Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-7.
Lecture-18.3.
Simplification of Time-Bandwidth formalae.

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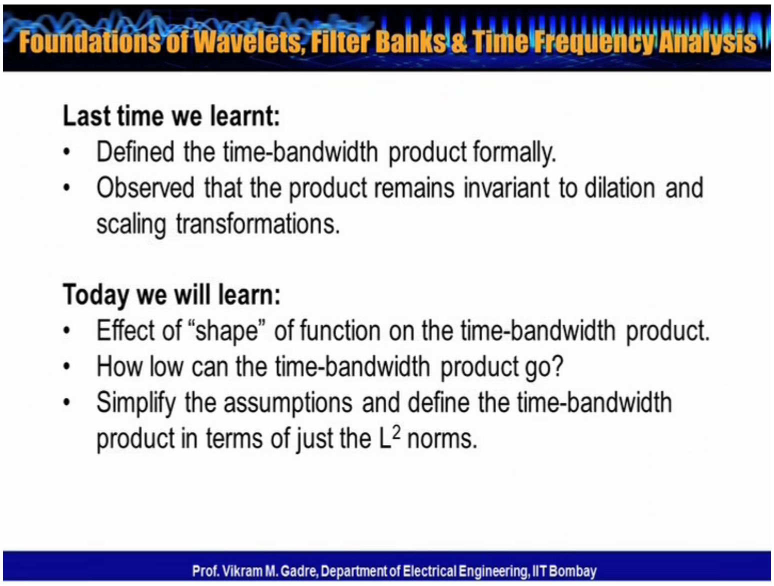

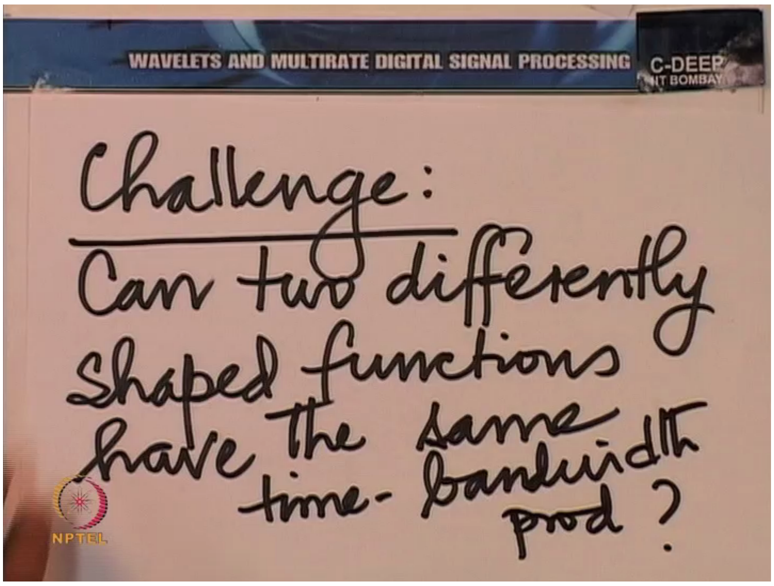

So the time bandwidth product is something very fundamental about a function. What is it then variant to, what does it change with? It changes essentially with the shape. So different shapes have a certain time bandwidth product associated with them. that is interesting. Now I put before you a question for thought. I like to put certain questions to challenge your imagination and this is one of them. Can 2 different shaped functions have the same time bandwidth product?

And so as not to leave you entirely in the dark, I give you a hint. What would happen if you took the Fourier transform of a function? Can one employ the property of duality somewhere and with that slight hint which I am sure should take you rather far, I shall leave the rest of the reasoning to you to answer this question. Anyway, that would also bring forth one more kind of invariance, so it is in some sense giving you 2 gifts all at once and answer to this question and one more kind of invariance that this time bandwidth product exhibits.

Anyway, so now what we have seen is that this time bandwidth product is a very important quantity in the context of functions. It in some sense is characteristic of the shape, though I must remark not unique to the shape but characteristic of the shape. Now, we saw what was the time bandwidth product of the pulse in the previous lecture.

