**Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.**
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**Week-7.**
**Lecture-17.3.**
**Example: Haar Scaling function.**

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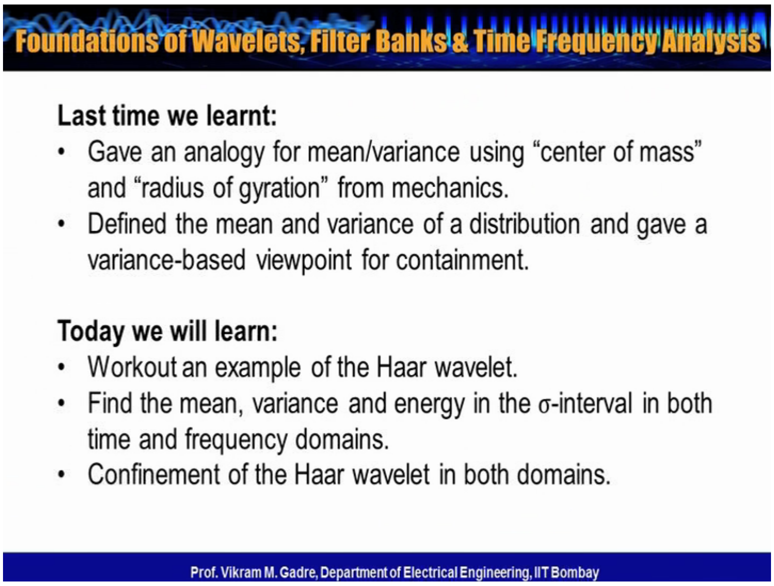


