Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-6.
Lecture-16.3.
Some thoughts on ideal time-frequency domain behaviour.

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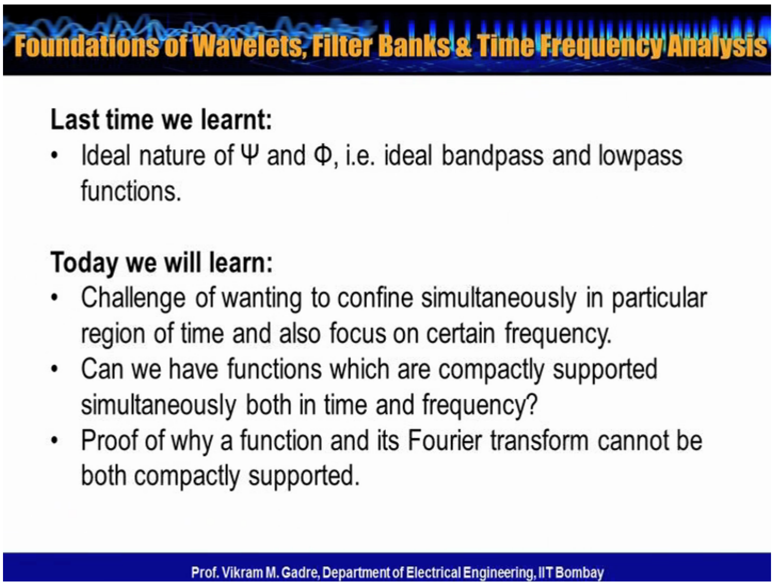

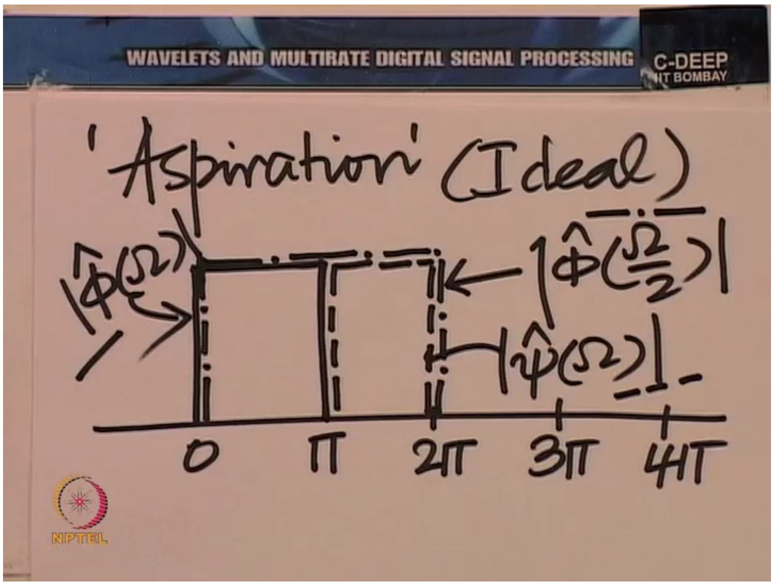

Let us draw the ideal situation. So the aspiration, the ideal is the following. This is the aspiration for phi, at least in terms of magnitude. I will show the aspiration for the corresponding phi of omega by 2 and then I used a kind of dashed line here to show the aspiration for psi. This is the ideal towards which we are going. This one dot dash, this one solid, this one only dashes. Now things have begun to fall into place.

In fact now we can also see what we mean when we forget about phi entirely and use only psi, what are we doing in the frequency domain rather what are we aspiring to do. So you know, you can, what I am saying is instead of thinking of all the shells up to a point in removing one shell, think of the whole onion as only shells, only psis. So what is happening then, the following is happening in the frequency domain.

