Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-7.
Lecture-19.2.
Evaluation of Time-bandwidth product.

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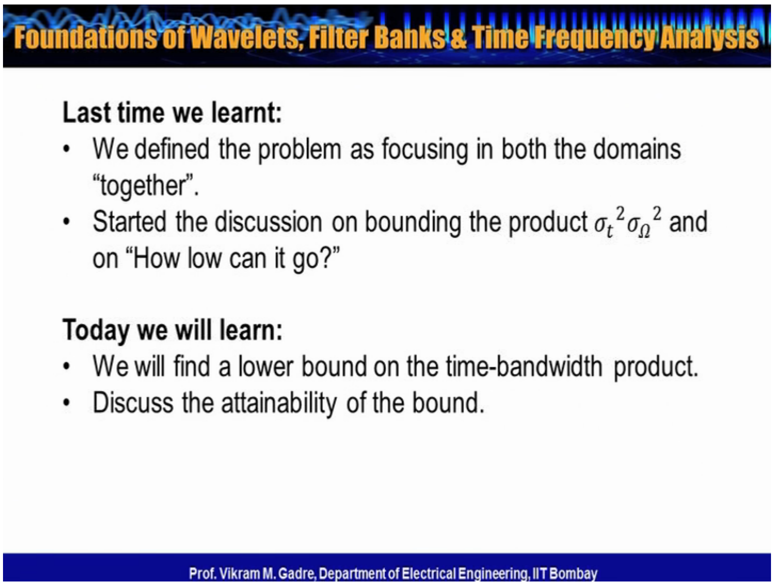

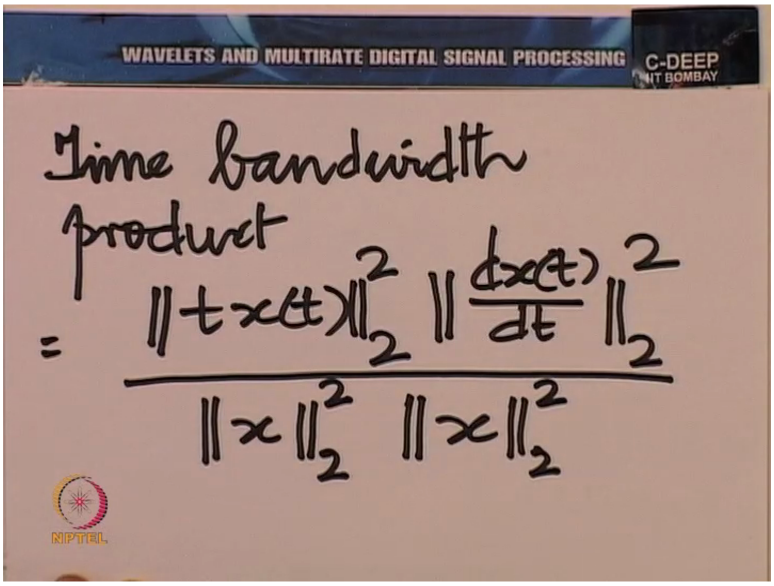

To answer that question, let us recall some of the exhibition that we had derived in the previous lecture. We had looked at an expression for the time bandwidth product based on entirely time domain quantities and let us would that expression down clearly once again. We had shown that the time bandwidth product without loss of generality can be written ultimately as follows. It is the L2 norm of t xt the whole squared times the L2 norm of the derivative, again the whole squared divided by the L2 norm of x squared multiplied by the same.

So in other words you could write the L2 norm of x raised to the power of 4 if you like by combining these 2 terms. Anyway, I am writing them separately to emphasise that this is associated with this and this with this notionally. Now of course I must also again mention the slight abuse of notations, here when we write t times xt, we are referring to the whole function as an object and not to a specific value of t, the same holds here.

Anyway, this was just to recall what we did yesterday and remember we had done this after doing a little bit of preparation on the functions so to speak. Having noted that translation in time and in frequency has no effect on the time bandwidth product we have said we could shift the function in time so that it is centred in time. Subsequently we can shift it in frequency without affecting the time function so that it has Centre in frequency.

In all this, the time bandwidth product does not change, so here we have a function which had been assumed to be centred in time and frequency and now we are working with that function and this is without any loss of generality. Anyway, coming back then to the calculation of these quantities, let us once again gave a vectorial interpretation to this. Now you see, look at the numerator, the numerator is a product of 2 squared norms, let us focus our attention on the numerator.

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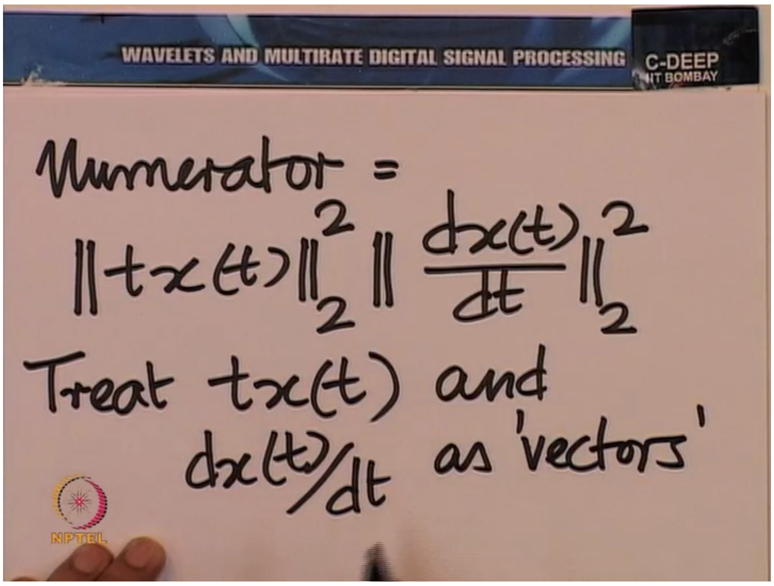

