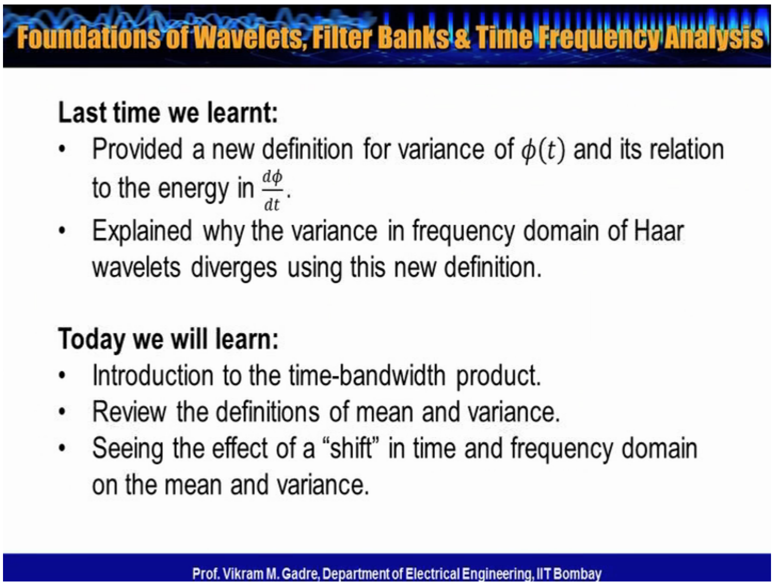
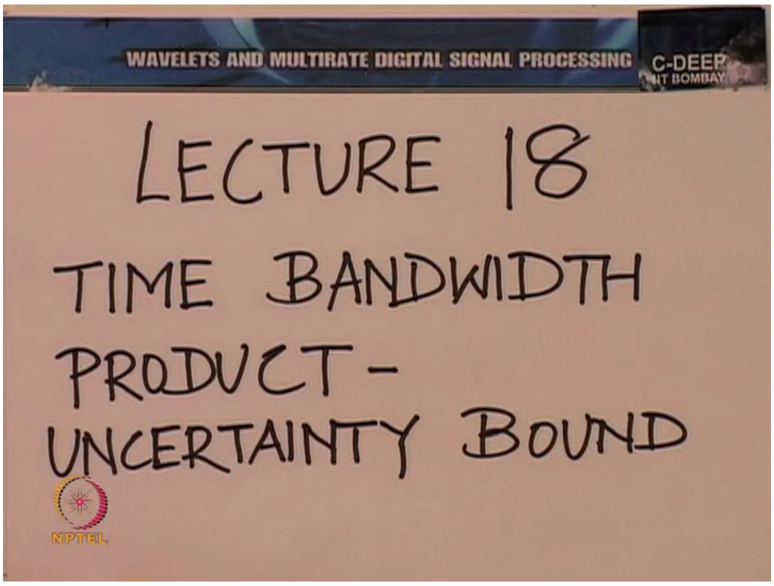
Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
Professor Vikram M. Gadre.
Department Of Electrical Engineering.
Indian Institute of Technology Bombay.
Week-7.
Lecture-18.2.
Signal transformations: effect on mean and variance.

(Refer Slide Time: 0:17)