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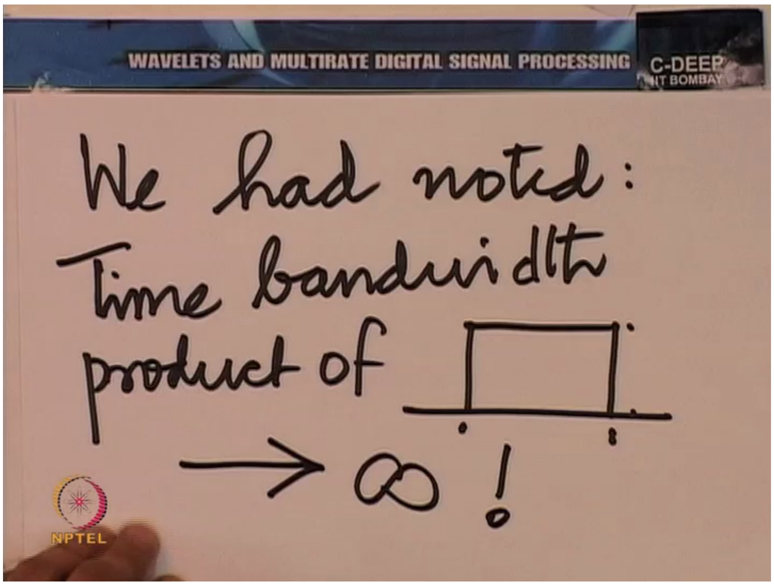

Last time we had noted that the time bandwidth product of this pulse, now you will agree with me that I do not really need to write the limits here, nor do I need to give you the extent of this, nor even do I need to give you the height of this, just the shape would do because whatever be this extent, whatever be this height, the time bandwidth product is all the same. But as you noted, the time bandwidth of this product tends to infinity. And we had asked, well, this is terrible.

It told us why we were not content with the Haar. When you are dealing with Haar in terms of the scaling function or the wavelet function, you are essentially working with functions of an infinite time bandwidth product. Having a large time bandwidth product is cap. It means that I cannot localise nicely in time and frequency. So the whole objective or the whole game is to see how small you can make this time bandwidth product.

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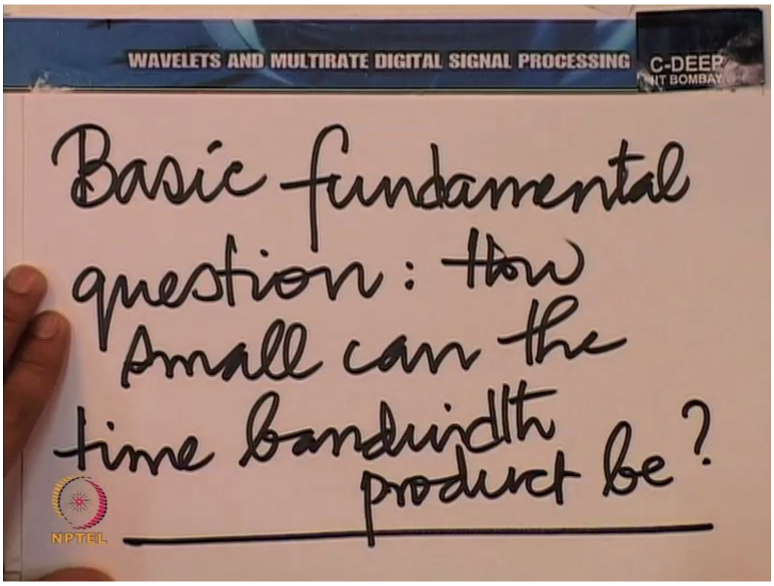

So that is the question that the uncertainty principle is going to ask and we shall formulate that question today. A basic fundamental question… How small can the time bandwidth product be? the Haar, terrible, infinity. Well, we shall have good news very soon, we shall be able to come up with functions whose time bandwidth product is not cap at all. But then we will again see what nature often does to us were scientists and engineers.

You can reach within a certain level of this bound that we are going to come to soon but reaching the boundary itself is impossible. Anyway, that is just a prelude, now what we need to do is to establish that bound and to establish that bound let us 1st simplify our work, as we should always do. try and take away unnecessary trappings from the problem, identify the essence the core of the problem and then try and solve the problem. So what we are trying to establish is the fundamental lower bound on the time bandwidth product.

