Foundations of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-1.
Lecture - 3.1.
Piecewise Constant Representation of a Function.

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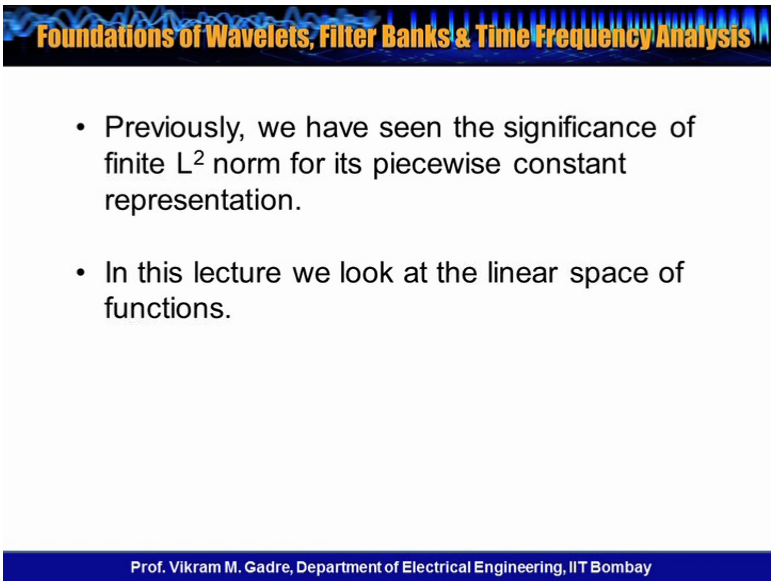

A warm welcome to the lecture on the subject of wavelets and multirate digital signal processing. Let us spend a minute on what we talked about in the lecture. We have introduced the idea of a wavelet in the 2nd lecture and we have done so by using the Haar wavelet. Essentially, where piecewise constant approximation are refined in steps by factors of 2 at a time. In today's lecture we intend to build further on the idea of the Haar wavelet by introducing what is called a multiresolution analysis, or an MRA as it is often referred to in brief.

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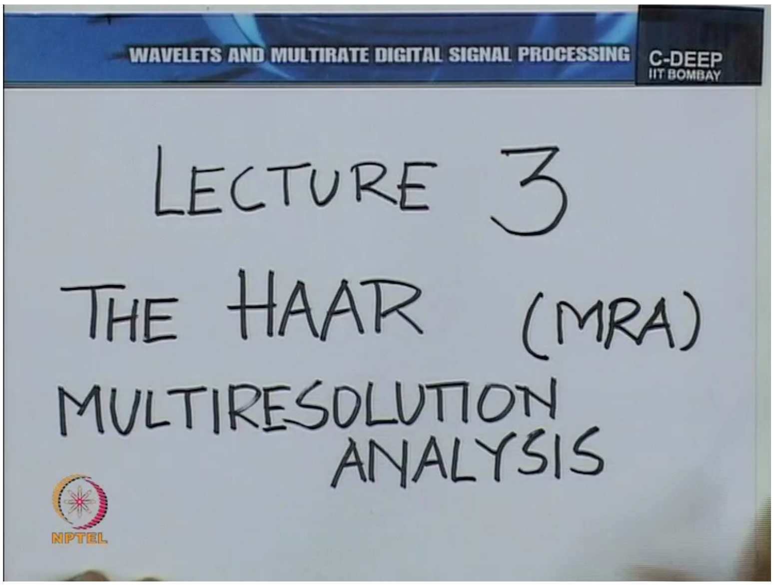

So let me title today's lecture. We shall title today's lecture as the Haar multiresolution analysis. And in fact let me also put down here the abbreviation for multiresolution analysis, MRA. You see the whole idea of multiresolution analysis has been briefly introduced in the context of piecewise constant approximation. So recall that we said that the whole idea of the wavelet is to capture incremental information. Piecewise constant approximation inherently brings in the idea of representation at a certain resolution.

We took the idea of representing an image at different resolutions, in fact we use the term resolution when we represent images on a computer. 512 cross 512 is a resolution lower than 1024 cross 1024 and one way to understand the notion of wavelets or to understand the notion of incremental information is to ask, if I take the same picture, the same two-dimensional scene or same two-dimensional object so to speak and represent it 1st at a resolution of 512 cross 512 and then at a resolution of 1024 cross 1024, what is it that I am additionally putting in to get that greater resolution of 1024 cross 1024 which is not there in 512 cross 512, the Haar wavelet captures this.

