Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-7.
Lecture-17.1.
Defining Probability function.

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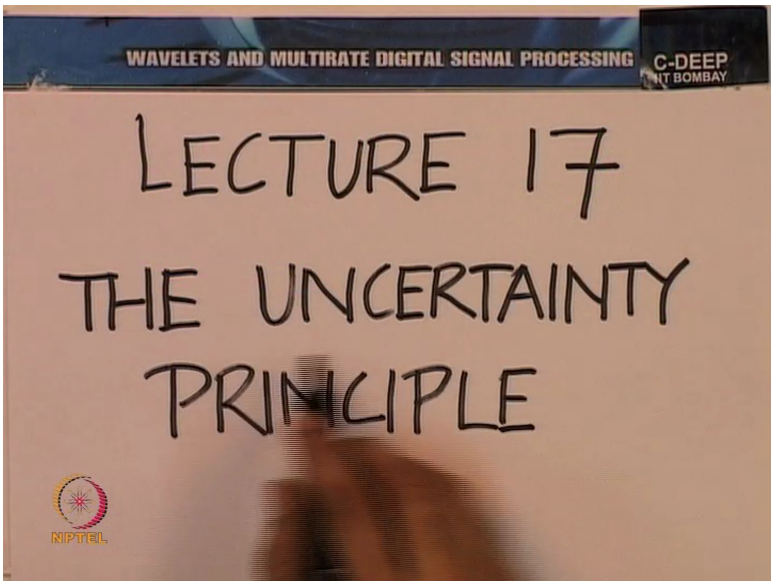

A warm welcome to this lecture on the subject of wavelets and multirate digital signal processing. We build in this lecture a very important principle, in fact in some senses the principle that lies at the heart of the subject of wavelets and time frequency methods, namely the uncertainty principle. Therefore as you note today we shall devote the whole lecture to a discussion of the uncertainty principle, laying the foundation of what uncertainty means 1st and then proceeding to obtain certain numerical bounds on confinement in 2 domains simultaneously.

Let me 1st give an informal or a diffused, non-formal introduction to the idea of containment. Well, we did a little bit of that yesterday in the previous lecture. But what I intend to do now is to say a little more in terms of formality and then proceed to write down the mathematical relationships or definitions. Recall that we said that there is of course a very tight or a very strong kind of notion of confinement. that would ask that you have compact support in both domains, time and frequency.

the function must be nonzero strictly do over a finite interval of the real axis and must be nonzero strictly over a finite interval of the real axis in the frequency domain as well. So both in time and in frequency you demand that the function be nonzero only over a finite part of the independent variable or the real axis. this is a very strong demand and of course yesterday we mentioned that it cannot be met ever.

And in fact I also hinted at the idea behind the proof. It related to the fact that if you noted that the function was finitely supported, compactly supported on the real axis, there were certain properties of that function, specifically the existence of an infinite number of derivatives which made it impossible for that function to be compactly supported or nonzero only on a finite interval of the independent variable in the natural domain.

Natural domain can be in time, can be in space, whatever. Anyway, this was what we called the strong version of containment. And we said that this of course was not possible but we had asked whether a weaker notion of containment could be admitted. Namely, we do not insist that the function be strictly nonzero over a finite interval but that most of its energy, most of its content so to speak in some sense, be on a finite interval of the independent variable. Which index is it?

And simultaneously in the transform domain, in the frequency domain, we insist that most of the content be in a finite interval of the frequency axis. this seems like a more reasonable requirement and to a certain extent this requirement can be met. And as I said, to give a diffused or non-formal presentation of how it can be met, I shall begin this whole discussion by saying that we are finally going to come out with certain bounds on how much you can contain in the 2 domains simultaneously.

So there are several steps to reach this destination. the 1st step is to put down in a non-diffused come in a formal way what you mean by containment, what do you mean by most of the container being in a certain finite range? And we had also hinted at the approach that we would take to do this, briefly, in the previous lecture. We have said that there are 2 ways of looking at it. You could think of the magnitude squared of the function and the magnitude squared of the Fourier transform as a one-dimensional object.