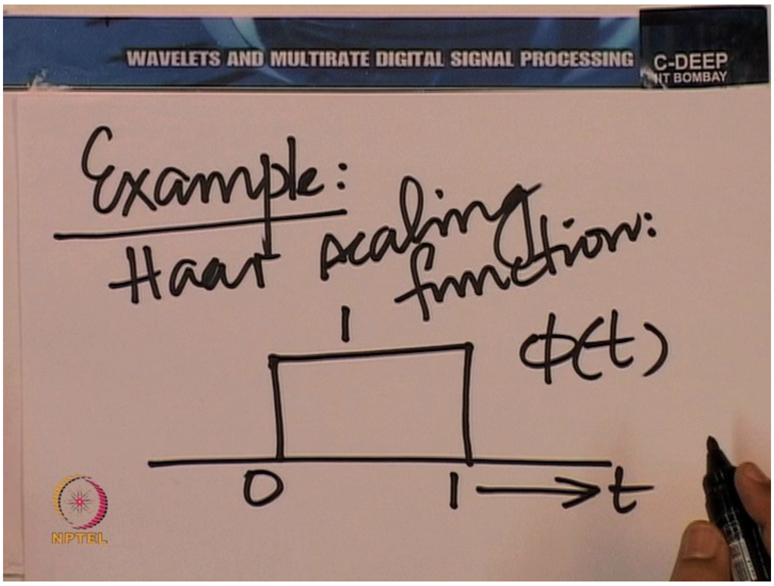

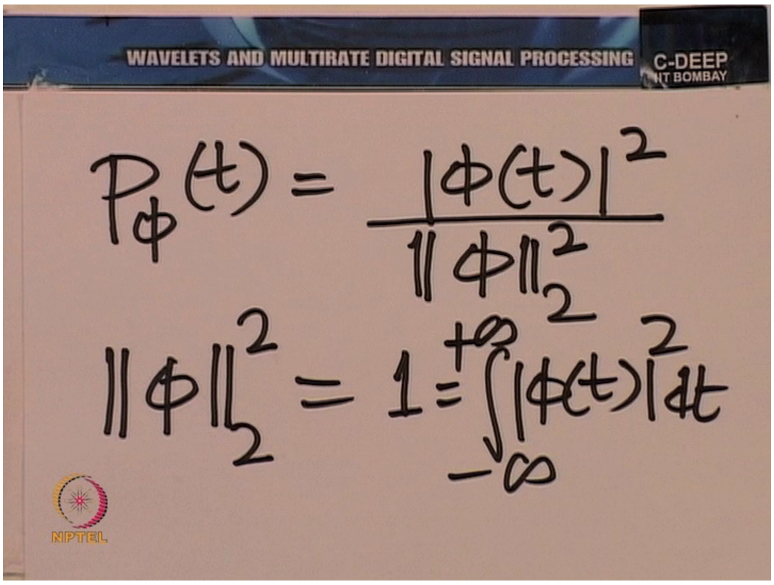


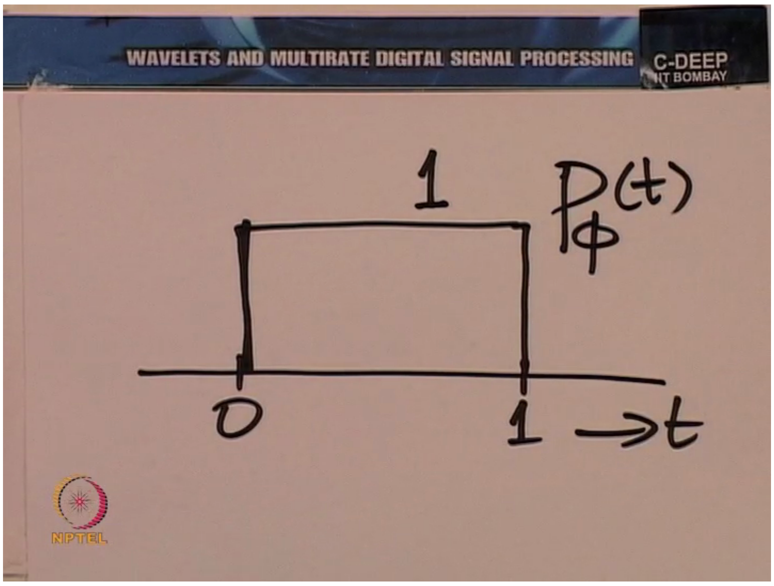


In fact let us take the Haar scaling function as an example, let us calculate the variance. You see, the Haar scaling function Phi of t is 1 between 0 and 1 and 0 else and then of course it is very easy to write down the density here. It is very easy to see that the squared norm in L2R of Phi is one, it is essentially the integral Phi t mod square dt over all t, easily seen to be one. And therefore very likely P phi t looks very much like Phi t. This is how P phi t looks.

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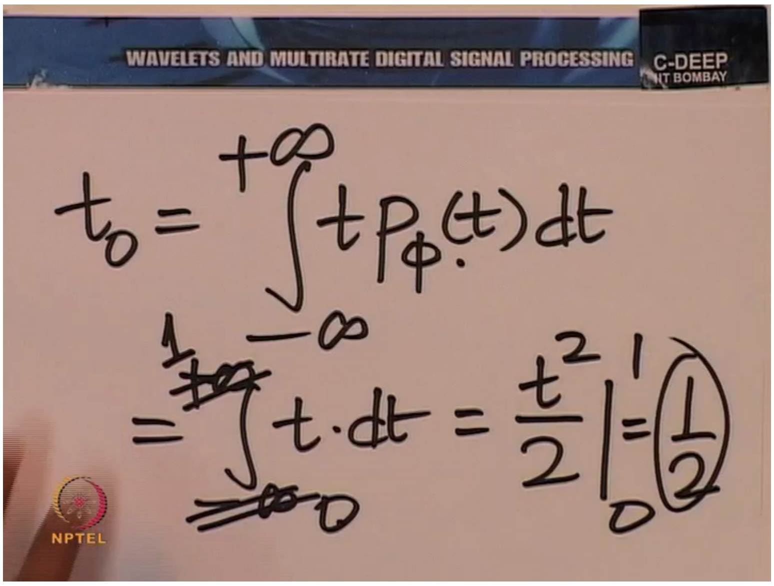


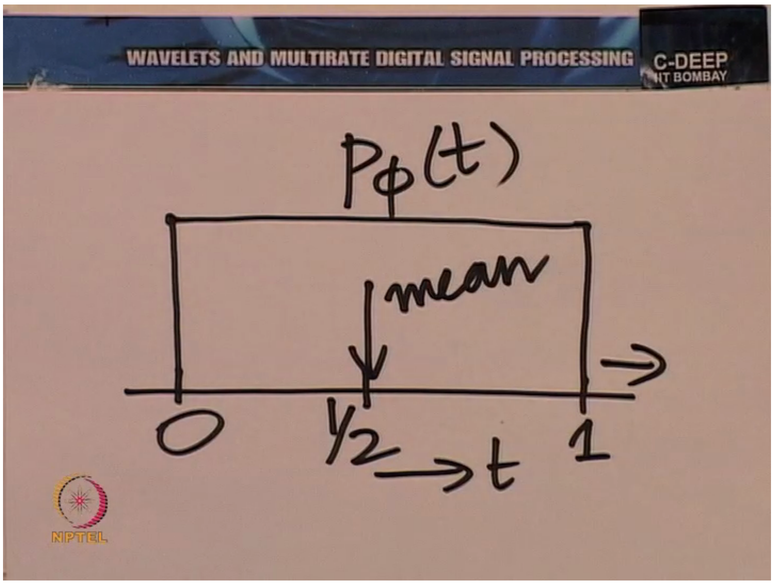


Our job is easy, let us find the mean. In fact even before I formally set out to find the mean I can estimate the mean graphically. The mean is going to be in the centre, at half, that is obvious. But let us do it formally. So t0 would be integral t P Phi t dt over all t in this essentially amounts to integral t dt from 0 to 1, so I have replaced t P Phi t by 1 and I have replace the limits by 0 to 1 and this is obviously t square by 2 evaluated from 0 to 1 which indeed is nothing but half, as we expected.