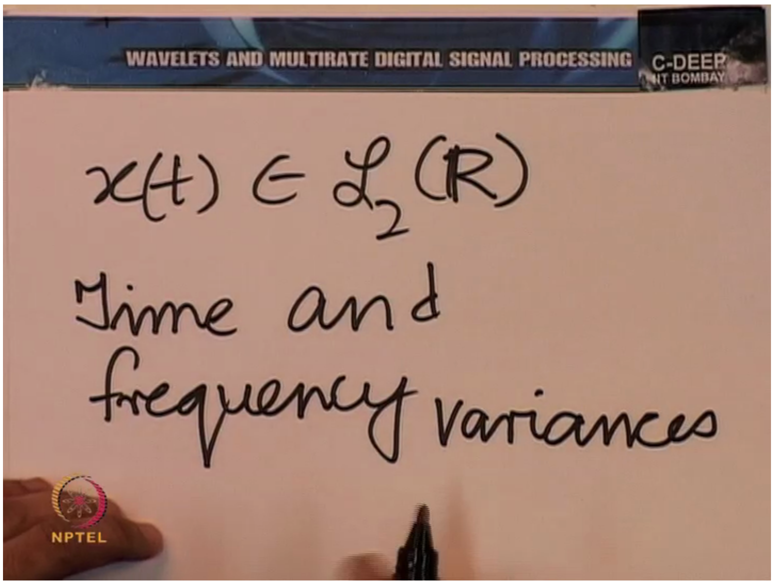


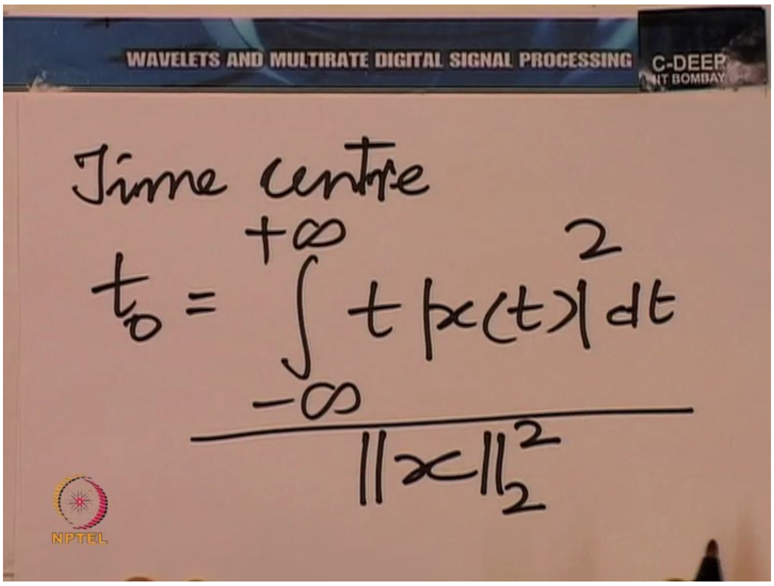


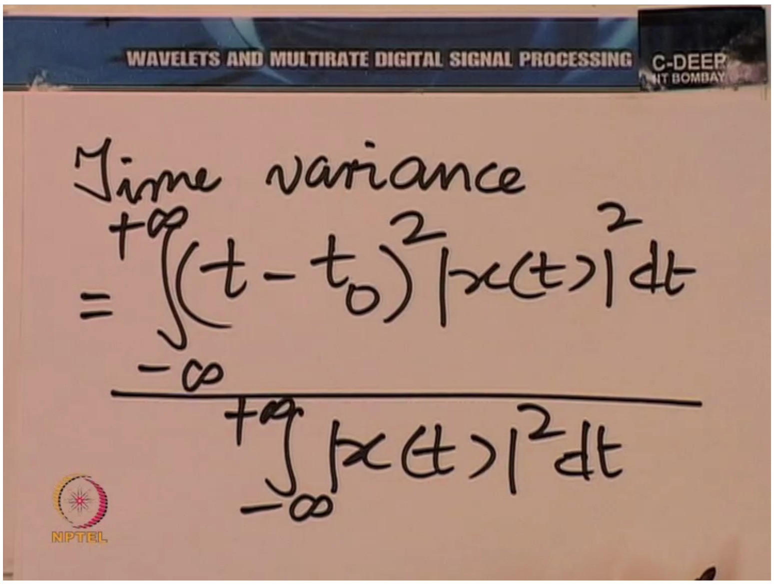
A very warm welcome to the 18th lecture on the subject of wavelets and multirate digital signal processing. We build further in this lecture the idea of uncertainty and what is called the time bandwidth product, the product of Sigma t square and Sigma omega squared, as we defined them the last time. Now we shall just recall a few ideas from the previous lecture to put our discussion in perspective.