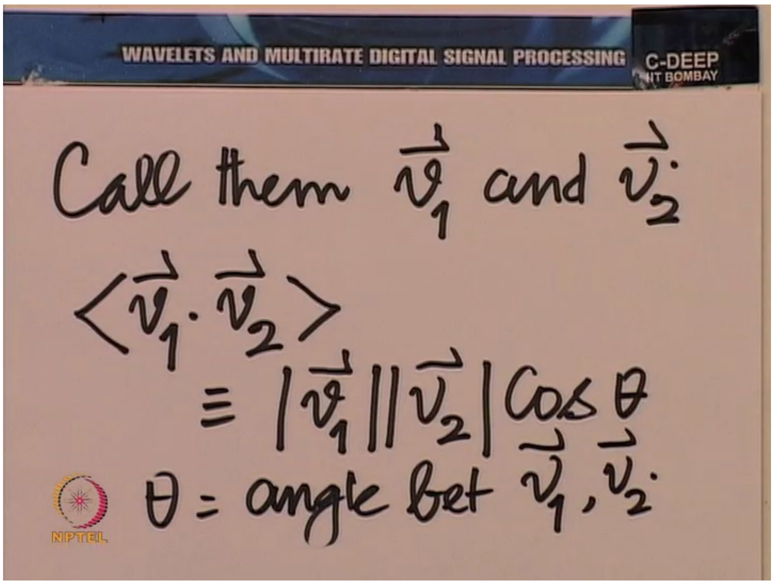

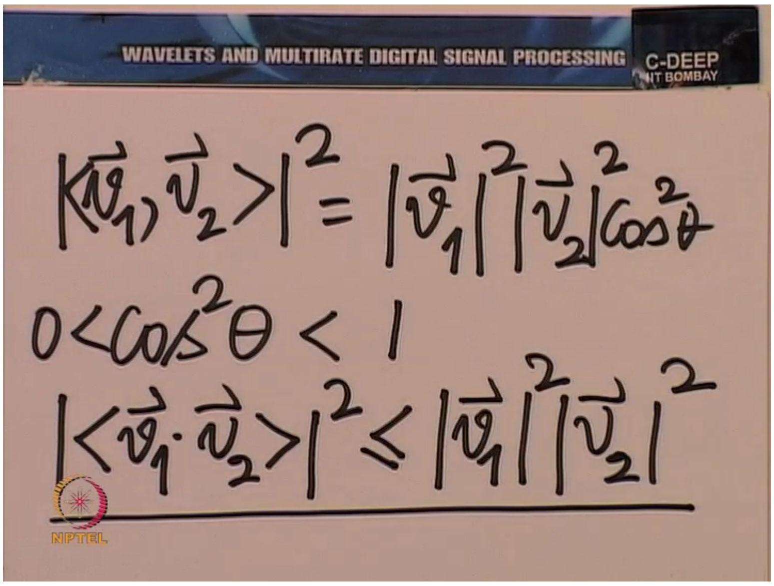

So numerator here is the norm of t xt the whole squared multiplied by the norm of d xt dt the whole squared in L2R. Now let us take recourse to a very fundamental principles that we know in vector analysis. Treat t xt and d xt dt as vectors as we often do, generalised vectors. So let us call them V1 vector and V2 vector. Now if we recall a basic principle of inner products, the inner product of V1 with V 2 is essentially of the form, the magnitude of V1 times the magnitude of V2 times cosine of the angle between V1 and V2.

Theta is the angle between V1 and V2. And therefore it is obvious in very low dimensional spaces which can of course be taken to higher dimensional spaces also. That if you consider the magnitude squared of the dot product, let me use the dot product notation, it is of course the product of the magnitude squared of V1 and V2 multiplied by cos squared theta. And cos squared theta is always less than 1, in fact between 0 and 1.

Here of course we are talking about theta real and that is the interpretation which we always have even if we are talking about complex functions, the angle here is assumed to be real. You see, what it means therefore is that the modulus of the dot product squared is always less than or equal to the modulus of V1 square times the modulus of V2 square. This is a very simple but a very important principle. And in fact this principle can be generalised to the generalised vectors that we are talking about, functions viewed as vectors.

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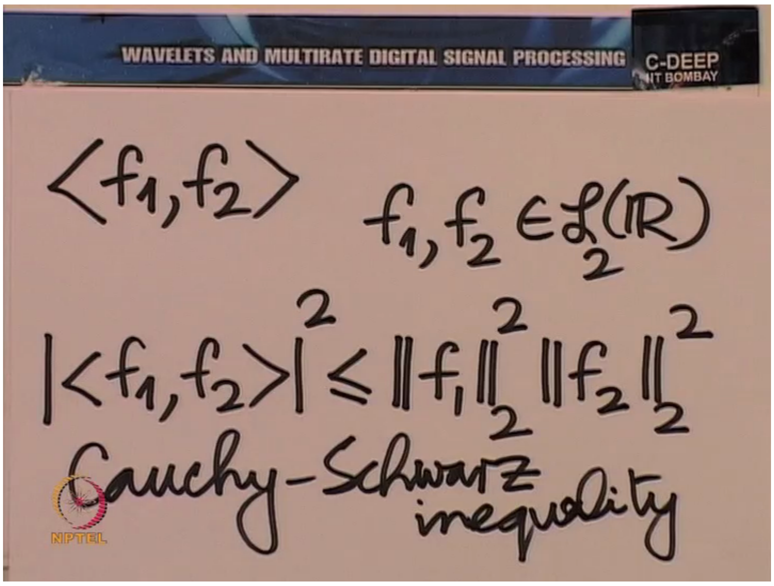

In fact this is a very important property or a very important theorem in function analysis. It is often called the Cauchy Schwarz inequality. It says and let me write that down in formal language. It says the inner product, the magnitude squared of the inner product of 2 functions. So let us resume there are 2 functions F1 and F2 in L2R. The magnitude squared of the dot product of F1 and F2 is less than equal to the product of the squared norms in L2R of F1 and F2.

the Cauchy Schwarz inequality as it is often known. And of course this vectorial interpretation makes this property obvious but one can also prove this formally in functional analysis without taking recourse to the visualisation in the language of vectors. However since it is not our objective to review this basic concept from function analysis, I mean to give a detailed proof, we shall give it this vectorial interpretation and be satisfied. So we will take the Cauchy Schwarz's inequality as true and proceed from there.