And then you could talk about the centre of that object, centre of mass if you like. You could talk about the spread of the object around the Centre of mass by using the notion of radius of gyration. Or if you prefer to speak in a language of probability densities, then you could employ the idea of a density built from the squared magnitude of the function and another density built from the squared magnitude of the Fourier transform. You could then look at the mean of these densities and the variance of these densities.

And the variances are indicative of the spread. So this was a non-formal introduction, now we need to formalise it. And that is what we shall do precisely to begin with, put down a formal definition, a formal explanation of the idea of spread. Now of course we have to define the domain in which we are going to work. We are going to work in L2R, we have agreed to that, it is always going to be the space of square integrable functions.

In fact I must mention that sometimes we are actually going to work in the intersection of the space of square integrable functions and absolutely integrable functions. to be on the safe side, let us put down that requirement right now and let us put down the tightest of the requirements, namely that the function belong to the intersection of these 2.

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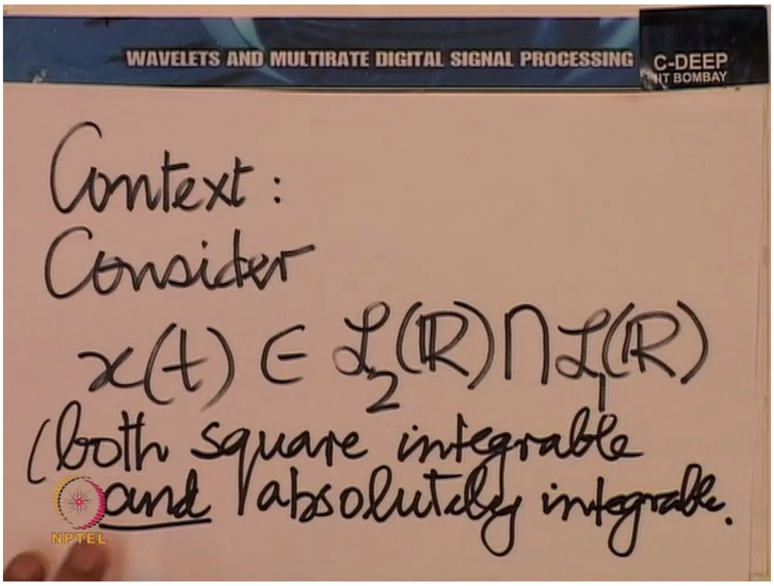

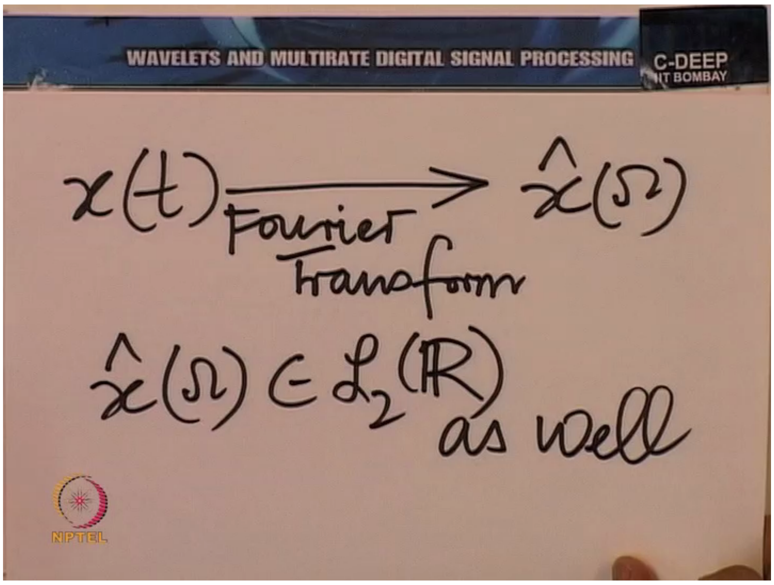

So the context, consider a function, let us say x of t which belongs to the intersection of L2R and L1 R, which means it is both square integrable and absolutely integrable, I think we should note that. All right. Now because the function belongs to L2R, we assure that its Fourier transform also belongs to L2R. So let us xt have the Fourier transform x cap capital Omega. And we know that capital x, I mean x cap of capital Omega belongs to L2R as well.