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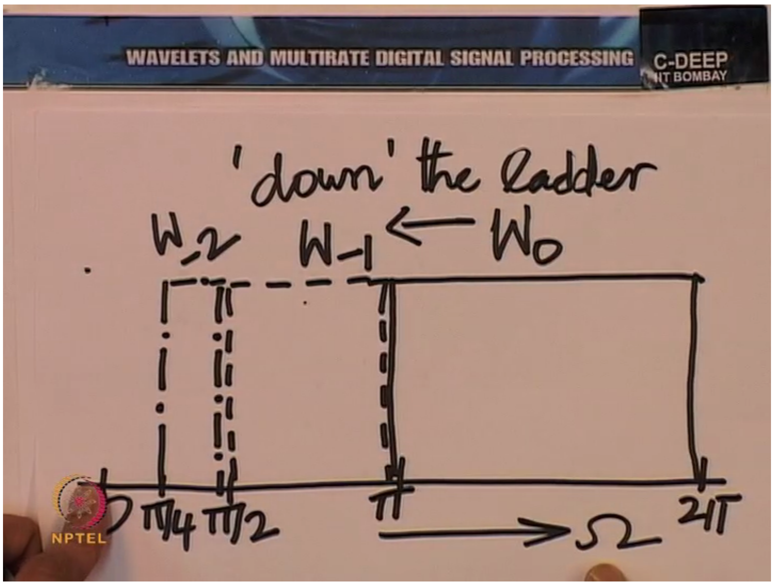

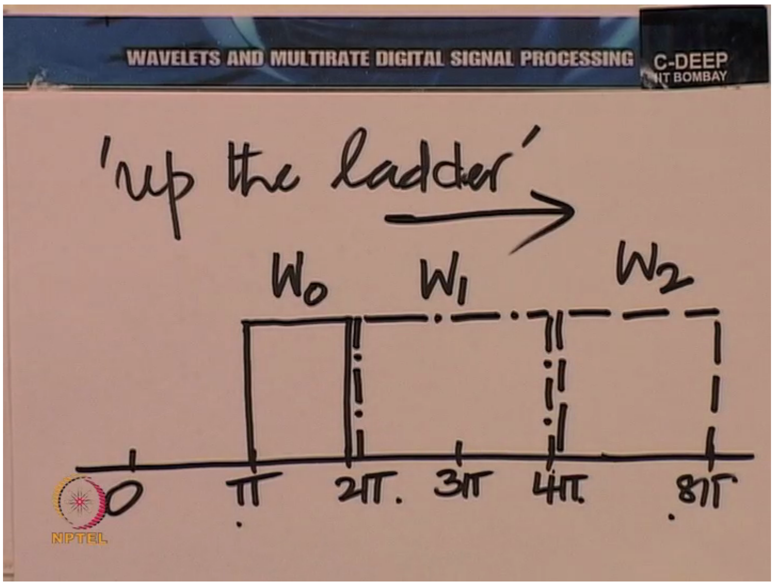

In fact now I need to work carefully around 0. So I will draw a big pie here and a big 2 pie there. So we will start with V1, so this is W0, W -1 will essentially do this ideally, W -2 would be between here, between pie by 4 and pie by 2 and so on. Each time you go towards

0, you are contracting this band by factor of 2 and therefore both the Centre frequency and the bandwidth have been reduced by a factor of half. And of course you can visualize going in this direction too.

So just for completeness I should draw W1 and W2, though not on the same graph, it is difficult to do, so we will draw it separately. So to be specific, we should say down the ladder here, and up the ladder here. So I will show 2 steps, not quite proportional but that is okay. This is W0, W1 will essentially take this, from 2 pie to 4 pie. W2 will cover 4 pie to 8 pie here, this is 8 pie, please note. Again as I said, forgive my drawing, it is not quite proportional but it is indicative.

Pie here, 2 pie here, 4 pie there, 8 pie there. Now we know what we are doing. As we go up the ladder, we are going to double the Centre frequency each time and double the bandwidth ideally. And once again let us show the behaviour as far as the spaces V go. So here we showed what happens with W, let us show what happens with V.