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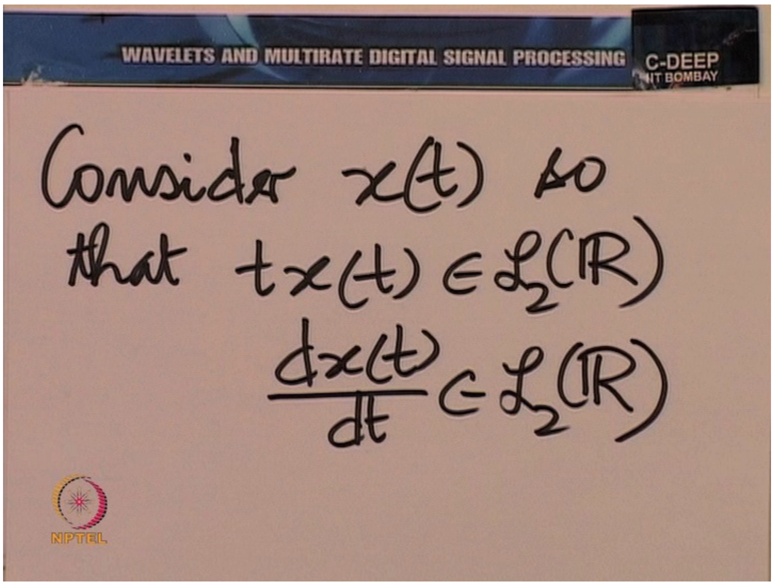

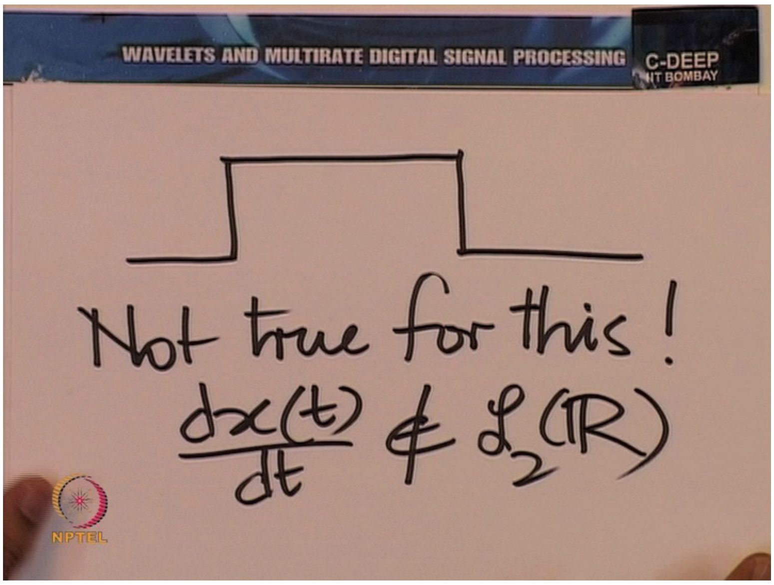

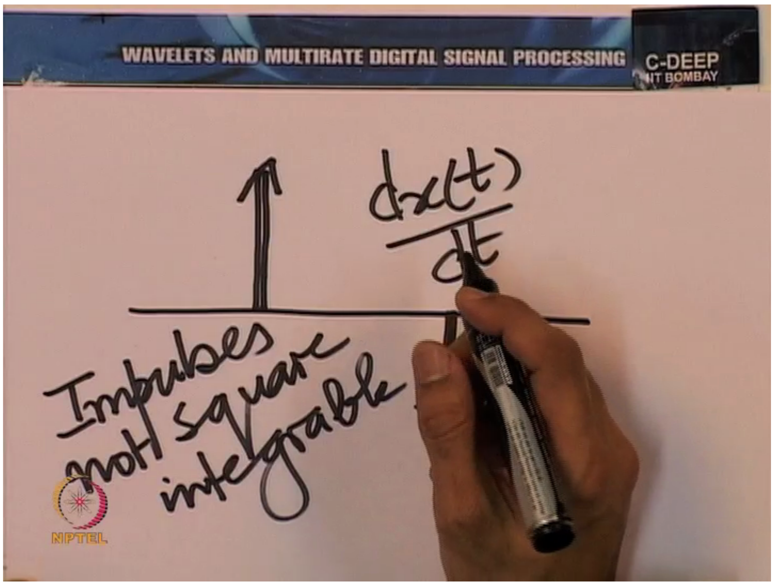

Now, please remember that there are assuming in this process that these functions belong to L2R. So let us go back to our uncertainty time bandwidth product. Consider then xt so that t xt belongs to L2R and d xt dt belongs to L2R. What if they do not? For example, we took the very simple case of a rectangular pulse a while ago. We saw that if we consider this function, not true for this function because d xt dt does not belong to L2R here, why so, d xt dt has 2 impulses, d xt dt would have an appearance like this, an impulse there and a downward impulse at the end of the pulse, this is what d xt dt would look like here.

And impulses are not square integrable, impulses do not have finite energy. I shall just spend a minute in justifying this because we have so far been informally saying it, we also proved this indirectly by looking at the frequency variance in this case. But we also must understand the direct proof here.

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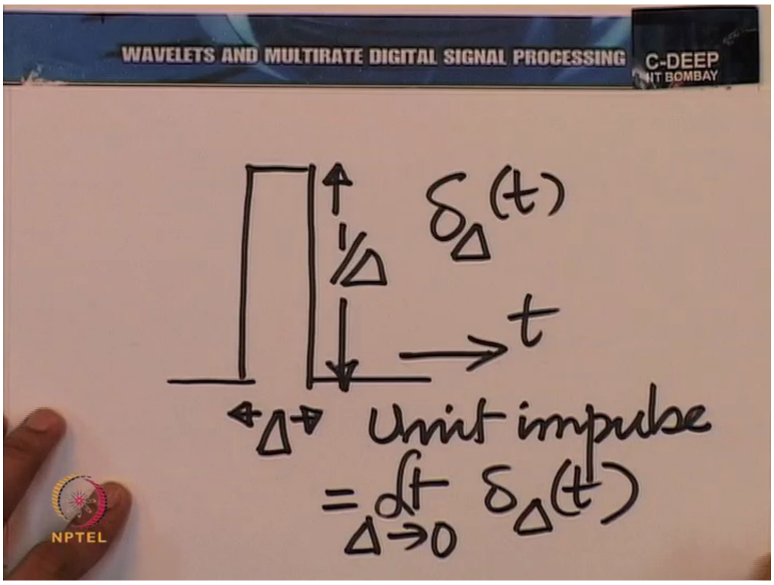

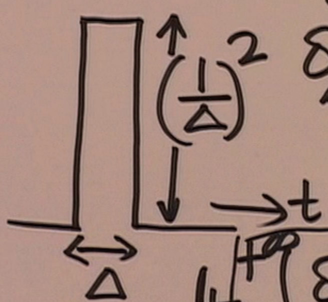

$\delta_\Delta^2(t)$

$$\lim_{\Delta \to 0} \int_{-\infty}^{+\infty} \delta_\Delta^2(t)\,dt \to \frac{1}{\Delta}$$

diverges

The impulse is not square integrable (infinite energy)

Time bandwidth product

$$= \frac{\|t\,x(t)\|_2^2 \left\|\frac{dx(t)}{dt}\right\|_2^2}{\|x\|_2^2 \, \|x\|_2^2}$$

After all what is an impulse, an impulse is the limiting case of this, an object which lies on a width of delta with a height of 1 by delta. This is delta, capital delta t if you please. And an impulse or unit impulse to be more precise is essentially something like a limit as delta tends to 0 of delta delta t. Now when you take the square of this, so delta delta t squared, it looks like this. And now if you take the integral of delta delta t squared dt over all t, then it diverges when you take the limit diverges.