So what we intend to talk about today is what is called the time bandwidth product, essentially, as I said the product of the time variance and the frequency variance. And we wish to build a lower bound on the product based on some very fundamental principles that could be explained when we look at functions as vectors. So with that background let us recall some of the definitions that we have brought out in the previous lecture.

(Refer Slide Time: 1:55)



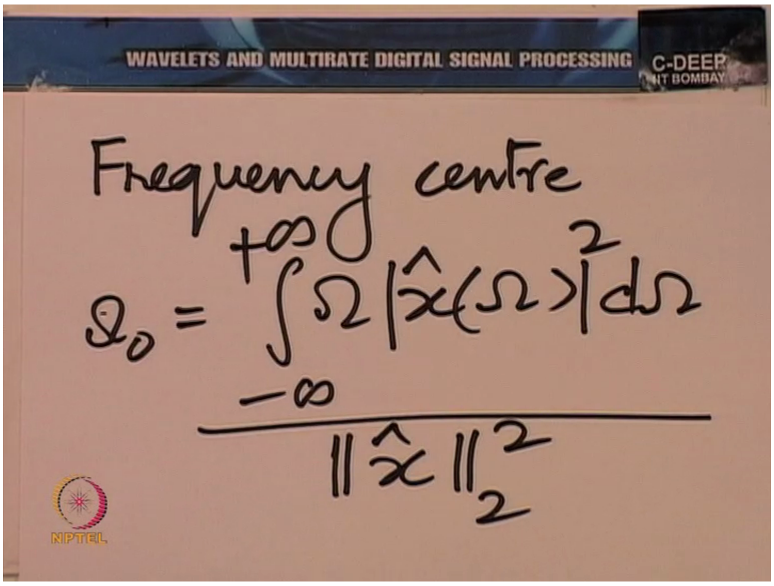


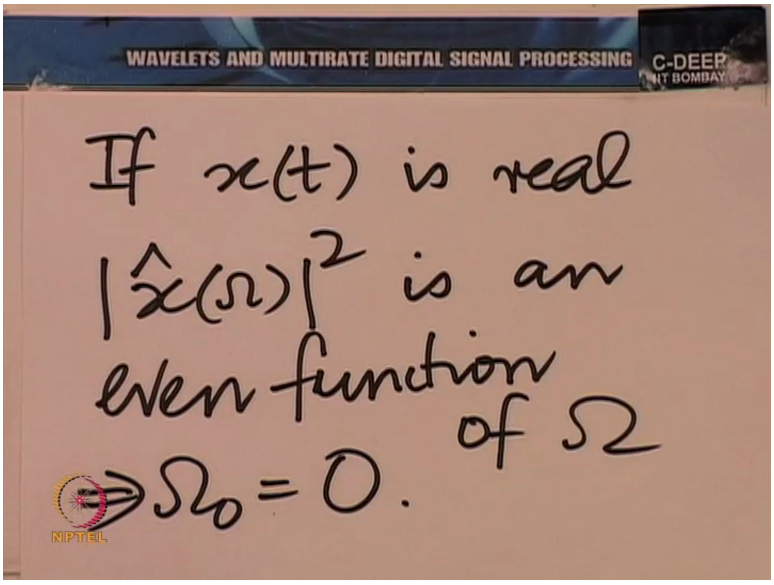


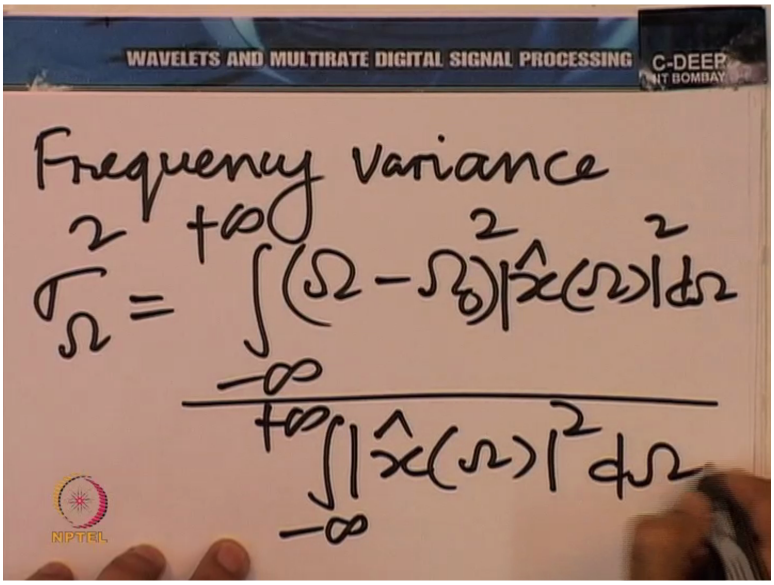
Last time we had agreed to confine ourselves to a function xt belonging to L2R and we said we could define its time and frequency variances. In fact, the time variance was 1[st] defined by choosing what is called the time centre, the time centre t0 was defined according to, according to this, essentially treating xt or mod xt squared as a mass in terms of t and then looking for the centre of mass, that is the interpretation of this.

And we assume the Centre was a finite number, I mean that is a reasonable assumption in almost all cases that we will talk about. From the time centre we define the time variance, the time variance was given by t - t0 the whole squared mod xt squared dt integrated over all time divided by the integral of mod xt squared. Essentially this time variance was indeed the variance of a density which we constructed out of that mass, mod xt squared divided by the norm of x the whole squared.

(Refer Slide Time: 4:23)