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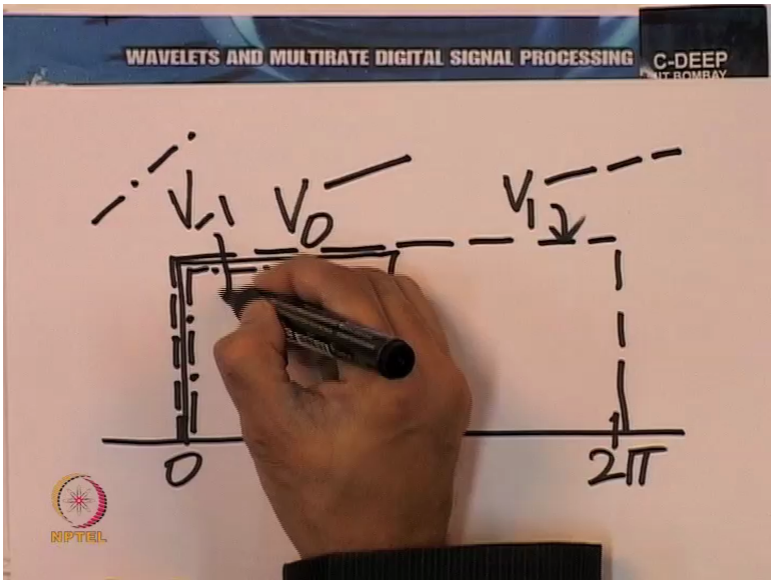

So I could show it just on one. So I will just show for completeness 3 of them. This is what V0 does, this is what V1 does, so V 0 is a solid line, V1 is just a dashed line and V -1 is the dot dash line I am drawing now. So these are what are called the complete subspaces, these are, well I should not use the word complete in the rigorous sense but I mean these are the entire set of shells up to that shell and the others which we drew a minute ago, the W's were just one peel or one shell at a time.

Now we understand perfectly what we are doing in frequency, we are trying to do. And now we also understand perfectly where the challenge lies. We are aspiring to do this and we also want to do something similar in time, we want to confine ourselves to a certain region of time and we also want to focus on a particular region of frequency. Ideally focusing means being only in that region and 0 outside. So the 1st question that we need to answer is it exactly possible?

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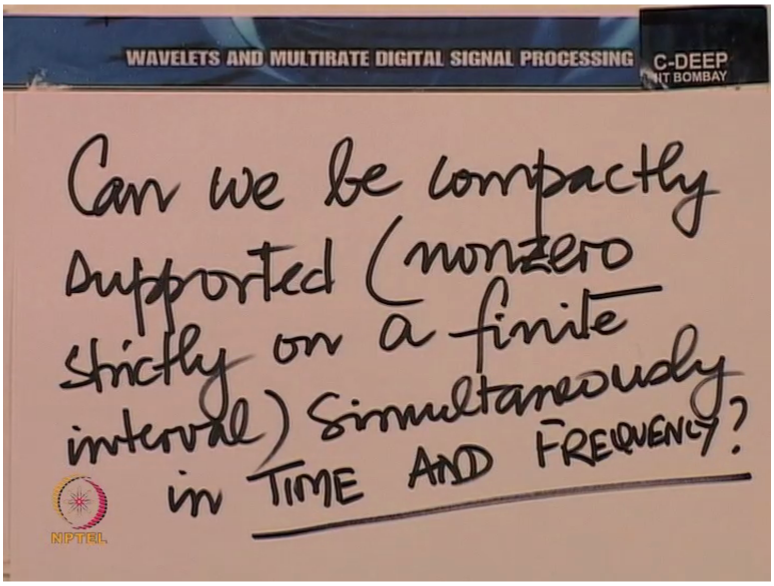

Can we be compactly supported in time and frequency simultaneously? Let us put the question, question is also important here. Can we be compactly supported, this is a technical term, compactly supported, I shall not spend too much of time on explaining the details but nonzero strictly on a finite interval is a simple way of saying it at the moment, simultaneously in time and frequency. And unfortunately or maybe fortunately because it brings up or opens up a whole new subject, the answer is no.