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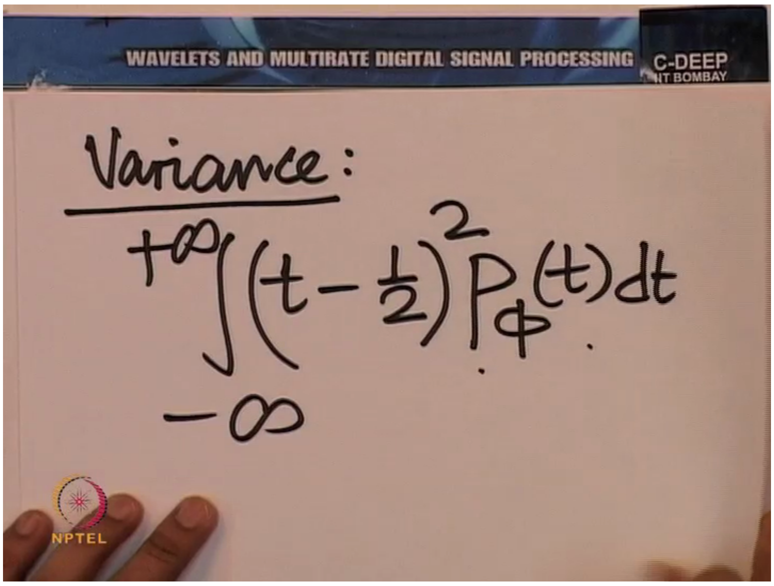


