Foundations of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-1.
Lecture -2.3
L2 Norm of a Function.

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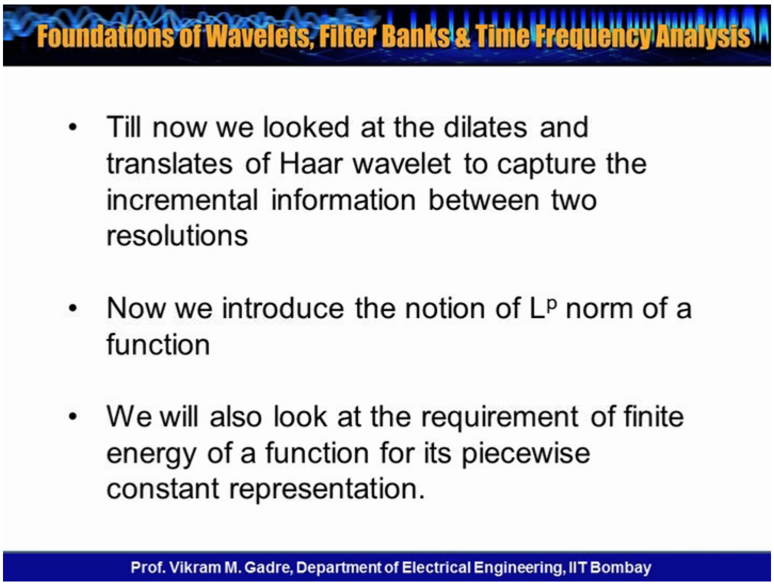

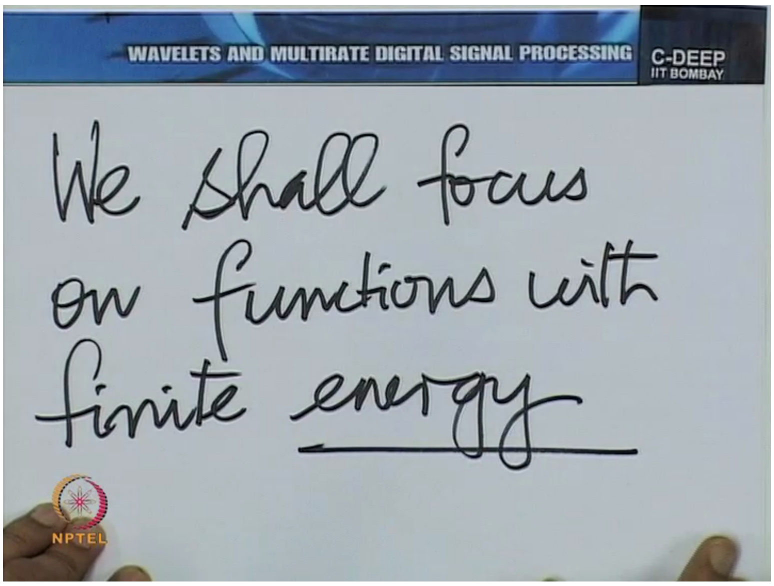

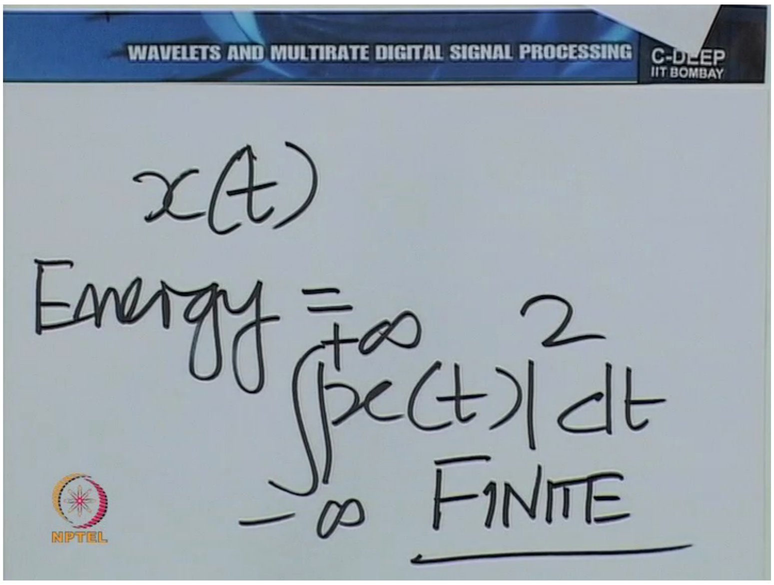

All that we ask for and that is not too unreasonable is that the function has finite energy. So let us at least put that down mathematically. What we are saying is we shall focus on functions with finite energy. And what does energy mean? Energy is essentially the integral of the modulus square. So if I have a function x of t, the energy in xt is the integral mod xt square over all T and this needs to be finite, all we are saying is this. Incidentally this quantity has a name in the mathematical literature, or for that matter even in the literature of wavelets.