In fact this integral is essentially 1 by delta, 1 by delta squared into delta, so 1 by delta and this is diverging, this limit does not exist. So therefore the impulse is not square integrable, we must make a note of this, this is an important conclusion. Infinite energy, it contains infinite energy and that is manifested also in calculating the frequency variance for this rectangular pulse. Anyway, that apart, we have now agreed that if that is the case, anyway that lower bound does not arise, where these quantities diverge in spite of x belonging to L2R.

Remember, since x belongs to L2R, in the beginning we had this denominator. So, you know you had this denominator right in the beginning, norm x2 x in L2R to the power 4 effectively and this is guaranteed to be finite because we have confined ourselves to x2 into L2R. So if this is in L2R, the denominator is finite and positive obviously, otherwise it does not make sense, I mean you are not going to take a trivial function. So for a nontrivial function, this is strictly positive infinite.

And if anyone of these is infinite, there is no question of finding a lower bound, it anywhere, you know is the worst possible case that you can have. So it is no harm then that we have been considering finite quantities in the numerator. So with that little remark about our restriction, let us take them to be finite and proceed and use the Cauchy Schwarz inequality.

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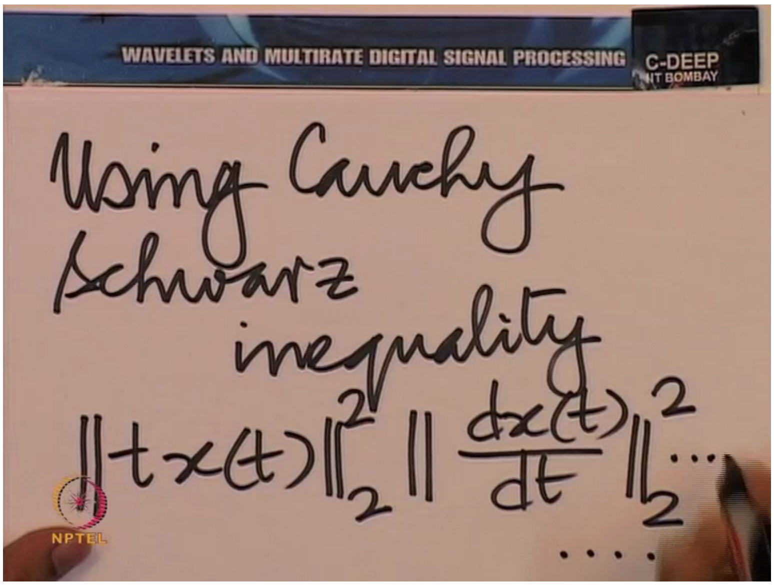

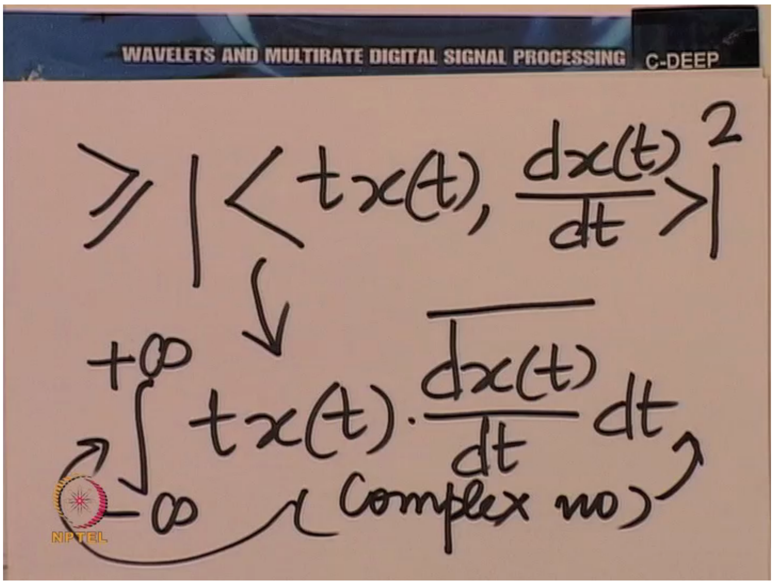

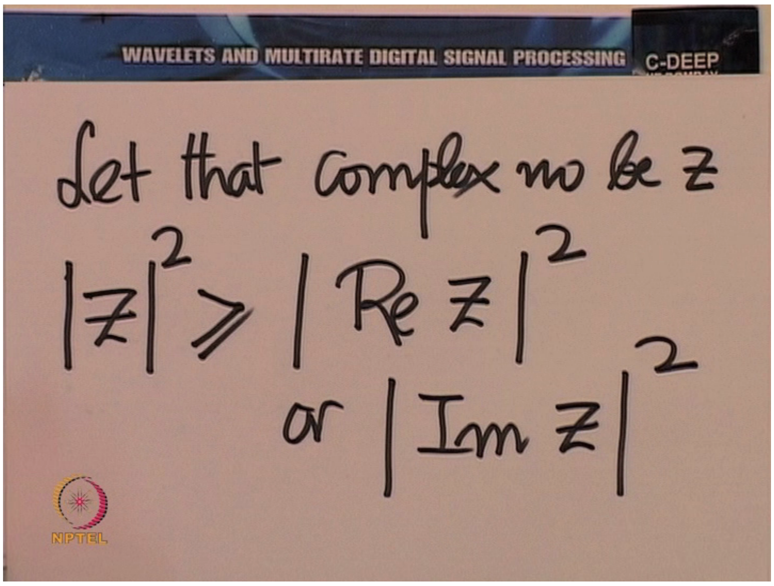

So using Cauchy Schwarz inequality, what do we have? Now we will use it the other way, what we have in the numerator is the norm squared in L2R of t xt times the norm squared of d xt dt in L2R. And from Cauchy Schwarz inequality we go further, this must be greater than or equal to the magnitude of the dot product squared. Let us write this down. This dot product is essentially the following integral. Remember, we need not confine ourselves to real functions, we should not because we are allowing a modulation in time, so we must allow complex functions here.

We have of course centred the functions, that is a different issue. We need to centre them, that we have done. Now, let us essentially look at this integral a little more carefully. Now we are talking about this as an entire complex number. This whole thing is a complex number, from here to here. If you take the magnitude squared of a complex number, so let that complex number be Z. It is obvious that the modulus squared of Z is greater than the modulus or greater than or equal to in general the modulus squared of the real part of Z.