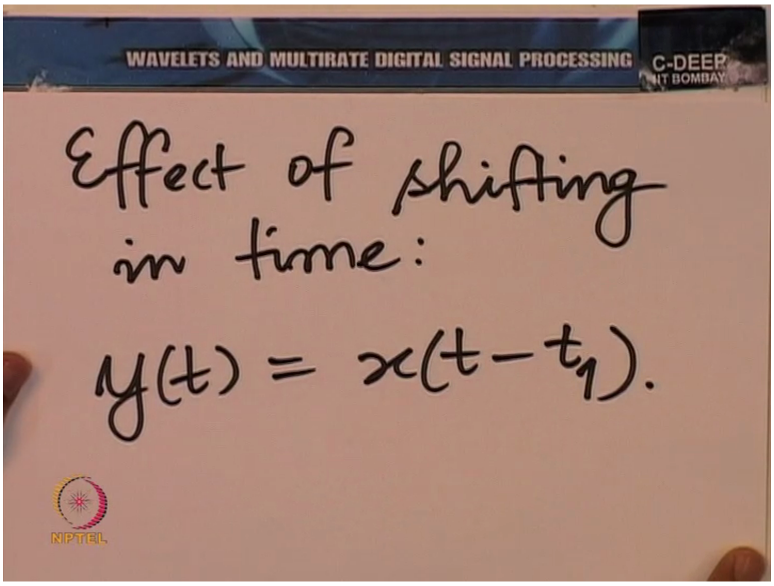


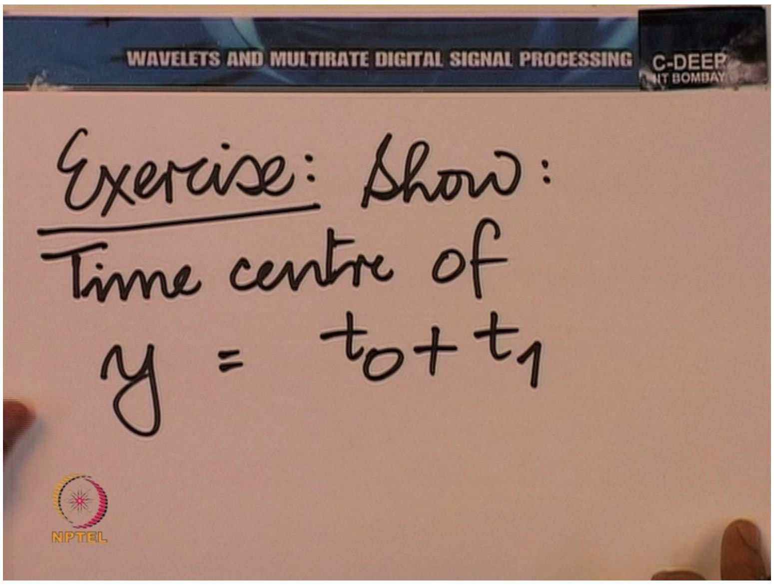
Now similarly we could talk about the frequency Centre and the frequency variance. So we define the frequency Centre to be capital omega 0 given by - to + infinity integral omega mod x cap omega the whole square divided by the norm x cap squared. Again we assume the frequency Centre is finite which is of course always, almost always the case, there is no problem. And we also noted that for the real xt, this frequency Centre is 0.

So we said if xt is real, then mod x cap omega square is an even function of omega, and that implies that omega 0 is 0 because of symmetry. Now we could similarly define the frequency variance. The frequency variance Sigma omega square so to speak is essentially integral omega - omega 0 squared, modular squared of x cap omega divided by a similar integral of mod x cap square.

So once again the frequency variance is like a variance on the density constructed out of the modulus squared of the Fourier of the Fourier transform. Now let us see what happens when we shift a function in time. As we expect, when we shift a function in time, there is no change in the magnitude of the Fourier transform. In fact all that happens is that the centre shifts.