If you talk about the exact behaviour, it is impossible to be compactly supported in both domains. That is not a very deep result in the theorems of Fourier analysis, though it is an important result. It is a relatively weaker, weaker in the sense not of requirement but in terms of the depth of proof or depth of implication. It is more easily proved, easier to indicate or to justify, you cannot be compactly supported in time and frequency simultaneously. Let us make that statement very clear, answer no.

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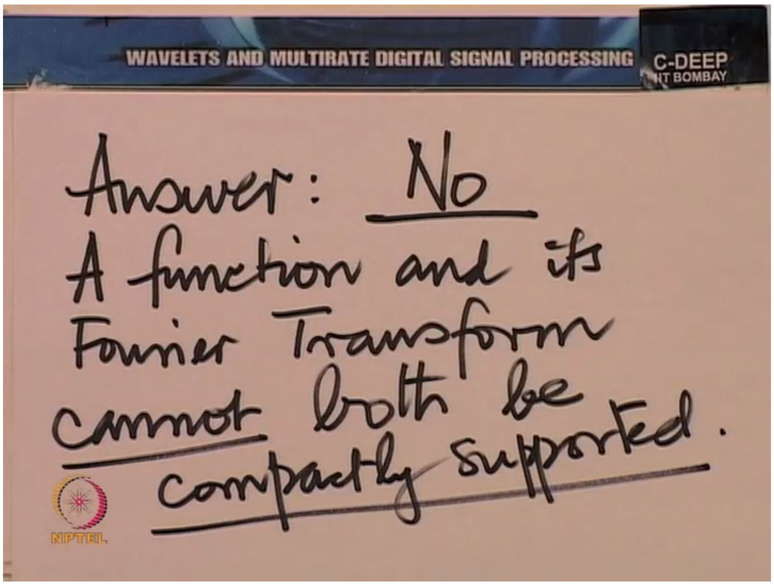