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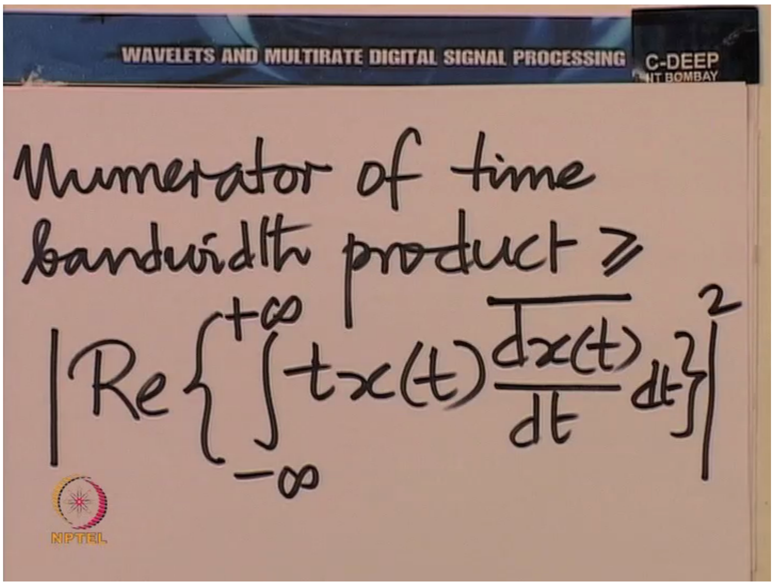

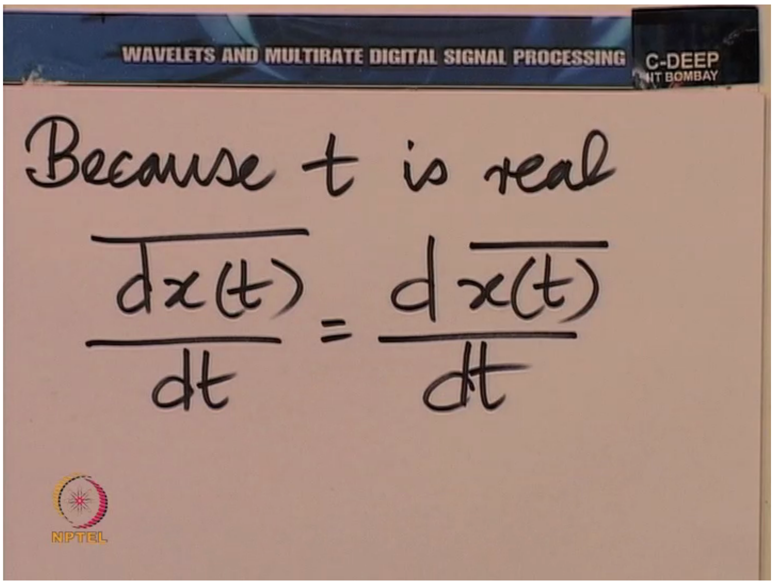

And we use that property here, of course the same thing holds good for the imaginary part too, but we are interested in the real part here. So in particular we have the numerator of the time bandwidth product is greater than or equal to modulus real part of this integral. the complex conjugate is above this entire thing, so this is what we have here. Now, a remark about this part, this complex conjugate.

We are taking the derivative of a possibly complex function x of t with respect to real variable t. So the complex conjugate of d xt dt is also the derivative of the complex conjugate of xt. What I am saying in effect is that because t is a real variable, d xt dt bar is also d x bar dt and I shall employ that in this expression 1^{st}.

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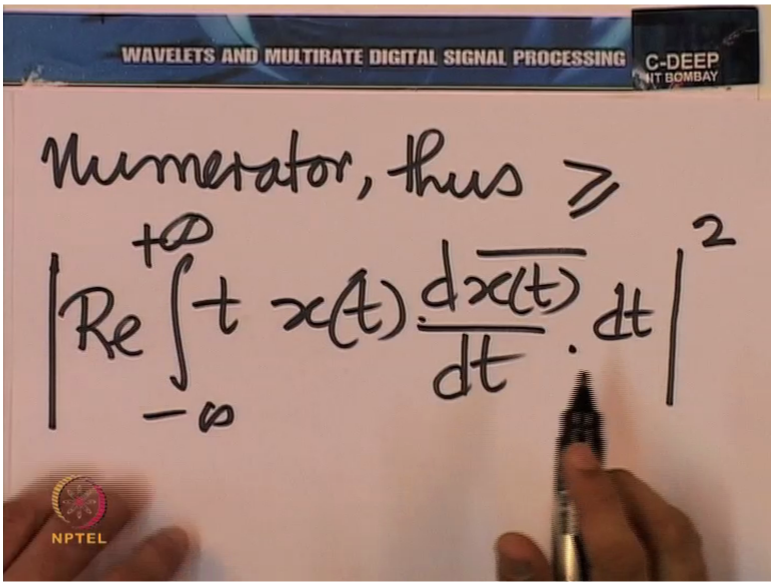

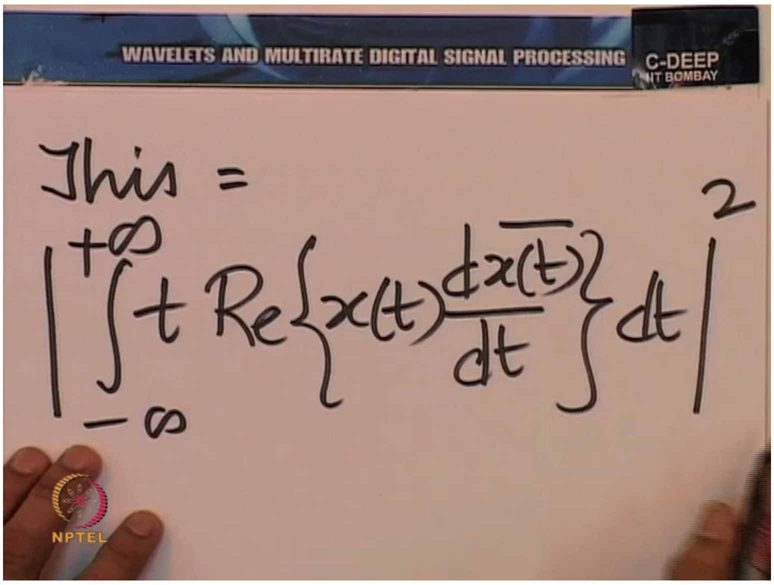

So, this quantity now is equal to the modulus of the real part of the following. The real part operates on the whole integral and now we again look at the real part. You see the real part is operating on integral with respect to t, t is the real variable. So the element of integration is real, this is a real function, this is possibly a complex function. So the real part can be taken right inside the integral and brought to operate on this only, the rest of it does not require you to say real part explicitly.