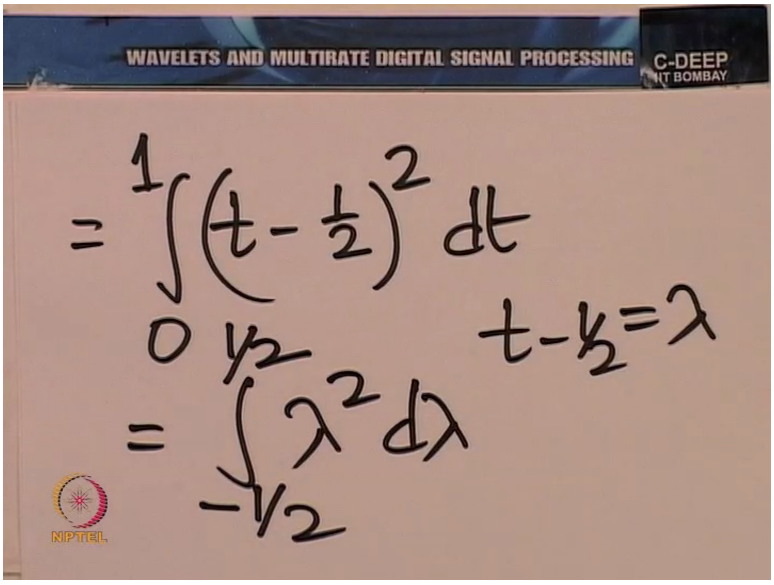


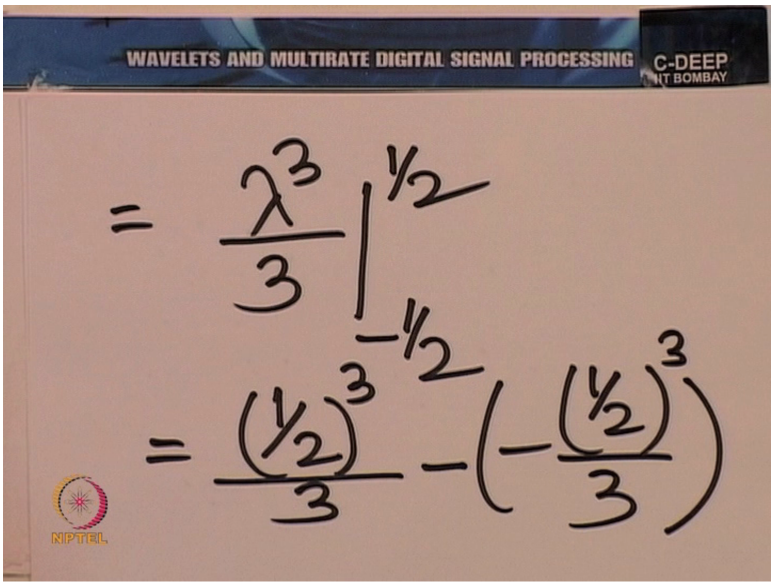

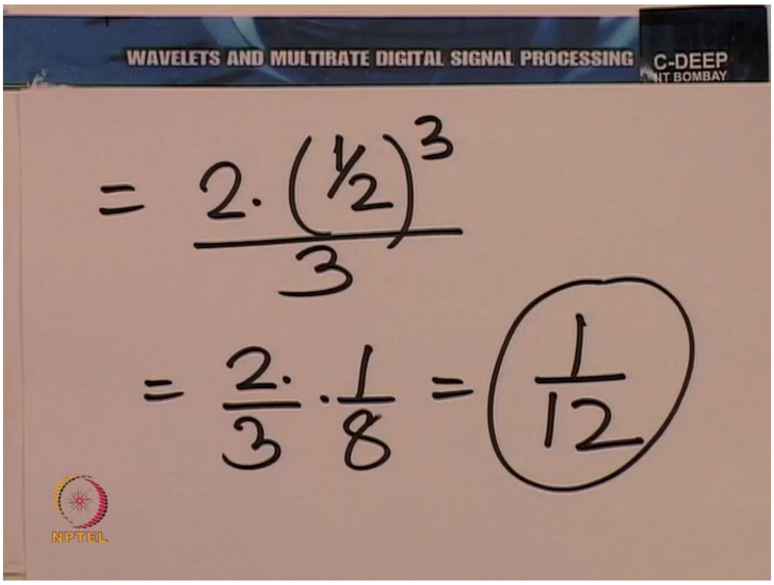


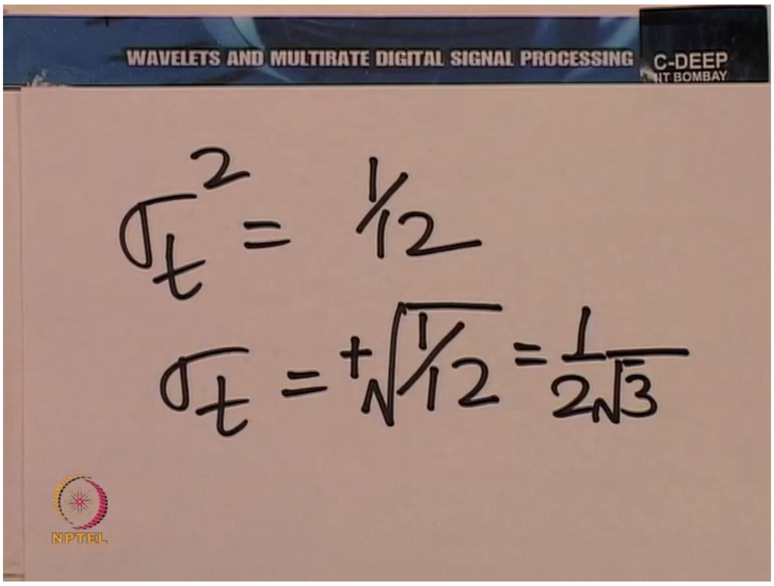


So the mean is indeed half, now we need to calculate the variance. That is a little more work but not too much. Variance, indeed, the variance would be given as t - half the whole square times P Phi t dt over all t. And once again noting that P Phi t is 1 only between 0 and 1 and 0 else, we can rewrite this t - half the whole squared dt. And if I care just to replace t - half by another variable lambda, I would get this to be, well when t is 0, lambda would take the value - half, when t is 1, lambda would take the value half and this would be lambda squared d lambda to be integrated.

And then we have an easy expression. So that is half cube by 3 - of - half cube by 3. And therefore you have 2 times half cube by 3 which is 2 by 3 into 1 by 8 or 1 by 12. So this is the variance. Now of course, Sigma t squared is 1 by 12 and therefore you may take Sigma t to be square root, the positive square root of 1 by 12, which is 1 by2 square root 3. As you can see, Sigma t is less than half.

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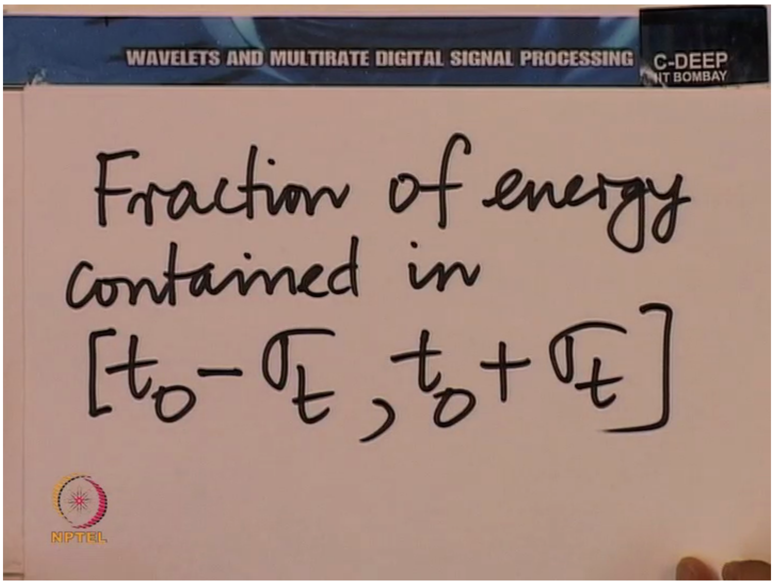


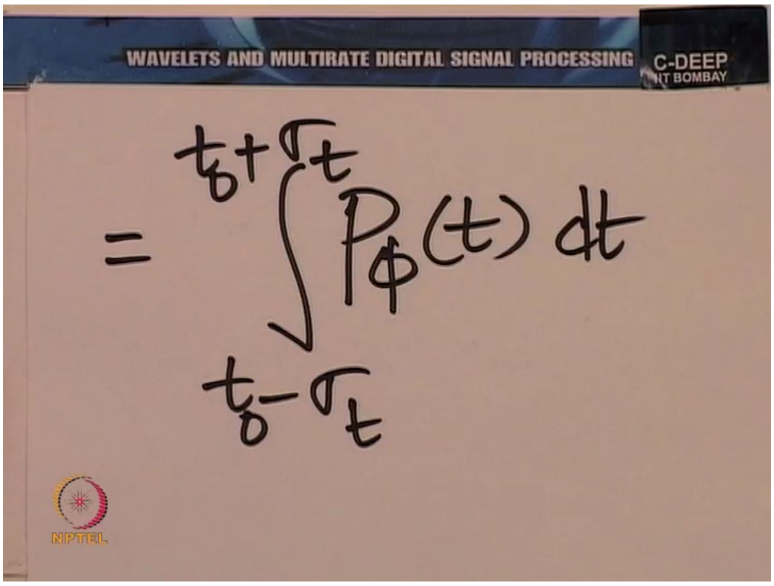


So in a certain sense, we do not really use the number half to denote the spread of Phi t around its mean. The variance does not say it goes all the way to half, it says the spread is a number slightly less than half. Most of the energy is contained in that region around the mean captured by the variance. In fact if you wish to be very specific, the fraction of the energy contained here would be, this actually would be the integral of the density between t0 - Sigma t to t0 + Sigma t.