We have already noted the invariance of the time bandwidth product to a number of different operations, namely shifting in time, shifting in frequency, multiplication by constant and scaling the independent variable. Now what we shall do therefore is to do away with the requirement of time and frequency Centre.

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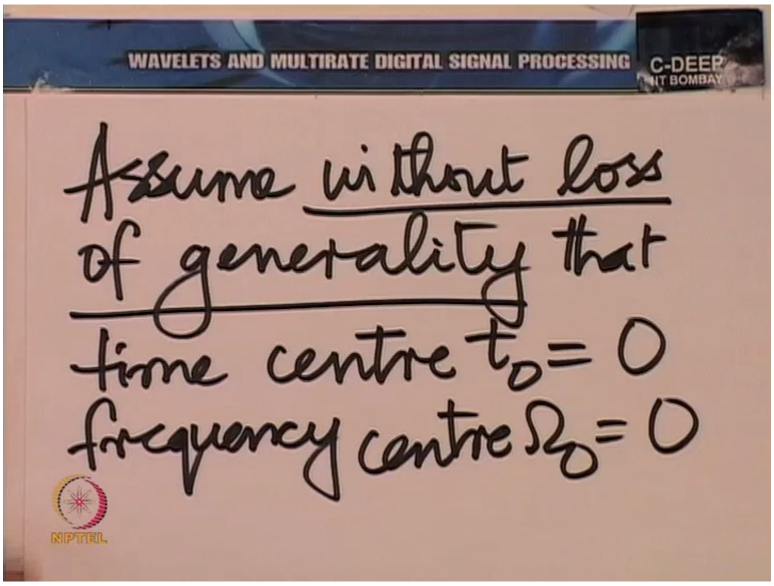

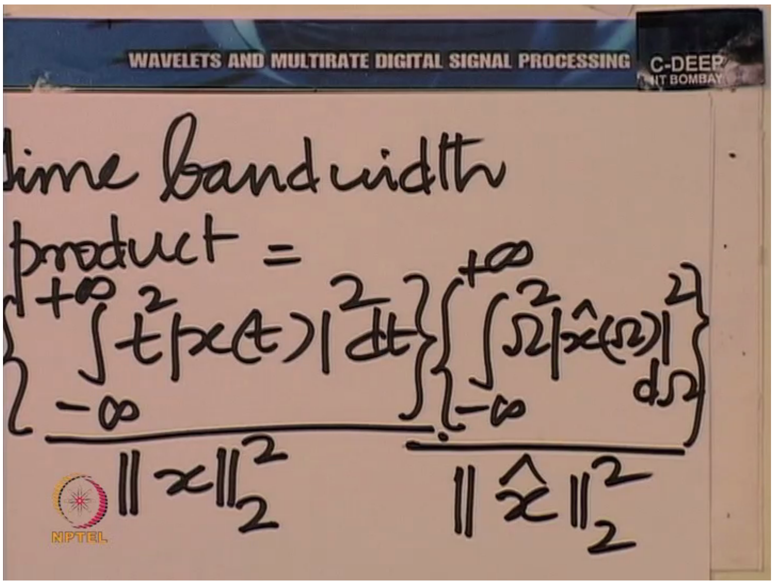

So let us 1st, to answer this question, assume without loss of generality and this without loss of generality is because of all that I just said, that the time Centre and the frequency Centre are both 0. After all, if the time Centre is not 0, you could always shift and make the time Centre 0 in time without affecting the time bandwidth product. If the frequency Centre is not 0, we can always shift in frequency which means modulate in time and bring the frequency Centre to 0.

Of course if the function is real, we do not even need to do that. So this is without loss of generality, let us make the time Centre and the frequency Centre 0, whereupon we then have the time bandwidth product to be essentially the following. It is integral - to + infinity the whole squared of x multiplied by t square dt divided by the norm of x squared times something similar in frequency. So we keep this here, - to + infinity omega squared mod x cap omega square d omega.