So this expression is equal to integral from minus to plus infinity t time the real part of xt d x bar dt the whole squared. And now we take note of this, how do we calculate the real part of a complex function, complex number in general? By adding the complex numbers and its conjugate and dividing by 2.

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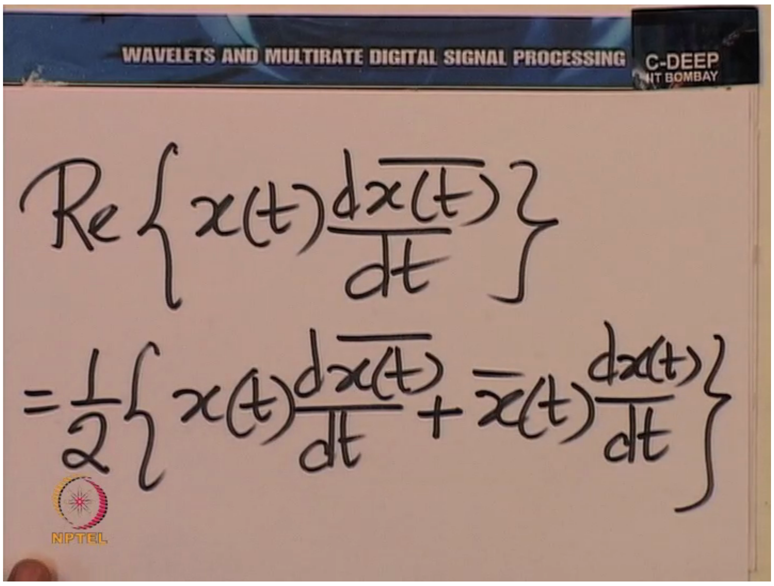

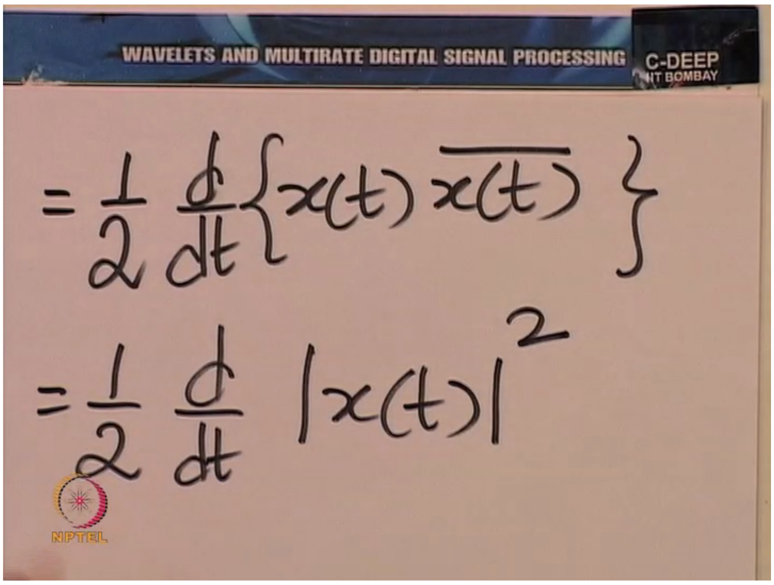

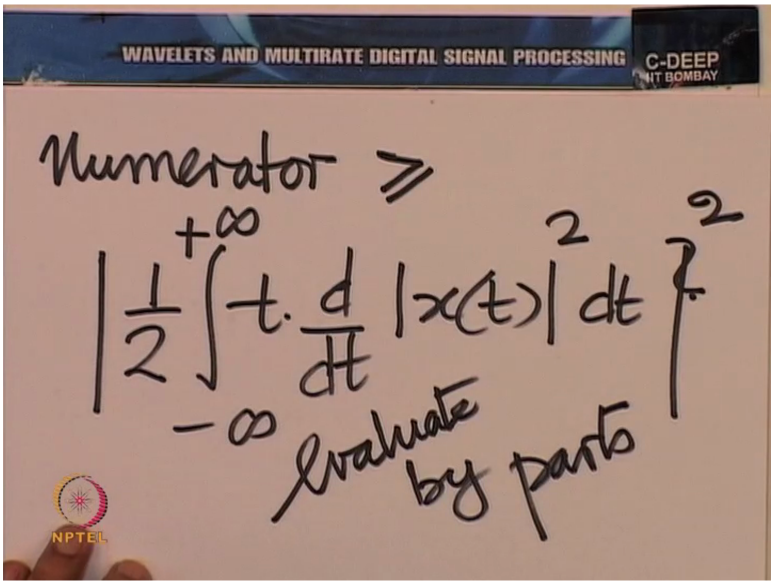

So essentially what we are saying is real part of xt d x bar t dt is half xt dx bar t dt plus the complex conjugate of this. And what is that complex conjugate, it is x bar t dx t dt, simple. And now we can see a product rule has been employed here. Essentially what we have here is the derivative of xt into x bar t, that is an important observation. So here this is equal to half d dt xt x bar t and xt x bar t dt is indeed the modulus of xt the whole squared.

A very beautiful observation and now we will put that observation into the time bandwidth product. So therefore the numerator of the time bandwidth product is thus greater than or equal to modulus half integral from minus to plus infinity t times d dt mod xt squared dt and this the whole squared. Let us take a minute to reflect on this, how do we evaluate this integral? Well, that is easy. We can evaluate this integral by parts and to evaluate by parts, we must 1st make the integral indefinite.