(Refer Slide Time: 7:21)

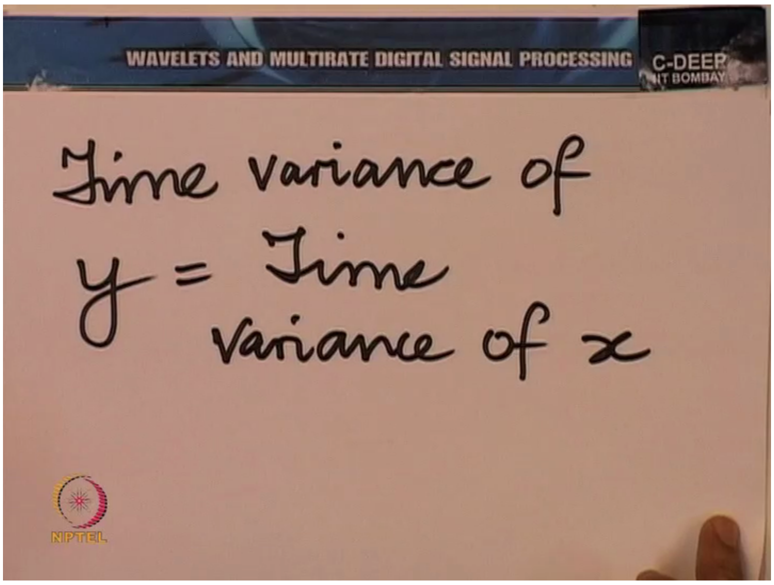




So effect of shifting in time. So let us consider y of t to be x of t - t1. Now we can easily write down the integrals corresponding to yt and they are not very difficult to evaluate. I shall not go through all the details of working but I shall straightaway put down some of the important results here. In fact I put it as an exercise. We can show that the time centre of y would essentially be, you see I will take an example.

Suppose for example you have a function centred at the point t equal to 5, so t 0 is equal to 5. Now if you shift the function backwards by 5 units in time you would get a function centred at 0, that is easy to visualize and if one just write the integrals down carefully, that is easy to prove. So essentially what it means is that when you shift backwards by the Centre, you are bringing it to be centred around 0 and with that we can generalise saying that the time centre of y is essentially t0 + t1.