So we 1st define a density or a one-dimensional mass if you like to call it. We know that both xt and x cap omega are square integrable and therefore if we take the magnitude squared of xt and the magnitude squared of x cap omega, they would enclose a finite area under them. In fact the 2 areas would be essentially the same but for a factor of 2 pie. Again if we chose to do away with angular frequency and used hertz frequency, that 2 pie factor would also go away.

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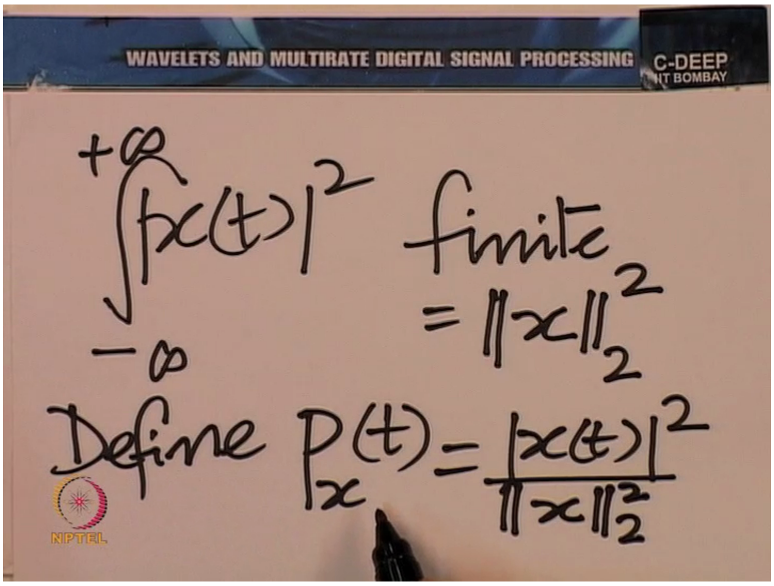

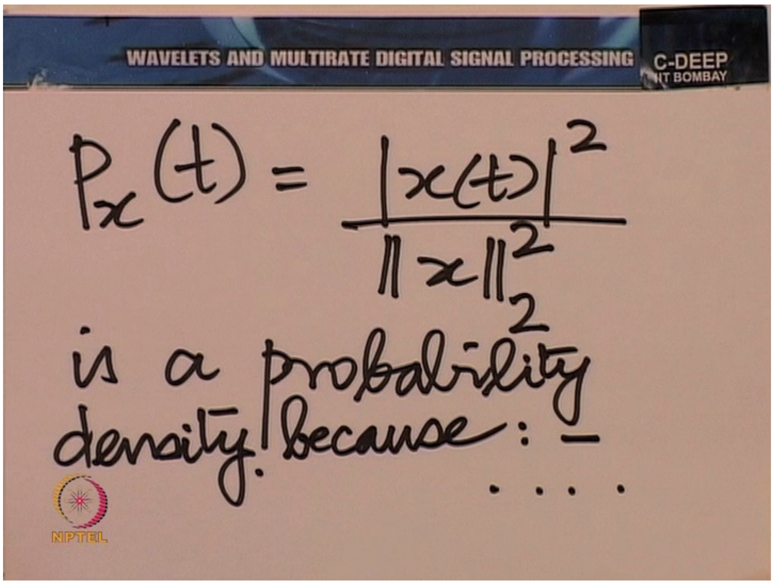

Anyway, what we are saying is mod xt squared integrated from - to + infinity is finite. Let us is fact use the standard notation for this, the norm of x in L2R the whole squared and therefore define a density Px given by mod xt squared divided by the norm again squared. Now a few remarks, and in fact we should write them down one by one. Pxt as we have defined it, namely mod xt squared by the norm in L2R of x the whole squared is a probability density.