So that would essentially be… Now I do not really intend to calculate this quantity for this case, it is a very simple calculation, of course integrated with respect to t. But what I am trying to emphasise is that we are not asking for hundred percent, we are not saying tell me the region over which hundred percent of the energy lies, that region could very well be the whole real axis. We are saying, well, a sinificant part of it.

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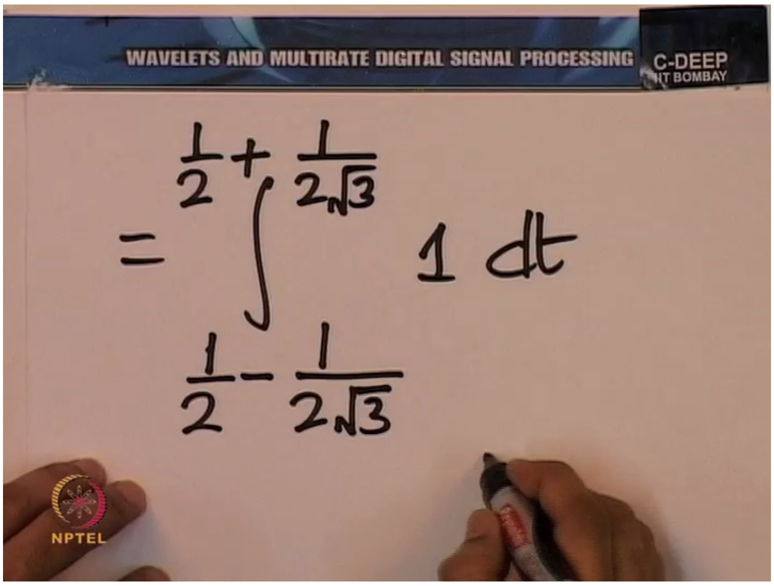


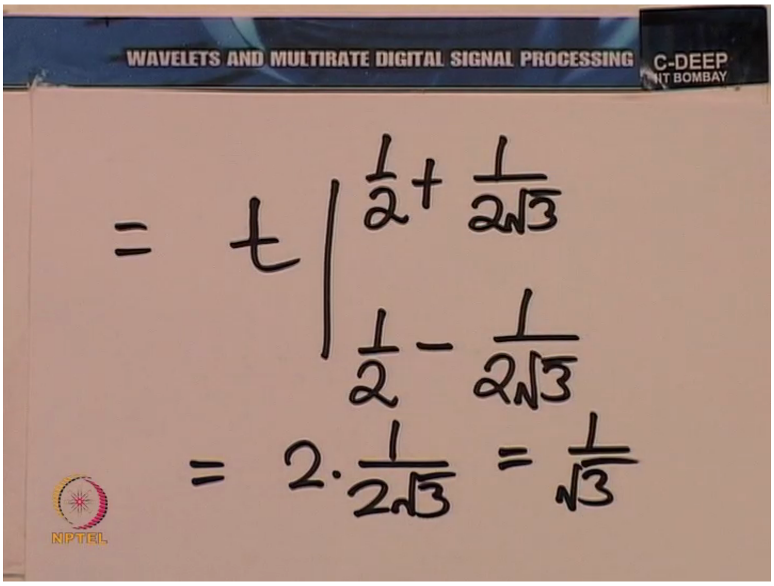


Now, in this particular case, maybe it is a good idea to actually calculate it how much is this really. So it is actually and this is essentially, so it is t evaluated from half -, so this is easily seen to be 2 times 1 by 2 under root 3 which is 1 by square root of 3. Now, certainly not a very large fraction like about 90 percent but it is about 1 by 1.7, more than 50 percent anyway. Incidentally this fraction is not going to be the same for different functions, it depends on the density.

But what we are trying to say that the variance is one accepted measure of spread and very often the variance actually tells us where most of the function is concentrated. Even in the case of this function, if you look at it carefully, what we are seeing is quite a bit of that function is contained between half -1 by2 square root 3 and half +1 by2 square root 3. So it is not an unreasonable range that we choose. Now we ask about the variance in frequency of the same function.