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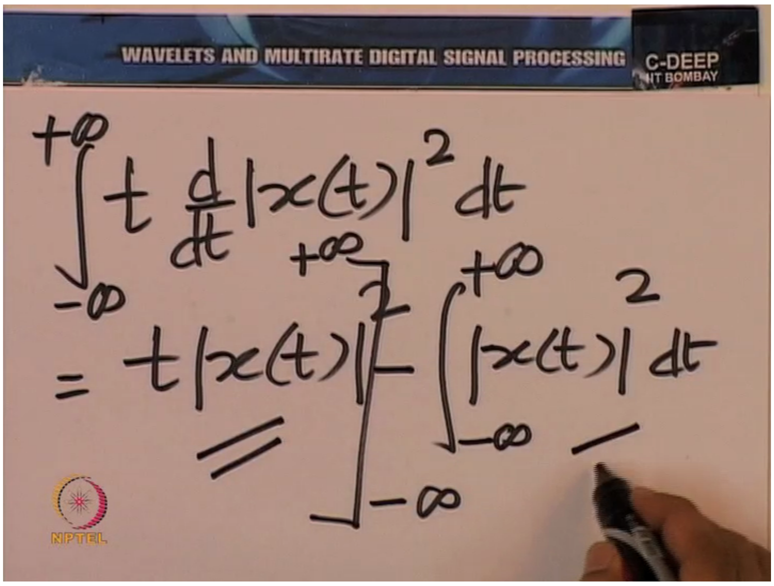

So let us simply consider the indefinite integral corresponding to this. Clearly, this is this term minus this term. And now we can substitute the limits. So we can put back the limits of minus to plus infinity here and put this from minus to plus infinity 1st and then this from minus to plus infinity as well. Now let us focus our attention on each of these terms individually. Let us take the 1st term t times mod xt the whole squared evaluated at plus infinity and then from it subtract evaluated as minus infinity.

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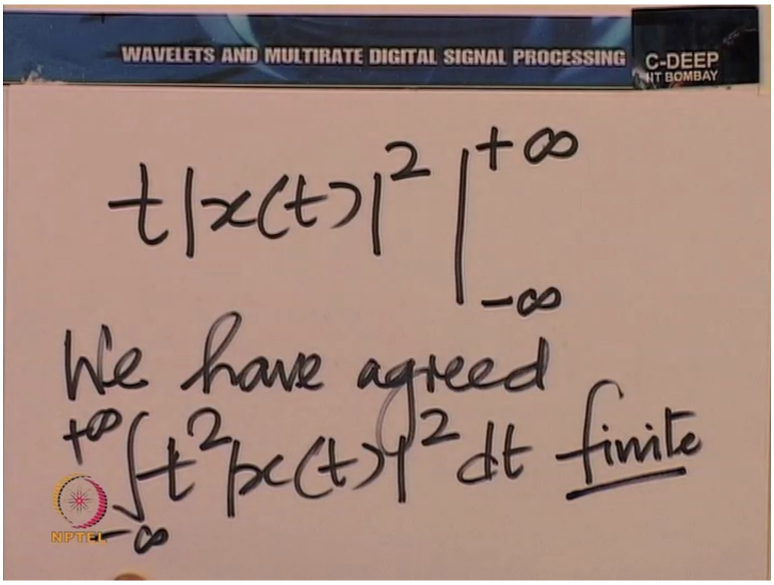

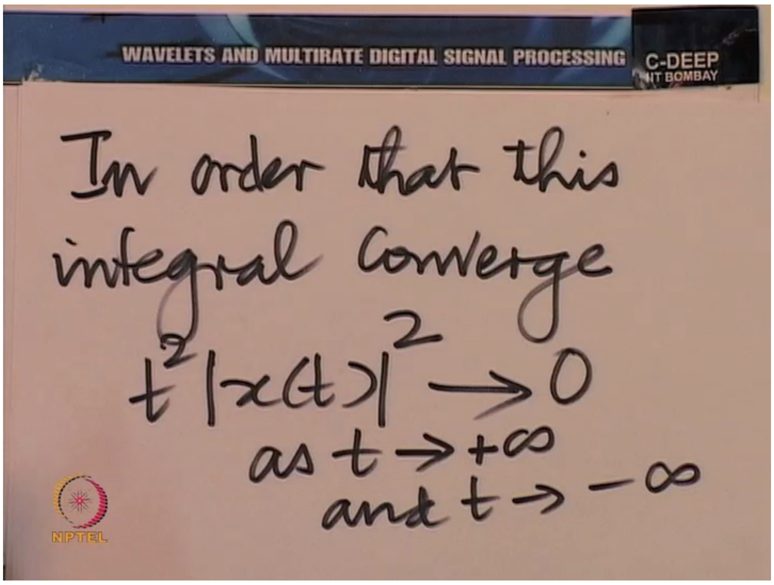

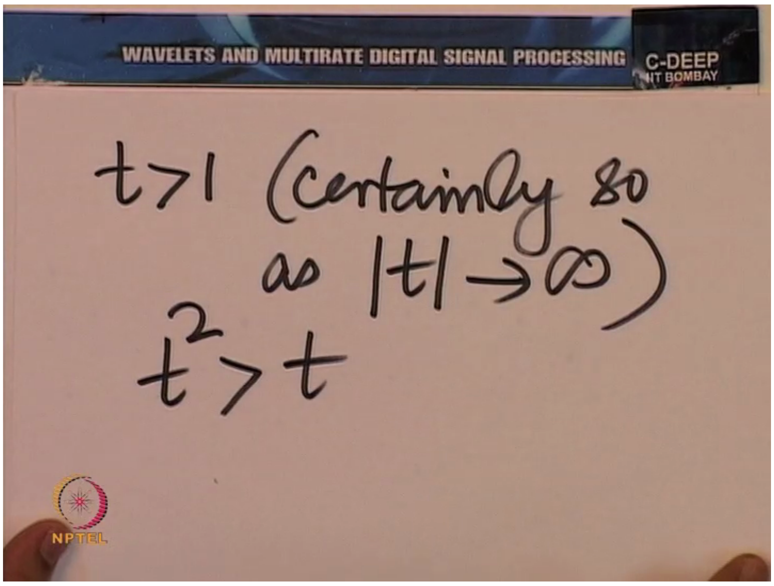

So let us focus our attention on t mod xt the whole square evaluated at the 2 limits and subtracted. Now we have agreed that t squared mod xt squared dt from minus to plus infinity is finite, we have agreed on that. We said if that is not true, anyway we have an infinite bound, there is no question of lower bounding then, it is a worst-case that we can deal with. So if this is finite, obviously you see the integral must be finite, the function must decay towards 0 at the end.

If the function does not decay towards 0 asymptotically as you go towards plus infinity and as you go towards minus infinity, you can see that there is going to be an infinite range over which the function has a finite positive value which would make the integral diverge. So in order that this integral converge, t squared mod xt square dt, the function t squared mod xt squared must decay as t tends to infinity and minus infinity. Let us make that observation.

In order that this integral converge t squared mod xt squared tends to 0 as t tends to plus infinity and t tends to minus infinity. Now it is also true that for t greater than 1 and of course as t tends to infinity, t is definitely going to be greater than 1. So for t greater than 1 and certainly so and I am talking about t squared actually or mod t you know. So certainly so as mod t tends to infinity, t tends to plus inifinity or minus infinity, you must have t squared greater than t.