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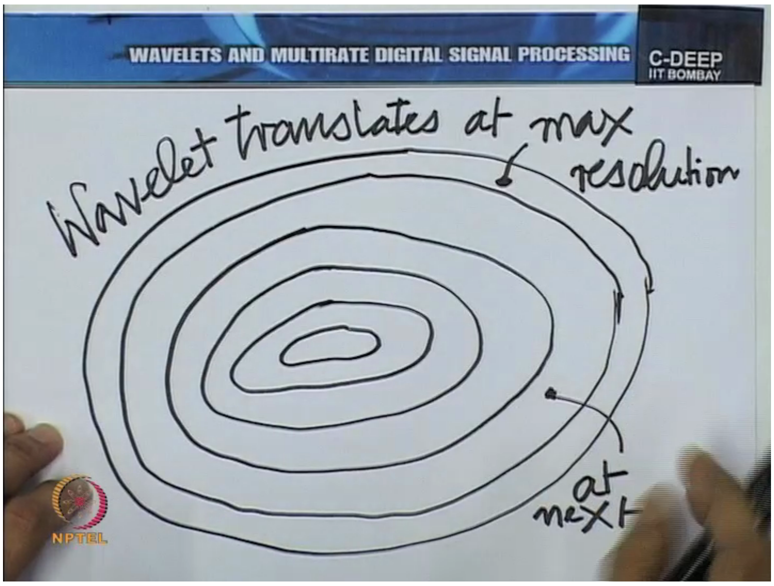

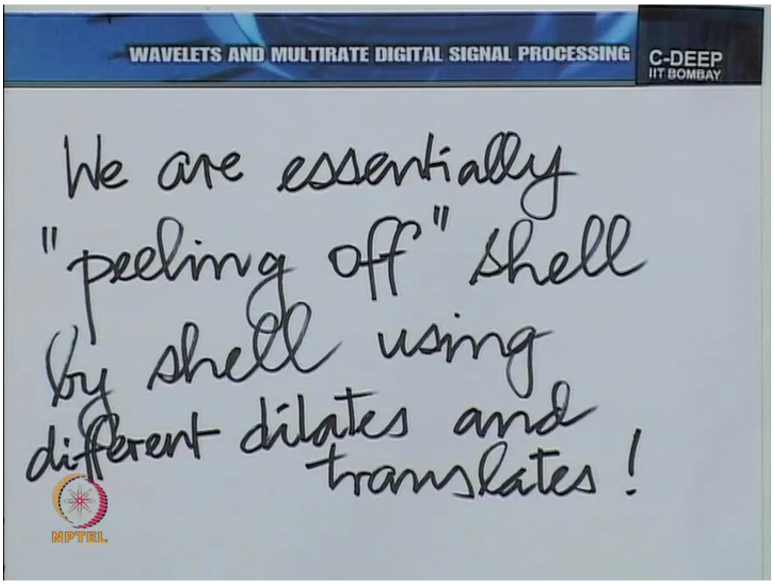

So in some sense you may want to think of the Haar wavelet as being able to capture the additional information in the higher resolution and therefore if you think of an object with many shells, so this is a very common analogy, you know, if you think of the maximum information, maybe as a cabbage or an onion informally. And if you visualize the shells of this cabbage or this onion like this, then the job of the wavelet is to take out a particular shell.

So the wavelet at the highest resolution, wavelet translates at highest resolution, at Max resolution, would essentially take out this, at next resolution it would take out this shell and so on. So when you reduce the resolution, what you are doing is to peel off shell by shell, in fact I think this idea is so important that we should write it down. We are essentially peeling off shell by shell using different dilates and translates of the Haar wavelet. And there again a little more detail, different dilates correspond to different resolutions and different translates essentially take you along a given resolution.

So that is the relation between peeling off shells and dilates and translates. Now all this is an informal way of expressing this, we need to formalise it and that is exactly what we intend to do in the lecture today. Again we would now like to talk in terms of linear spaces. So without any loss of generality, let us begin with the unit length for piecewise constant approximation. I say without loss of generality because after all what you consider as unit length is entirely your choice.