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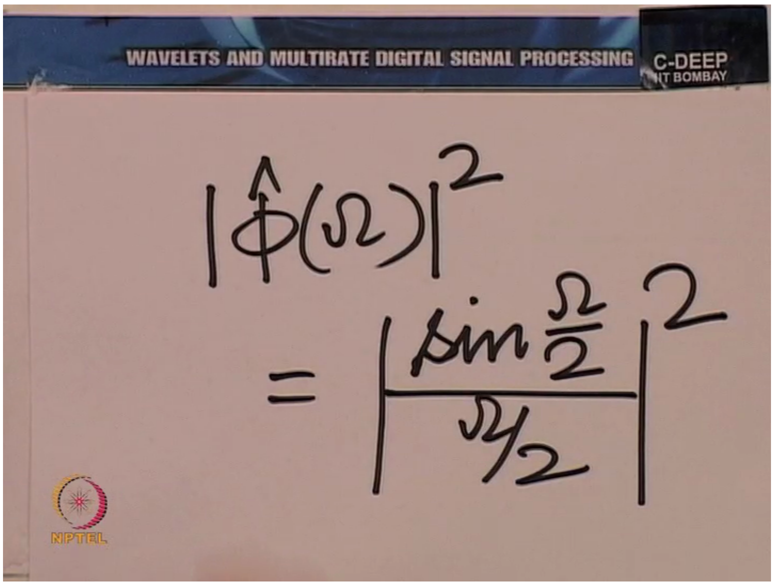


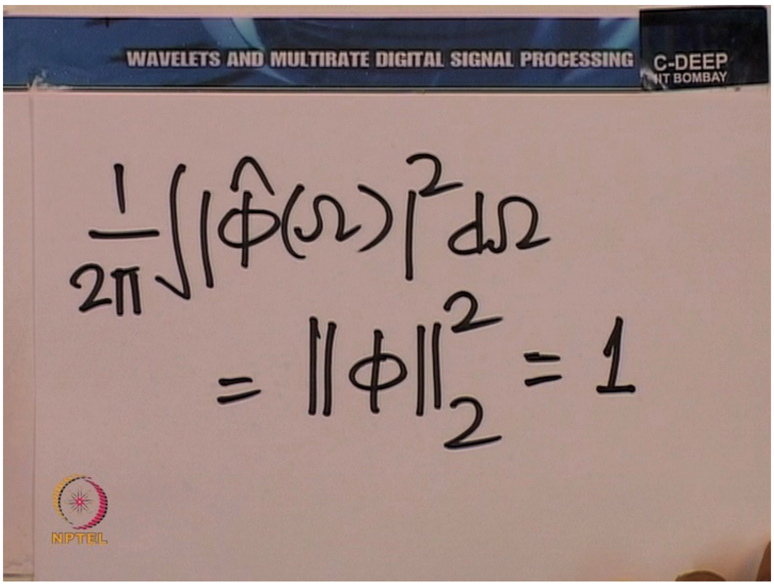

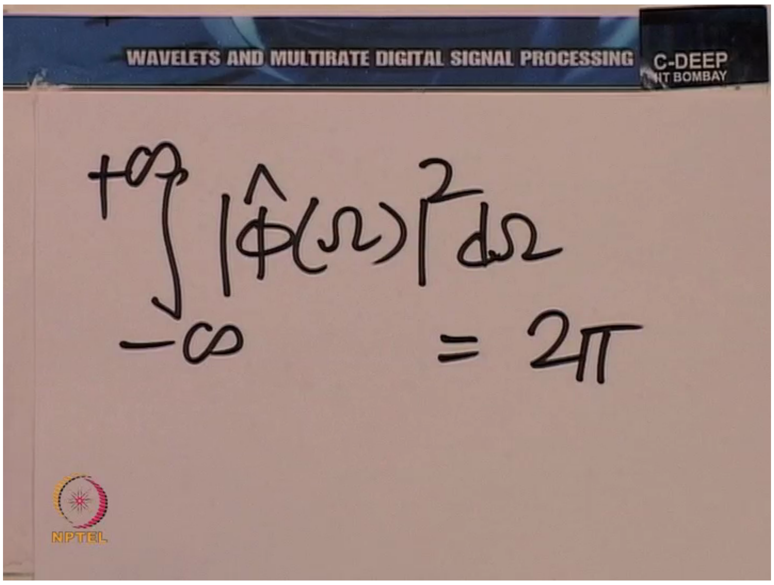


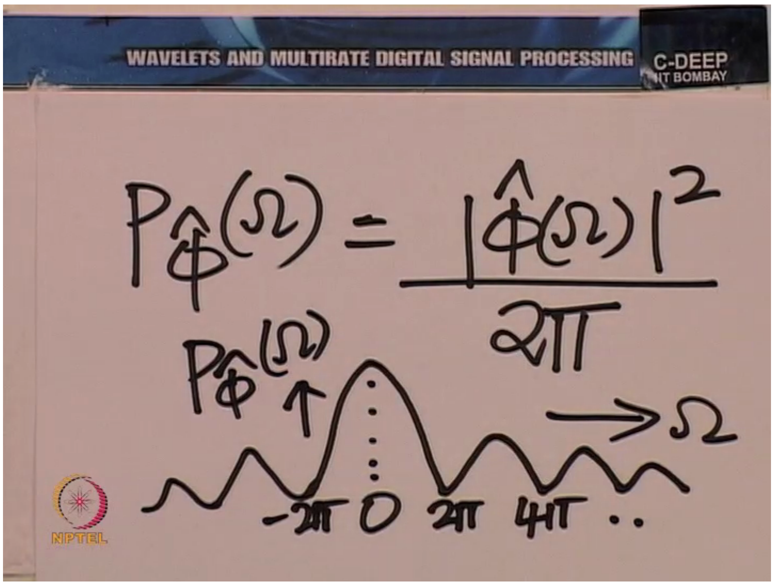


And there we are going to have a very pleasant or unpleasant surprise. So let us look at phi cap omega, in fact we are not so interested in phi cap omega, we are interested in phi cap omega mod squared. And that has the form, essentially sin omega by2 by omega by2 the whole squared mod and of course you could integrate this, you know. Indeed as you know the integral of phi cap omega mod squared d omega divided by 2 pie would essentially be the norm of Phi, L2R norm of Phi the whole squared, which is easily seen to be 1.