(Refer Slide Time: 9:34)

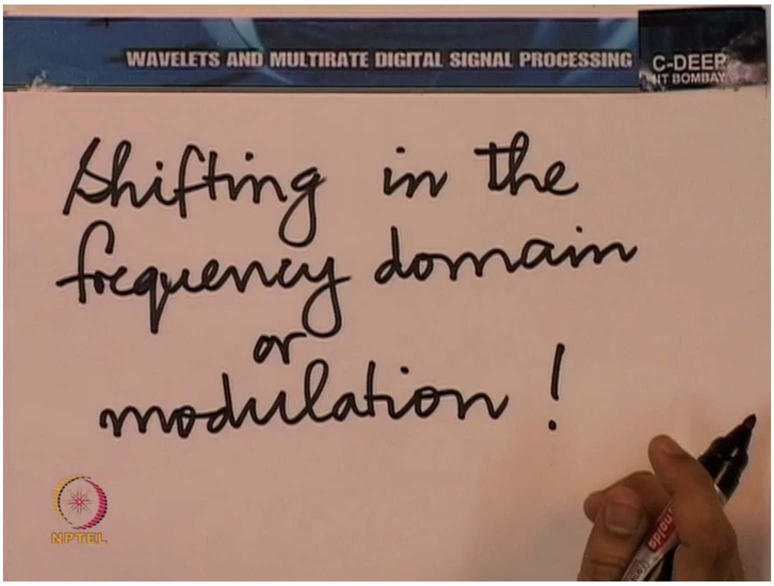


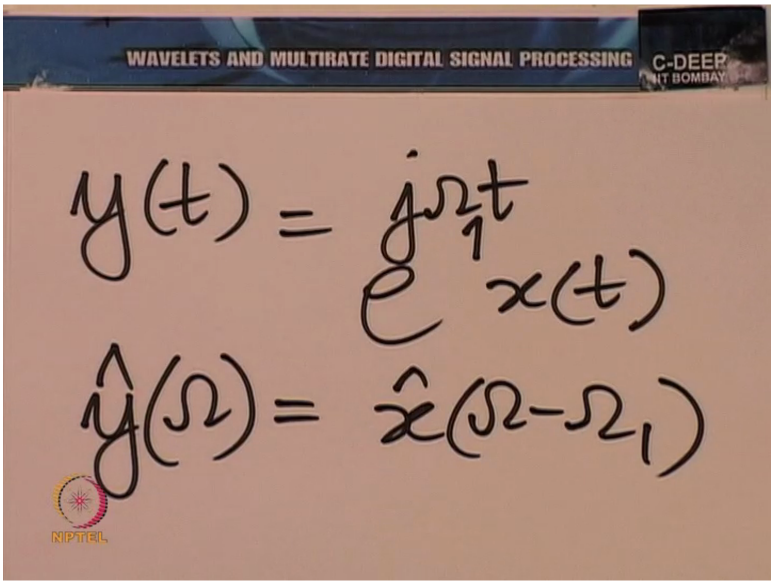
So for example, if t1 is equal to - t0, then the centre of y would be 0. Anyway, so one can show the time centre of y is 0, the time variance of y is equal to the time variance of x. Again this is a very easy result to show, just amounts to writing the integrals down carefully in making the transformation of variable, so I would not go into the full proof here. Similarly we can now analyse what happens in frequency. What happens to the Fourier transform, let us look at that 1^{st}.

So if y of t is equal to x of t - t1, the Fourier transform y cap omega is E raised to the power - J omega t1 times the Fourier transform of x. And very clearly mod y cap omega the whole squared is equal to mod x cap omega the whole squared, that is very easy to see here. So you see under a shift in time, the magnitude square or for that matter the magnitude of the Fourier transform is unchanged.