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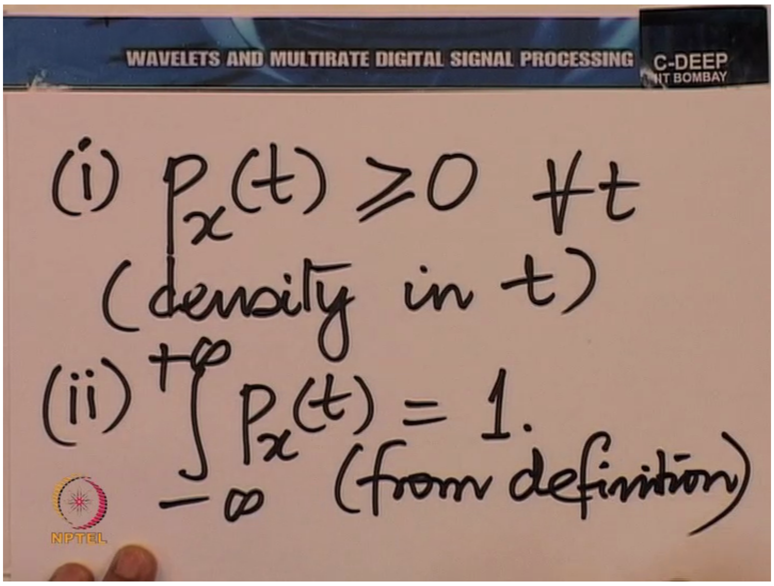

Why do we say this, well because of the following reasons, let us list them one by one. Number 1, P xt is greater than equal to 0 for all t, it is a density in t of course. So you may think of t as a random variable and this is the density on that. The integral over all t of P xt is easily seen to be 1 from the definition. Essentially the integral of P xt over all t would in the numerator again have the L2 norm of the function x and of course the denominator is indeed the L2 norm of the function x, of course both squared, the numerator and denominator and therefore they would cancel out to give 1.

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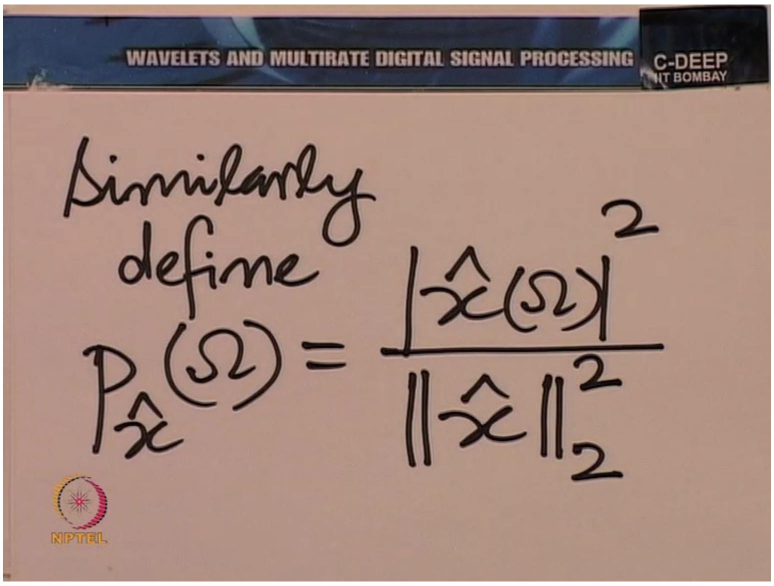

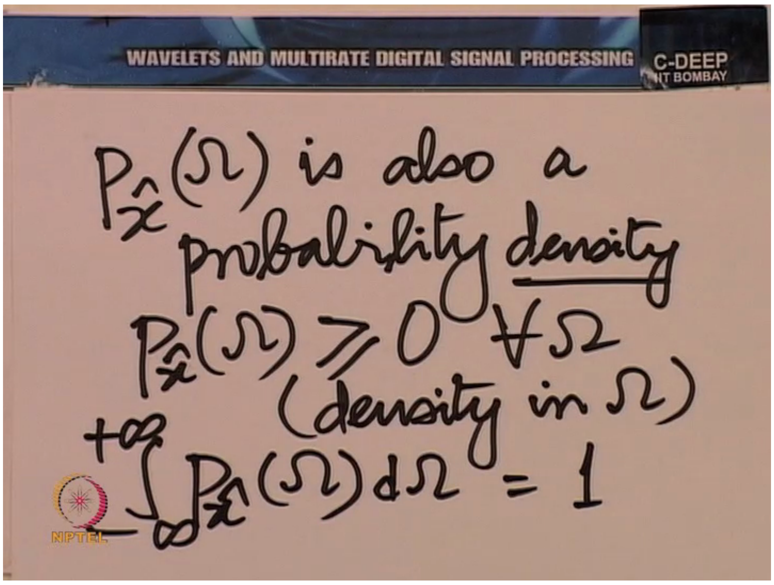

Similarly let us define a density in the Fourier domain, in the angular frequency domain. And then we shall write Px cap as a function of omega to be mod x cap omega squared divided by the norm of x cap square. Here again we are assured of the denominator being finite because of the L2R business. So, again we shall for completeness and formalism note that this is a probability density. Px cap omega is also a probability density. Indeed Px cap omega is greater than equal to 0 for all omega, it is a density in omega.