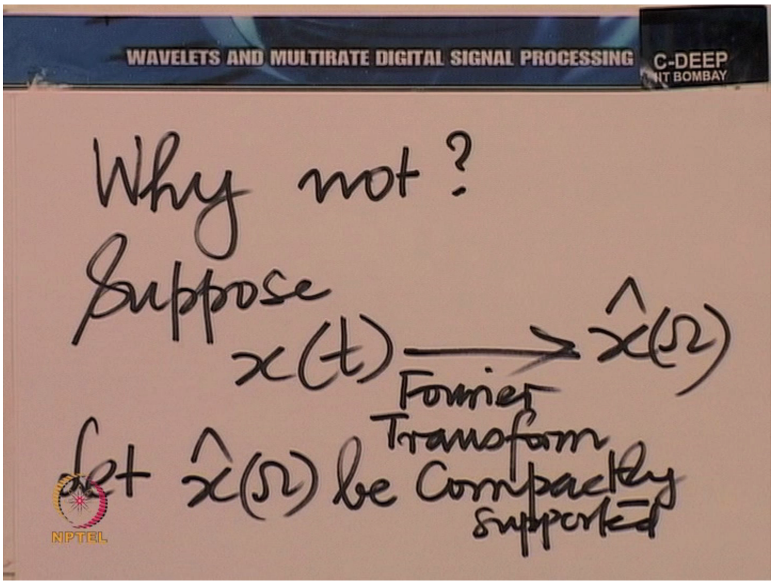

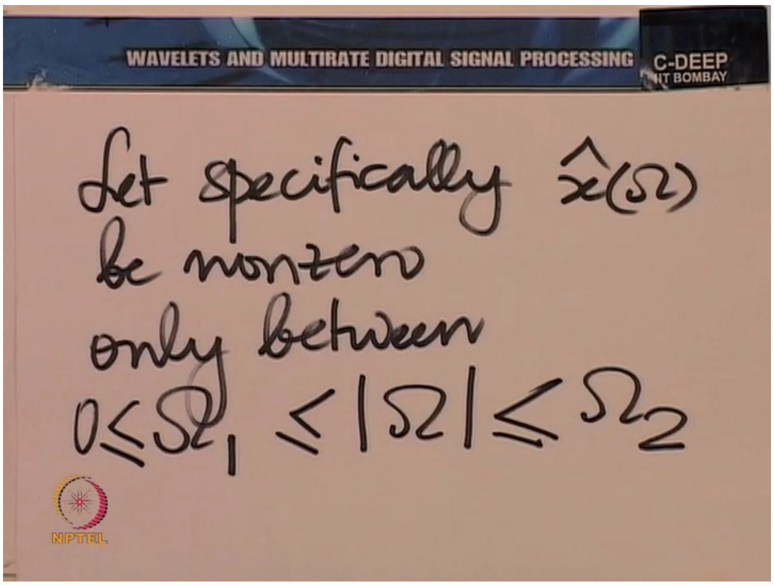

A function and its Fourier transform cannot both be compactly supported. In fact I shall give an indication of the idea behind the proof and I shall leave it to the class, I shall leave it to the students who are listening here to delve deeper. The idea behind the proof, why not? Well, suppose xf, suppose XT has the Fourier transform X cap F or X cap omega, let us take omega if you like. And let X cap omega be compactly supported, in other words, let us specifically X cap omega be nonzero only between omega 1 and omega 2 in magnitude.

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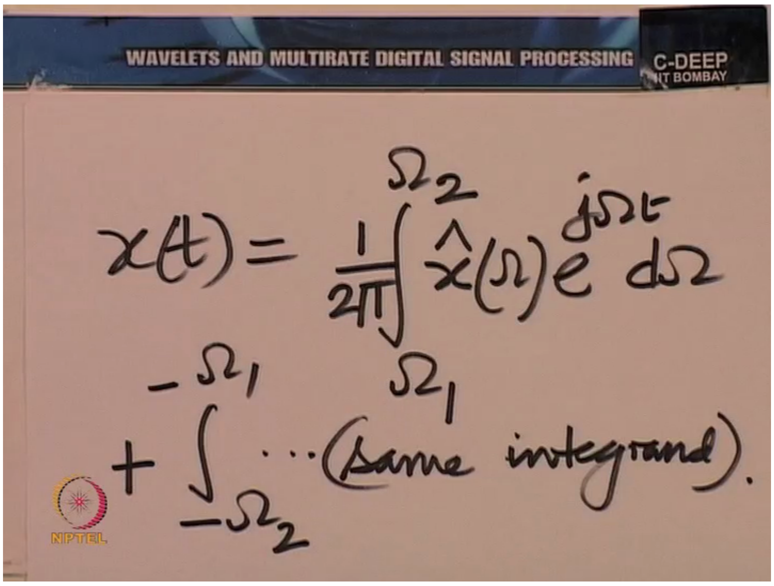

Of course, needless to say omega 1 is greater than or equal to 0 and therefore also omega 2. Then it is very clear that the Fourier , the inverse Fourier transform which gives us back XT is a finite integral. So X of T is then going to have a finite integration involved, + the same thing on the negative and the same integrand. Now the central idea in the proof is the following. I can take derivative on both sides and I remember I had a finite integral on both sides.

When I take a derivative with respect to time of XT, then if I look at the integral here, that derivative essentially acts only on E raised to the power J Omega T and that operation of taking the derivative into the integral is valid because this is a finite integral. The same thing holds good for the 2nd integral here. So in effect you are talking about a function which has an infinite number of derivatives because after all each of the integral is involved would be a finite integral here.

