequality becomes an equality. Now when would that become an equality? It would become an equality if the cos square theta term is 1, that is the cos square theta term is 1. And what do you mean by cos square theta term being 1, what really is theta? Theta is an angle between these so-called vectors, t xt and d xt dt. If you want the angle to be such that cos square is one, these 2 vectors must be collinear, they must be along the same lines so to speak.

So therefore in the language of functions we need the vectors, so-called vectors t xt and d xt dt to be collinear, one dimensional. What do you mean by them being collinear? They must be linearly dependent. And what do you mean by them being linearly dependent? Any of them must be a multiple of the other, in other words d xt dt is some constant, let us call that constant gamma 0 times t xt. The solution of this equation would give us the so-called optimal function, so we solve this.

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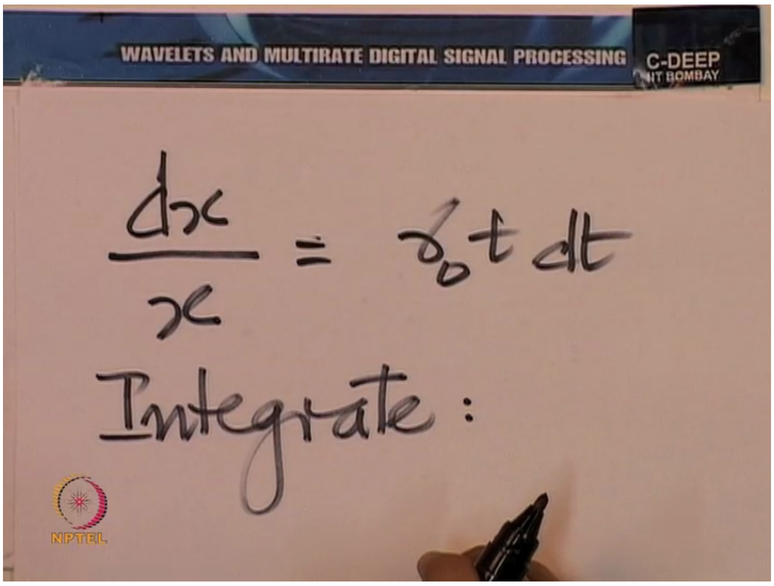

$$\int \frac{dx}{x} = e^{\left\{\gamma_0 \frac{t^2}{2} + c_0\right\}}$$

$\int \frac{dx}{x} \rightarrow \ln x$

c_0 ← Const of integ

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$$x(t) = \underbrace{e^{c_0}}_{\text{Constant } \tilde{c}_0} \cdot e^{\gamma_0 \frac{t^2}{2}}$$

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$$x(t) = \tilde{c}_0 \cdot e^{\gamma_0 \frac{t^2}{2}}$$

$$x(t) \in \mathcal{L}_2(\mathbb{R})$$

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How do we solve it, by simple change of the or the redistribution of the derivative. So essentially what we are saying is dx by x is equal to gamma 0 t dt and if we integrate, we get gamma 0 t squared by 2, a log natural x, well you must have a constant here, a constant of integration. And this is of course log natural x. So in other words we have log natural x is of the form gamma 0 t square by 2+ C0. Let us raise both sides using the natural base e. So e raised to the power ln x is e raised to the power this.

And therefore we have e raised to the power ln x is just x. So x of t is some constant, e raised to the power C0, this is a constant, let us call this constant C0 tilde times e raised to the power gamma 0 t square by 2. So let us write that down again and let us make a remark on gamma 0. xt is of the form C0 tilde times e raised to the power gamma 0 t square by 2.

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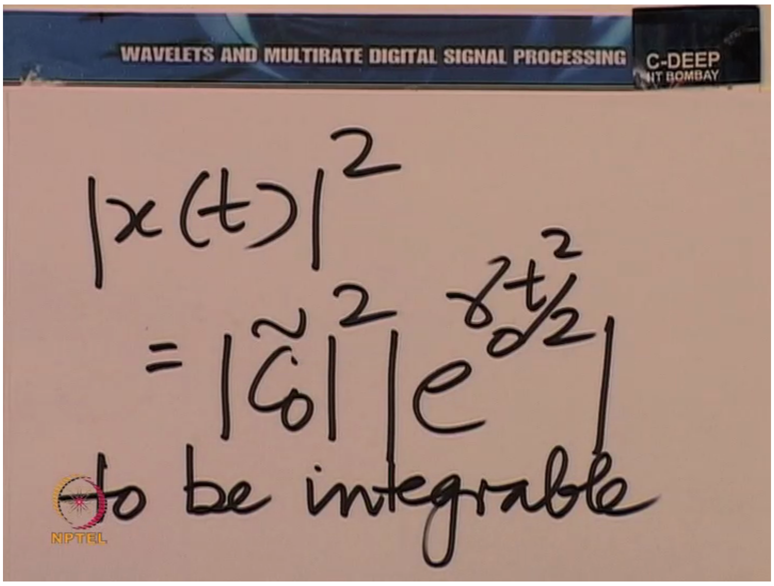