And the integral over all omega from - to + infinity of Px cap is 1. That is also easy to see by the very definition. Now, we have taken the probability density perspective but we could as well take the so-called one-dimensional mass perspective and let me also note that.

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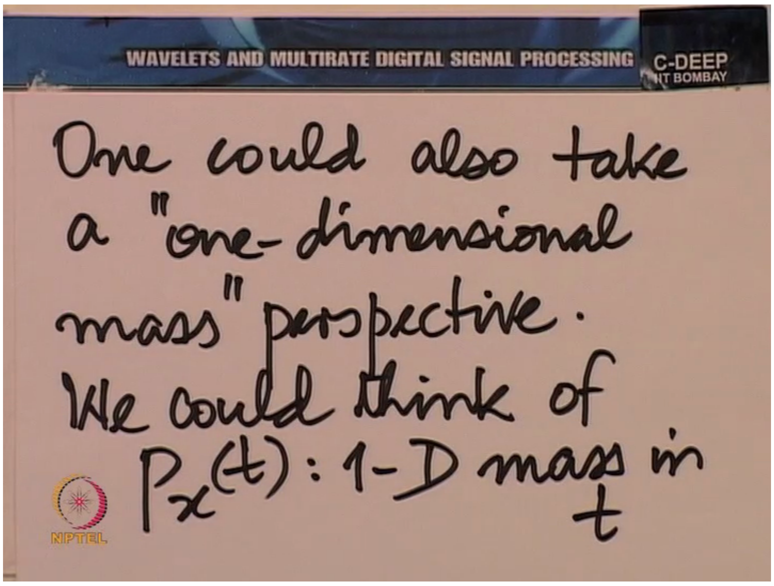

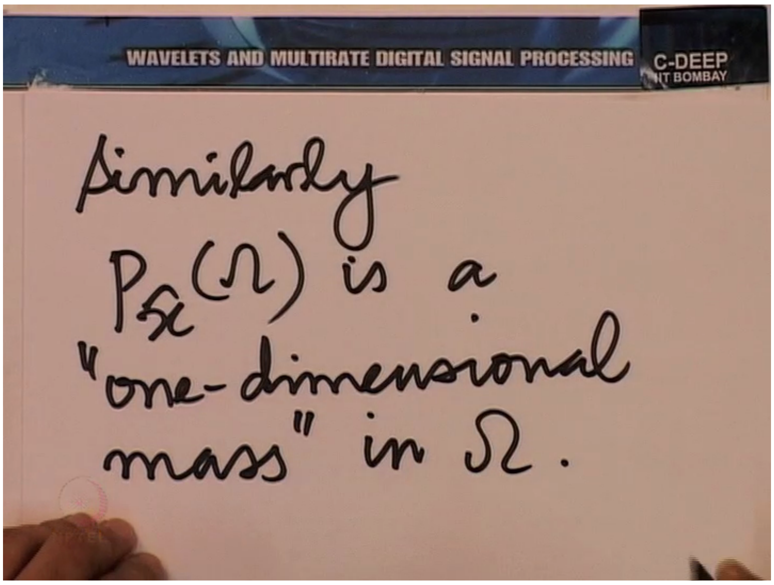

One could also take a one-dimensional mass perspective. That is, we could think of P xt as a 1-D mass in t and similarly you could think of Px cap omega as a one-dimensional mass in omega. So what I am saying is, all of us or all the objects around us are masses in three-dimensional space. So here in you have a simplified situation, you have a mass in one-dimensional space. That one-dimensional space can be the space of t or the space of capital Omega. Similarly as I said Px cap omega is a one-dimensional mass.