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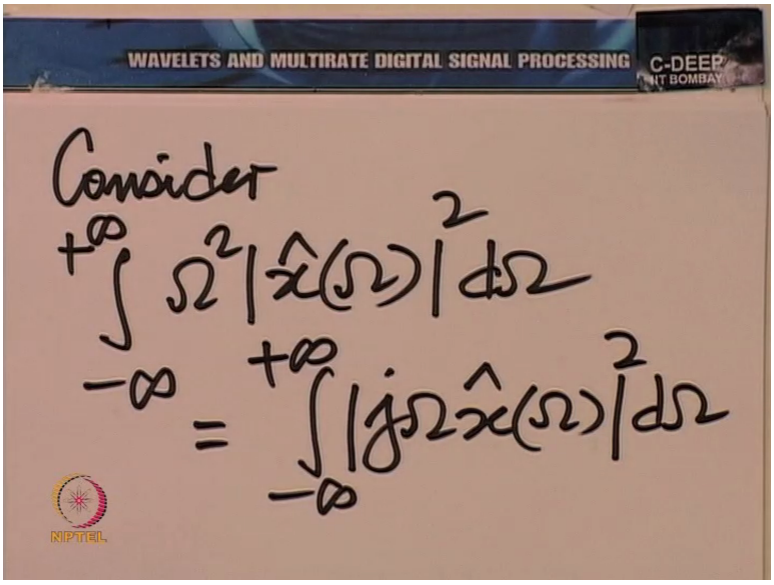

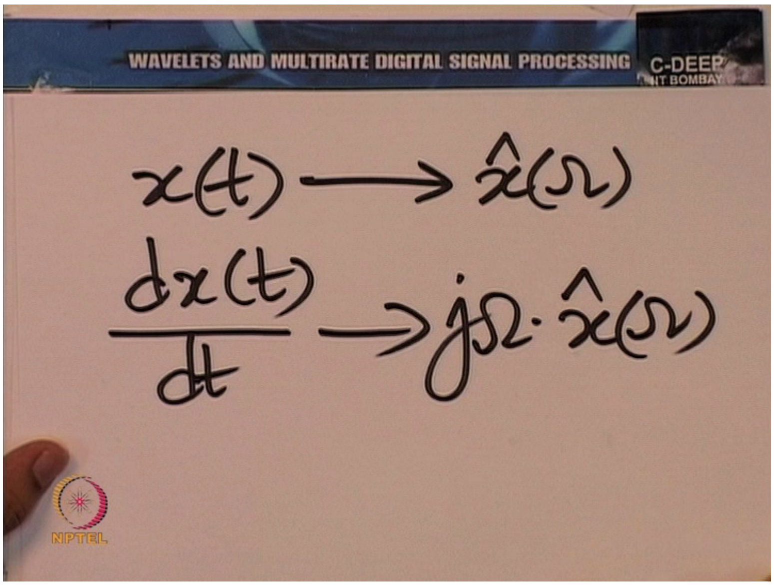

WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

C-DEEP IIT BOMBAY

$$\int_{-\infty}^{+\infty} \left| j\Omega \hat{x}(\Omega) \right|^2 d\Omega = 2\pi \int_{-\infty}^{+\infty} \left| \frac{dx(t)}{dt} \right|^2 dt$$

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WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

C-DEEP IIT BOMBAY

Time bandwidth product =

$$\frac{\left\{ \int_{-\infty}^{+\infty} t^2 |x(t)|^2 dt \right\}}{\|x\|_2^2} \cdot \frac{\left\{ \int_{-\infty}^{+\infty} \Omega^2 |\hat{x}(\Omega)|^2 d\Omega \right\}}{\|\hat{x}\|_2^2}$$

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Time bandwidth product =

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And here again divided by the norm of x cap the whole square. Now we have already made an observation about this and we formally make that observation again. We have already seen that this essentially is - to + infinity J omega x cap omega modern whole squared d omega. And that we have noted is essentially, you see, noting that if xt has the Fourier transform x cap omega, then d x t dt has the Fourier transform J omega times x cap omega.