The mass as seen on the frequency axis remains unchanged in the Fourier domain. So you can shift as much as you desire and it has no effect on the Fourier transform magnitude square, as you can see it affects the phase but not the magnitude. And the phase is not of consequence to us as far as the Centre and the variance are concerned. So anyway with that we have understood what happens when we shift. Now let us take the dual operation of shifting, namely modulation.

(Refer Slide Time: 11:47)

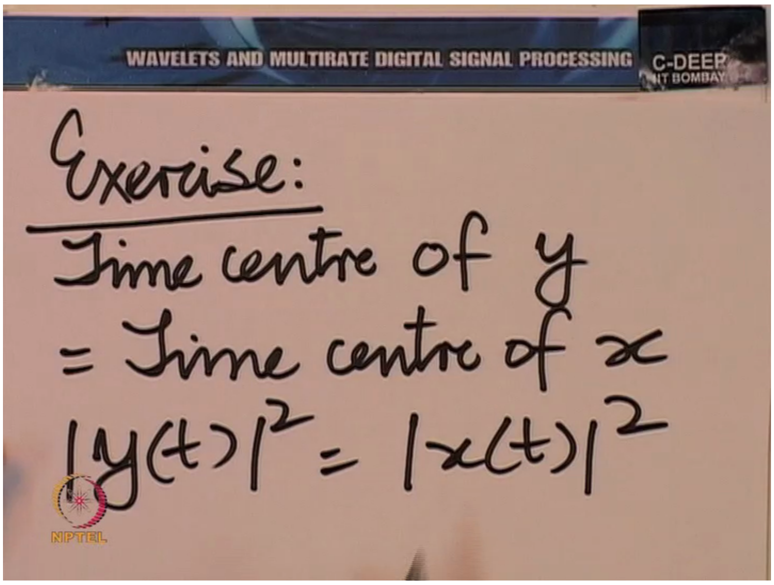


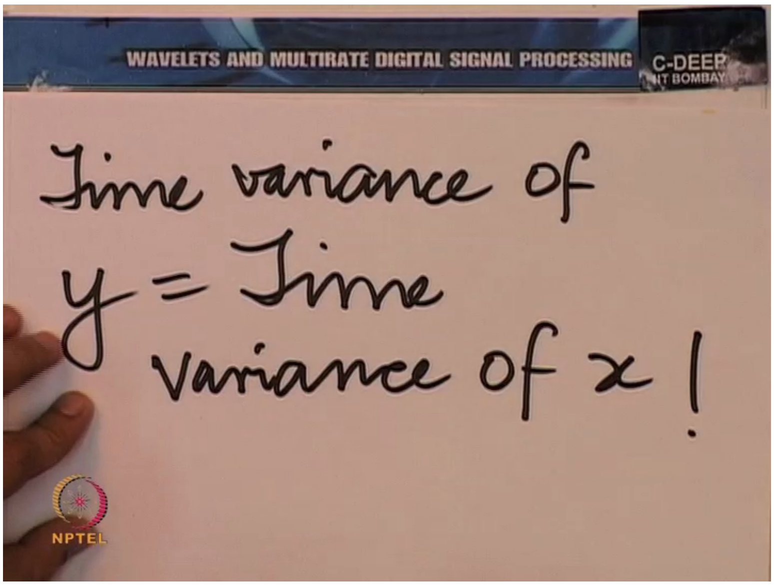


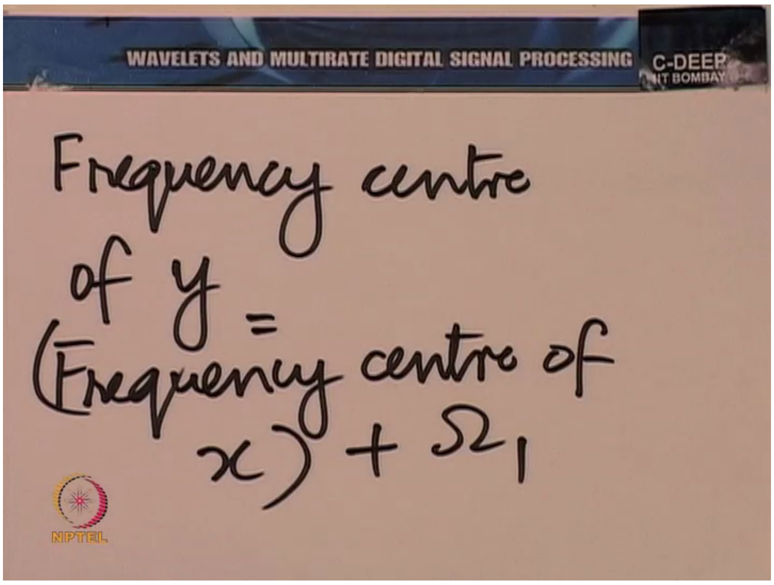
So in fact let us look at it as a shift in the frequency domain. So let us consider yt to be a modulated version of xt, let us consider yt equal to E raised to the power J omega 1 t times xt. Now course it is very easy to see the Fourier transform y cap omega is x cap omega - omega 1. You will recall that this idea is used in amplitude modulation or for that matter in frequency modulation in communication engineering.

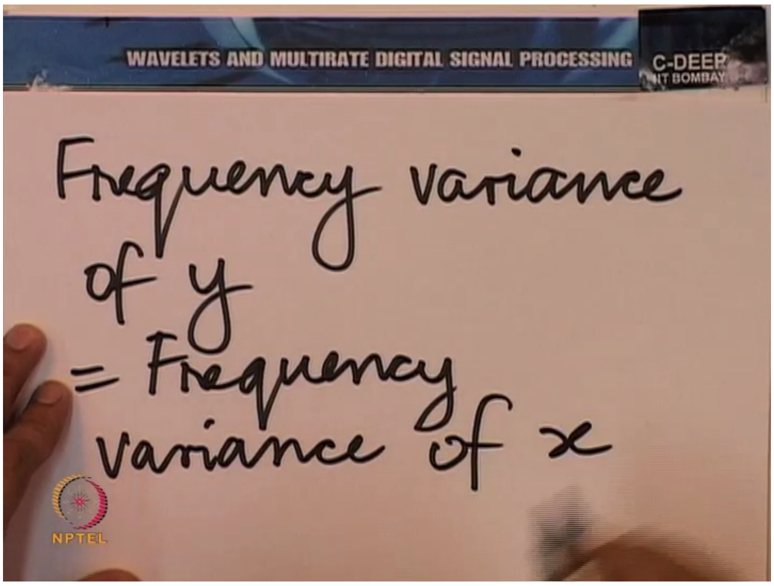
When we shift a signal on the frequency axis we are modulating in time. Now of course here we are talking about modulation by a complex exponential. If you modulate by a sine wave, we are essentially modulating by 2 complex exponentials and adding up these 2 modulating signals, just to put the idea of modulation in perspective. Anyway, coming back to this, so when you modulate with E raised to the power J Omega 1 t, we essentially shift on the omega access as we have done here.

(Refer Slide Time: 13:43)









So y cap omega is x cap omega - omega 1 and of course it is again easy to work out the following exercise. The time centre of y is equal to the time centre of x. In fact that is very easy to see mod yt squared is equal to mod xt squared and so as far as the time the name is concerned, as far as the variable t is concerned, there is no change in the mass that is placed on t. So it is not surprising that the time centre is unchanged as also the time variance which we will now show.

The time variance of y is equal to the time variance of x. What about the frequency variance and the frequency centre? That goes exactly dual to what happened when we shifted in time. So frequency centre, the frequency centre of y is now going to be the frequency centre of x + omega 1, I will put the bracket there. And what happens to the frequency variance? Well, shifting on the frequency axis has no effect on the variance.

Again I leave it to you to show this by writing down the integral carefully, it is a simple exercise, I shall put down the final result. So the frequency variance of y is equal to the frequency variance of x.