So I am just, I am not really giving you a rigorous proof, I am just indicating the central idea in the proof. It relates to the fact that the function which is compactly supported in the frequency domain must have a certain kind of smoothness as seen in time. No matter how many derivatives you take here, you do have an expression for the derivative there. That means the derivatives exist and in fact can also be shown to be continuous. So there is a, there is the quality of infinite smoothness in that function XT in some sense.

As I said, all this is only indicative of the proof. Now I encourage those of you who are more mathematically minded to take this proved to completion. Show that because of this finite integral here and the fact that the function must be smooth as much as you desire in terms of derivatives, it cannot be compactly supported in time as well. In effect, what you are saying is, you are asking for an analytic function, a function which has an infinite number of smooth derivatives to be compactly supported in time, there is a problem there.

Well, that was indicative of the proof, that was indicative of the central idea as to why you cannot have compactly supported functions, both in time and frequency together. And this is where the whole challenge starts. But now we need to ask is slightly more relaxed question and that will be the issue that we shall discuss in much greater depths due course now. The question is suppose we do not ask for strict compact support.

That means suppose we are not saying that a function must be nonzero outside, sorry, nonzero only inside a certain compact interval, only inside certain finite interval and 0 everywhere else. We do not mind a certain amount of energy of that function or most of the functions in a certain sense being concentrated in a certain region in time and also in frequency. Then can we get a function which is both compactly, or not compactly but in that sense restricted in time and frequency.

And of course as we expect, the answer is yes, if we are willing to give up a little bit, we can get something. If you are willing to give up exact compact support, so if you are willing to allow some leakage outside that reason of time and therefore also outside the certain region of frequency. But be content with the fact that in a certain weaker sense the function is concentrated in a certain region of time and in a certain region of frequency, then can one 1st have this kind of broad concentration in time in frequency together?

Well the answer is yes because it depends on what you mean by that weaker sense of concentration. In fact phi and psi are in that since concentrated, both in time and frequency in a weaker sense. If you focus only on the main lobe, and of course the main lobe has a certain amount of the energy, then yes, indeed, of course, Phi is simultaneously localised in time and frequency. So what is the general sense that we are going to allow?

Well, that sense will come from essentially either what we might call the statistical property of variance or if we want to use a mechanical analogy, the idea of Centre of mass and radius of gyration or the volumetric occupancy of a body. So we will think both of the function and its Fourier transforms as one-dimensional bodies. And we can think of their centre of masses and then we could think of how much the body spreads around the centre of mass by using what is called the idea of radius of gyration.

Another perspective is, if you think of probability density functions based on the function and its Fourier transform, you could ask what is the mean of that density either in the time domain or the frequency domain. And then you could ask what is the variance of the density, again either in the time domain or in the frequency domain. And now there is a clear way to formulate. Can we have finite variance both in time and frequency? And there, as we expect the answer is going to be yes, that is not a problem.

Now, the more difficult question, how small can the function be simultaneously in time and frequency in this broader sense? So, how small can you make the variance in time and frequency simultaneously, that is the deeper question. And that is the whole idea behind the uncertainty principle. In fact, now we are beginning to understand why we needed to go to better and better multiresolution analysis. Why could we not be happy with the Haar?

The Haar is somewhat concentrated in frequency but well concentrated in time. I at one point asked you to find out the Fourier transform of the Dobash functions as well. So, you know if you look at the Dobash functions as you go from length 4 to length 6 to length 8 and if you look at their Fourier transforms, you would find that they are slightly better approximations to that ideal lowpass filter with pie and ideal bandpass filter with band between pie and 2 pie as we desired.

So, what we are going to do subsequently now is essentially to essentially bring out this concept of uncertainty more deeply and then to investigate whatever we have been doing in the language of uncertainty starting from this point onwards. Thank you.