Whereupon integral - to + infinity J omega x cap omega Mod squared d omega is essentially integral from - to + infinity d xt mod, d xt dt mod squared dt but there is a factor of 2 pie, so it is 2 pie times this. And of course we also see that norm of x cap square is 2 pie times the norm of x square. And therefore if we now put back all together here, when I all these together, I keep these as they are in the time bandwidth product and I know that this is essentially the norm squared or the energy in the derivative divided by the energy in the function, the factor of 2 pie cancels from the numerator and denominator here.

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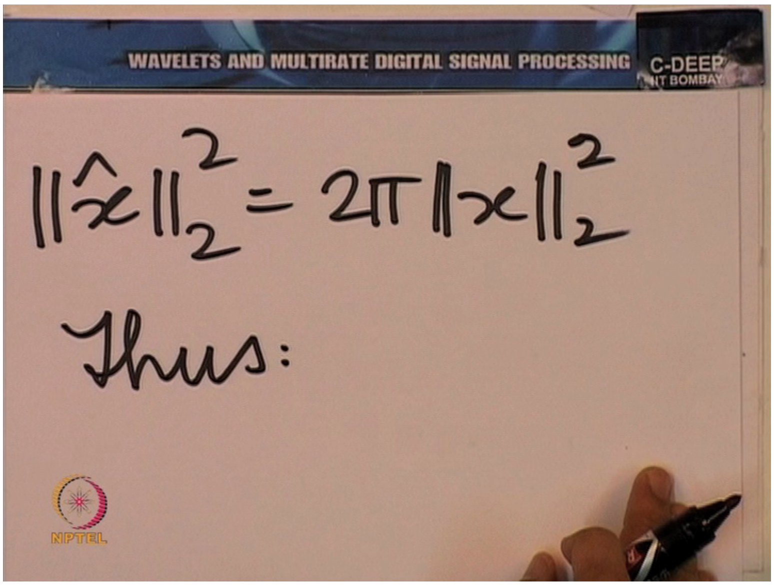

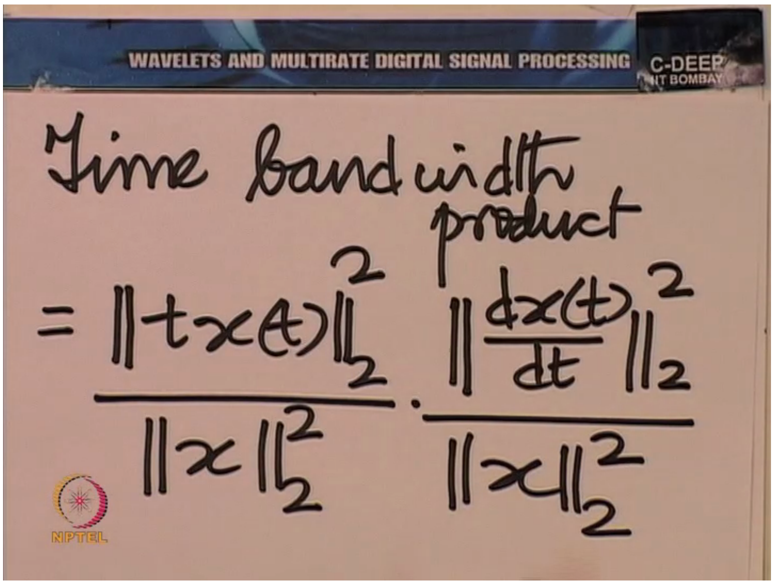

So we could write down all in all, thus the time bandwidth product, the time bandwidth product can be written as essentially the energy of the L2 norm of t xt whole squared divided by the L2 norm of x multiplied by the L2 norm of d xt dt whole squared divided by the L2 norm of x the whole squared, a very interesting result. I will just remind you that when we put back the expression for the time bandwidth product, this is essentially the L2 norm of t xt and that is how we come to this conclusion.

So here of course there is a bit of abuse of notations, we are talking about a function t xt and then taking its L2 norm. So this is a ratio of 2 products of L2 norms and now we are all set to minimise this product. In fact that would be our next step, in the beginning of the next lecture. What is the minimum value of this product, minimum over all x that can exist in L2 R?

And that would give us a very fundamental bound in nature which is called the uncertainty bound. We shall derive that uncertainty bound in the coming lecture, thank you.