And therefore integral phi cap omega the whole squared d omega over all omega of course is essentially 2 pie. Therefore, we essentially look at the quantity mod phi cap, P phi cap rather, which is essentially of the form phi cap omega the whole squared divided by 2 pie. And let me sketch this, in fact we are familiar with it. It would have an appearance like this, this is 0, this is 2 pie, 4 pie and so on and so forth. We know this, we have been doing this more than once.

Now it is very easy to see what the mean of this function is. The function is symmetric around omega equal to 0 and therefore the mean is 0. By the way this is not a surprise. For many real functions we would find the mean, in fact for all real functions the mean of the density on the frequency axis is going to be 0. The Fourier transform for a real function is magnitude symmetric and therefore it is not surprising that for a real function, the mean as understood in this sense is always going to be 0 on the frequency axis.

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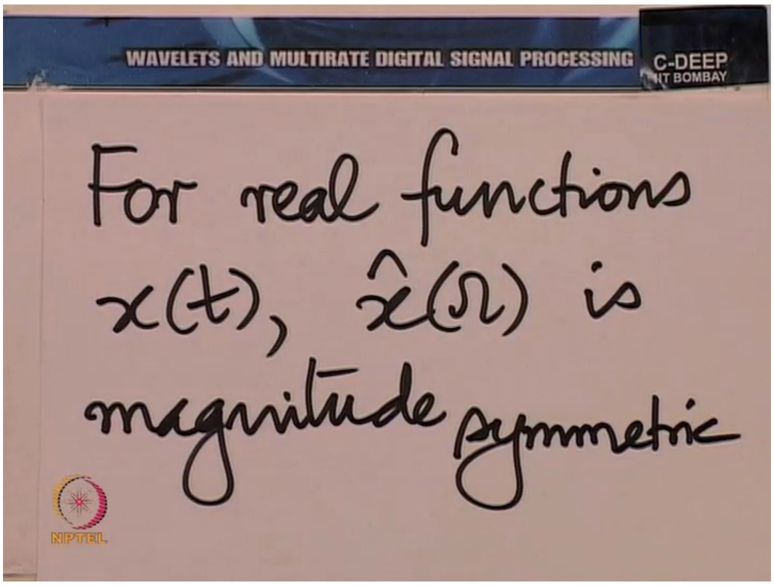


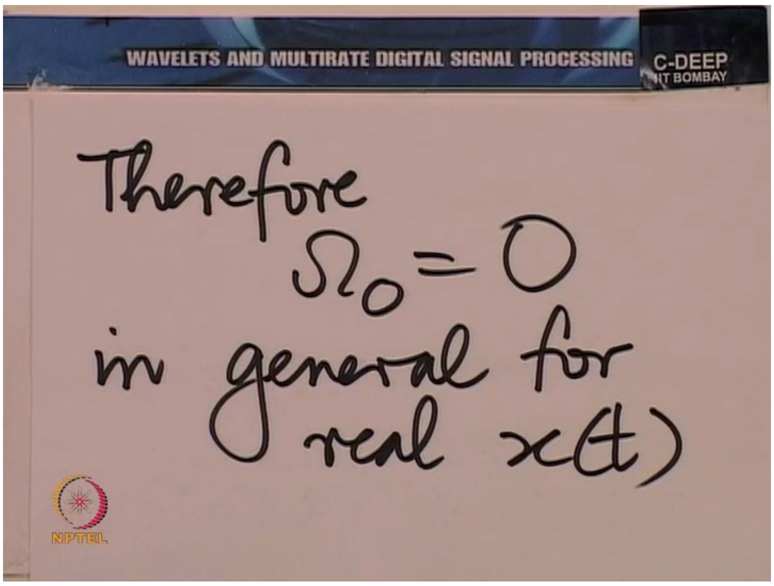

Variance of $\hat{\phi}$:

$$\int_{-\infty}^{+\infty} (\Omega - \Omega_0)^2 P_{\hat{\phi}}(\Omega)\, d\Omega$$



$$= \int_{-\infty}^{+\infty} \frac{\Omega^2 \cdot}{2\pi} \left( \frac{\sin \Omega/2}{\Omega/2} \right)^2 d\Omega$$



$$= \int_{-\infty}^{+\infty} \frac{4}{2\pi} \cdot \sin^2 \frac{\Omega}{2}\, d\Omega$$

not important

trouble!

Let us make a note of that, it is a very important conclusion. For real functions, x cap omega is magnitude symmetric, therefore the mean is 0. Now comes the variance and here we have a very unpleasant surprise waiting for us. I say unpleasant because maybe we should have had something better. So the variance of phi cap could be calculated as follows. Integral over all omega, omega - omega 0 the whole squared P phi cap omega d omega.