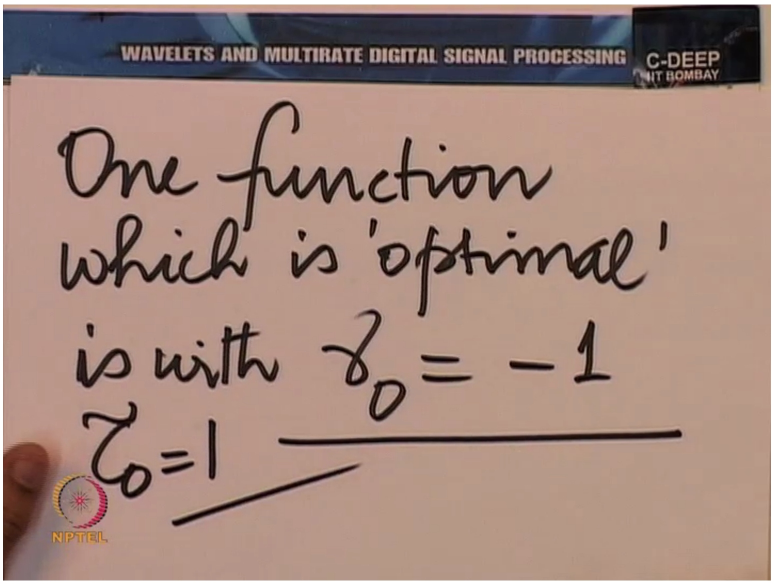

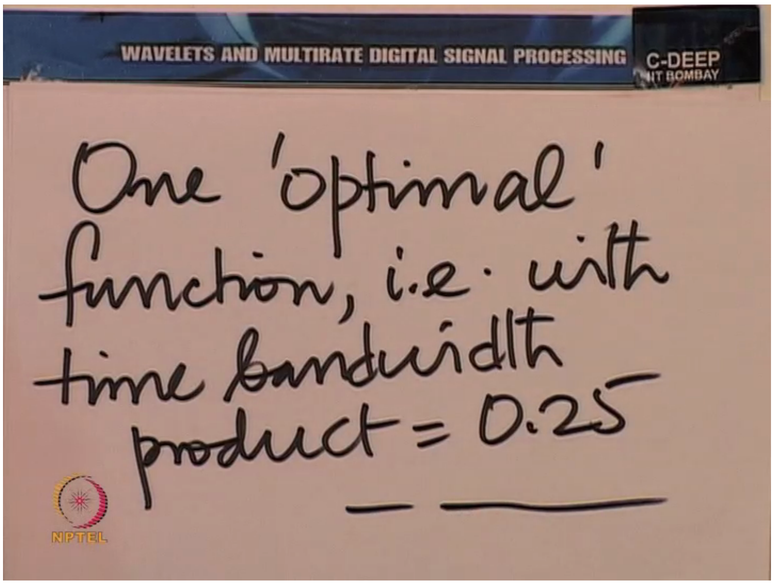

Now, you want xt to belong to L2R and therefore we want mod xt squared, which is essentially mod C0 tilde squared times mod e raised to the power gamma 0 t square by 2 to be integrable. Now this is possible only if gamma 0 has a negative real part. If gamma 0 has a positive real part, this is going to be a Gaussian, so-called Gaussian that grows in time, it is not going to be square integrable. And therefore we must choose gamma 0 with a negative real part.

And we can choose any one example of that. So in other words one function which reaches the lower bound which is optimal in this sense is with gamma 0 equal to minus1. And there we have, we can also choose of course C0 tilde to be 1 because it does not matter if you scale a function, the time bandwidth product is unaffected. And therefore one optimal function in the sense of time bandwidth product, that is with time bandwidth product equal to 0.25 is the Gaussian.

A very interesting conclusion, the Gaussian is optimal. The Gaussian seems to arise in many situations. It has arisen in this, having noted this we shall conclude today's lecture and proceed in the next lecture to delve further into this issue of how close we can get to the optimal with other functions, thank you.