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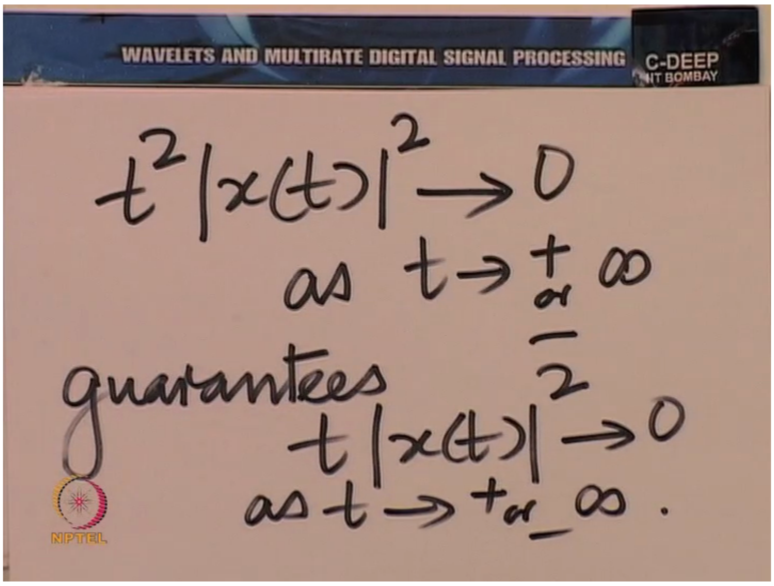

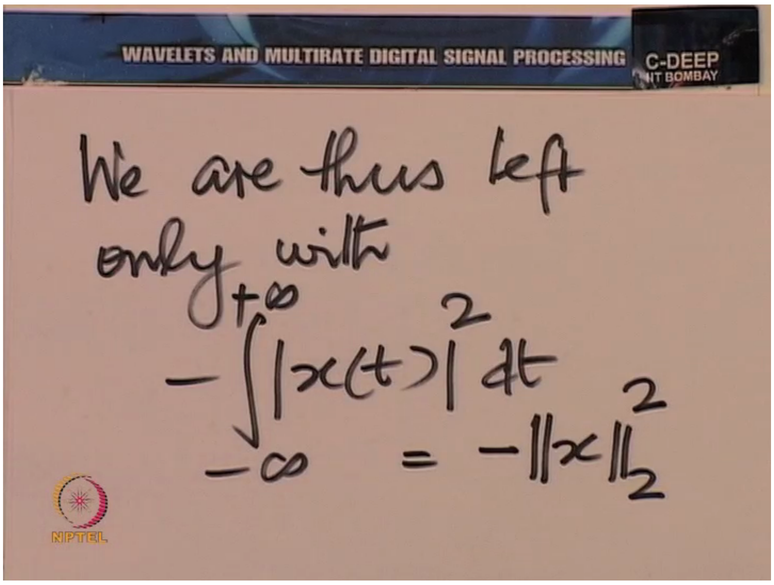

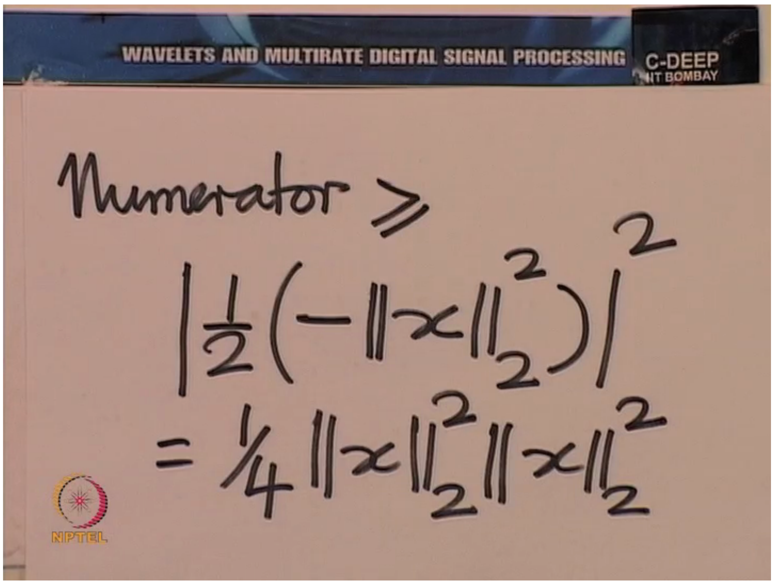

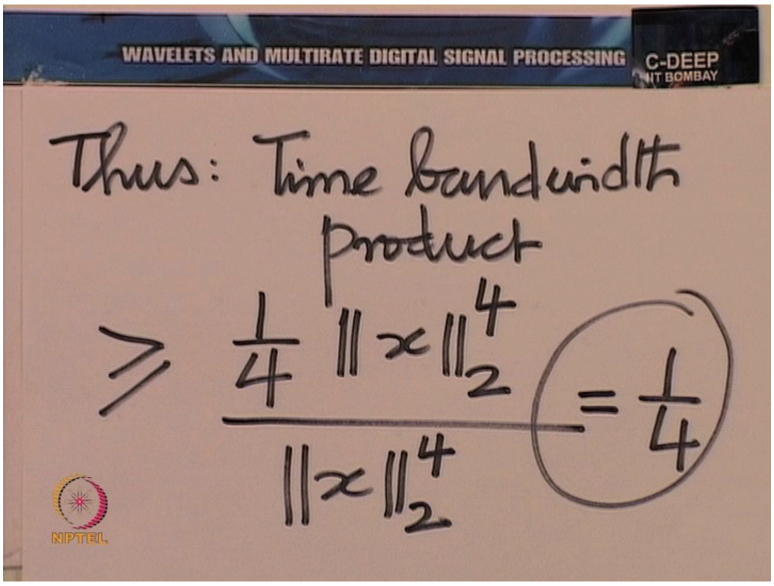

What this means is that t squared mod xt squared tending to 0 and t tends to plus or minus infinity guarantees t mod xt squared tending to 0 as t times to plus or minus infinity. And therefore that 1st term has vanished, so we are left only with the 2nd term. And the 2nd term is very familiar to us. In fact this is minus the norm of x in L2R the whole squared, so simple. And therefore we have a very beautiful conclusion here. We are saying the numerator is always greater than or equal to modulus half into minus the norm of x in L2R the whole squared the whole squared which is one fourth times this times this.

And therefore overall the time bandwidth product is greater than equal to one 4th norm x in L2R to the power 4 divided by norm x in L2R to the power 4 which is one 4th. A very very fundamental conclusion.