And if we make the required substitutions, we have, this is essentially omega square times sine omega by 2 divided by omega by 2 the whole squared divided by 2 pie here d omega. And here we are in serious trouble, this is integral - to + infinity 1 by 2 pie sine squared omega by2 times 4, so 4 goes up there the omega. And this is not important, just a constant but this is trouble, we are in serious trouble here.

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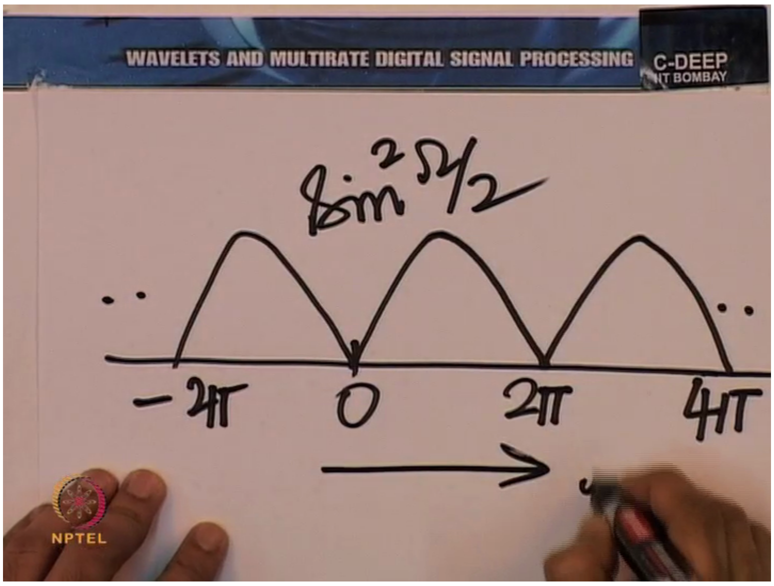


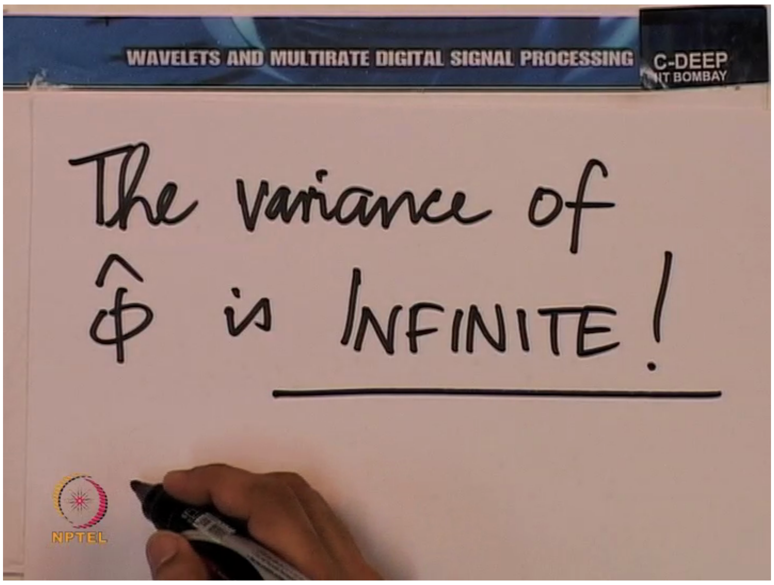

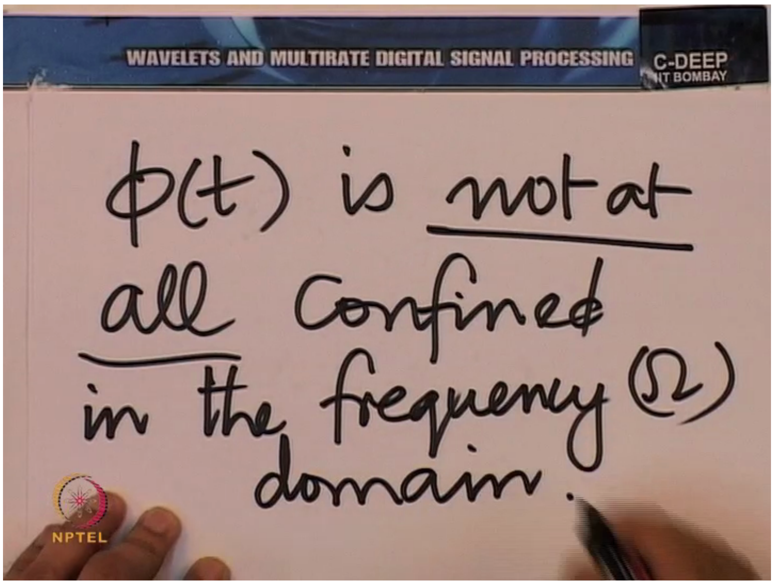


In fact let me sketch what we are trying to integrate here. We are trying to integrate this function, a periodic function with a period of 2 pie sine squared omega by 2, serious trouble as I said. We are trying to integrate a periodic function from - to + infinity and obviously that integral is going to diverge. So the fear that we have had when we started on our discussion with variances comes out to be true right in the very simplest case of a scaling function that we know. The variance of phi cap is infinite, in other words phi t is not at all confined in the frequency domain, at least in this sense.