You can call 1 metre unit length, you can call 1 centimetre unit length or if you are talking about time, you could talk about 1 second as unit length or unit piece and so on. So unit on the independent variable is our choice and in that sense without any loss of generality, let us start, make the focal point piecewise constant approximation at a resolution with unit intervals. So let us write that down formally.

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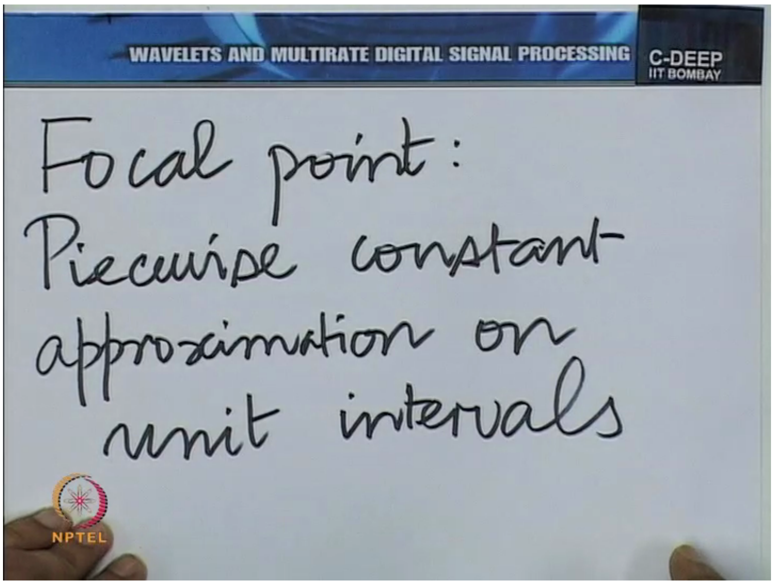

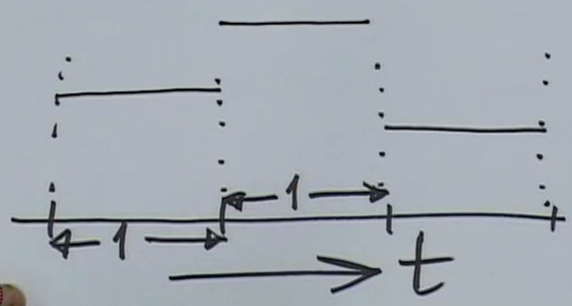

WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING C-DEEP IIT BOMBAY

What function $\phi(t)$ is such that its integer translates can span this space?

WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING C-DEEP IIT BOMBAY

The space of piecewise constant functions on the <u>standard</u> unit intervals!

$\forall n \in \mathbb{Z}$ $\quad]n, n+1[$

Piecewise constant, so you know the so-called fulcrum or focal point is piecewise constant approximation on unit intervals. And let us sketch this to explain it better. So what we are saying is, you have this independent variable, again without the loss of generality, let that independent variable be t, you have unit intervals on this. And on each of these unit intervals you write down a piecewise constant function essentially corresponding to the average of the original function on that interval.

So this is the average of the function on this interval, this one on this interval and this one on this interval. Now how can we express this function mathematically with a single function and its translates? So essentially we want a function, let us call it Phi of t now. So what function Phi of t is such that its integer translates can span this space, what space, 1st? The space of piecewise constant functions on the standard unit intervals. What are the standard unit intervals?

The standard unit intervals are the open intervals N, N +1 for all N over the set of integers. Now I wish to slowly start using notations which is convenient. So these notations, script Z would in general in future refer to the set of integers. So I think we should make a note of this.

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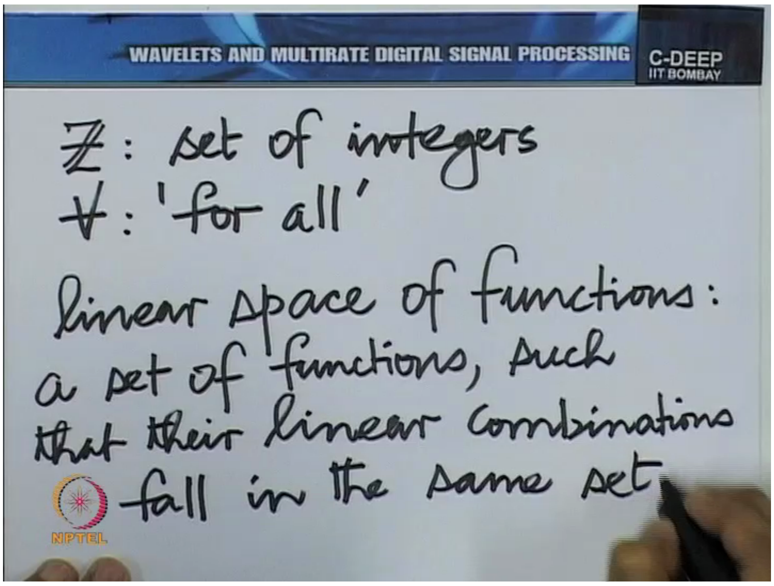

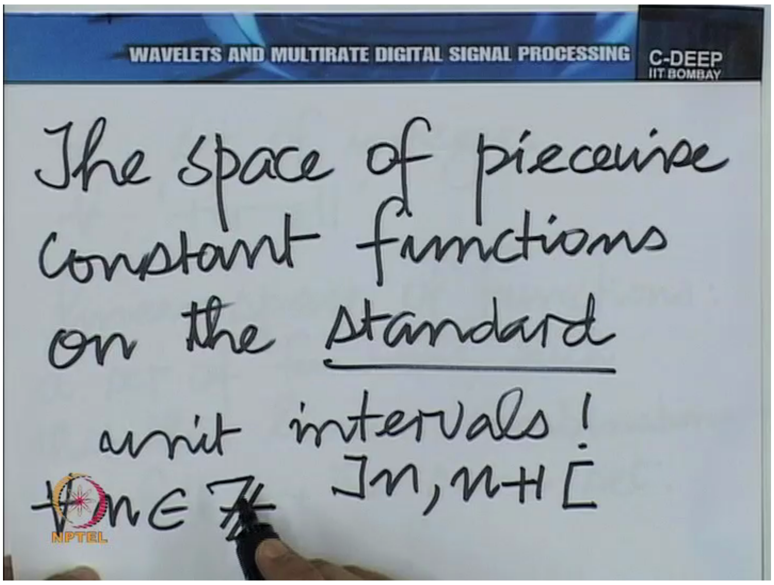

Script Z is a set of integers and this refers to "for all". So what are we saying here, let us go back. We are saying we have this space, now again I must recapitulate the meaning of space, linear space, a linear space of functions is a collection of functions, any linear combination of which comes back into the same space. So if I add 2 functions, it goes back into the same space, if I multiply a function by a constant, it goes back into the same space, if I multiply 2 different functions in that space by different constants and add up these resultants, it would still be in the same space.

In general, we would say any linear combination, so we say a set, a set of functions forms a space, a linear space, if it is closed under linear combinations. So we say linear space of functions is a set of functions such that their linear combinations fall in the same set. Now I am making it a point to write down certain definitions and derivations in this course and there is an objective behind that. I believe that a course like this is best learned by working with the instructor. So, although one could just listen and try and remember, that does not give the best flavour in a course like this.

It does require in-depth reflection in thinking and therefore I do believe that the student of this course would do well to actually note down certain things and work with the instructor, for it is then, that the full feel of the derivations, or the full feel of the concepts would dawn upon the students. Anyway, with that little observation and instruction, let us go back to what we are doing here. So you see a linear space of functions is one in which any linear combination of functions in that set fall back into the same set.

Now here there is a little bit of clarification required. You see in general if you consider the space of functions that we talked about a minute ago, namely the space of piecewise constant functions on the standard unit intervals, which are the standard unit intervals? The intervals of the form, open intervals of the form N, N +1 for all N over the set of integers, then there is an infinity of such functions. And naturally when you talk about linear combinations, you could have finite linear combinations and you could have infinite linear combinations.

Now for this point in time, when we talk about linear combinations, we are essentially referring to finite linear combinations. That is just a little clarification for the moment. Well, the idea could be extended to infinite linear combinations too, but I do not want to go into those niceties at this point in time, they would carry us away from our primary objective.