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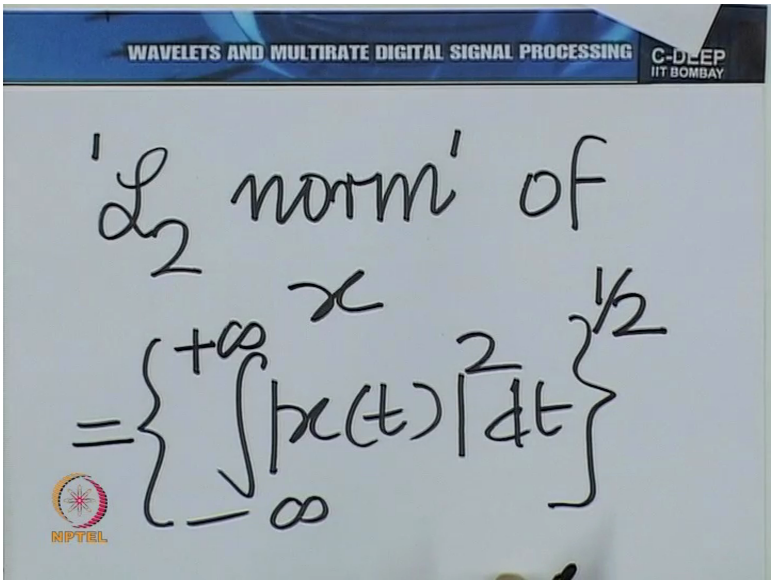
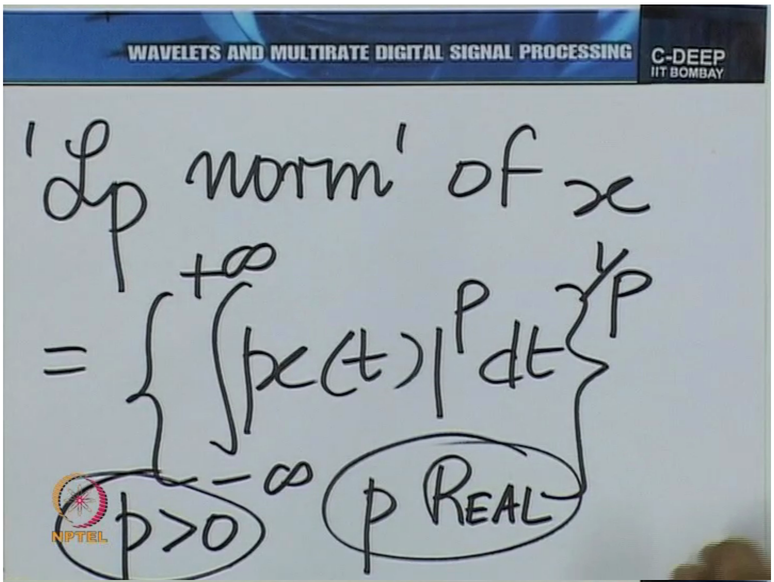
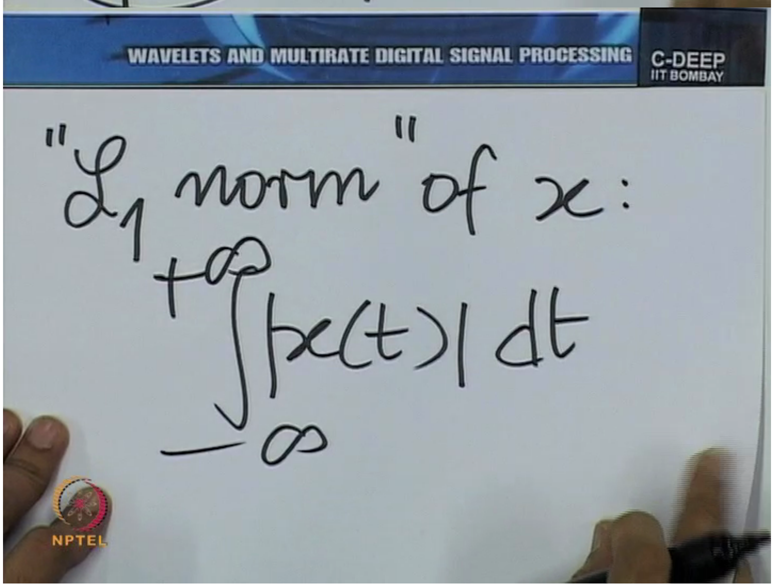



WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING
C-DEEP IIT BOMBAY
'$\mathcal{L}_p$ norm' of x
$= \left\{ \int_{-\infty}^{+\infty} |x(t)|^p dt \right\}^{1/p}$
$p > 0$
p REAL
NPTEL


WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING
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"$\mathcal{L}_1$ norm" of x:
$\int_{-\infty}^{+\infty} |x(t)| dt$
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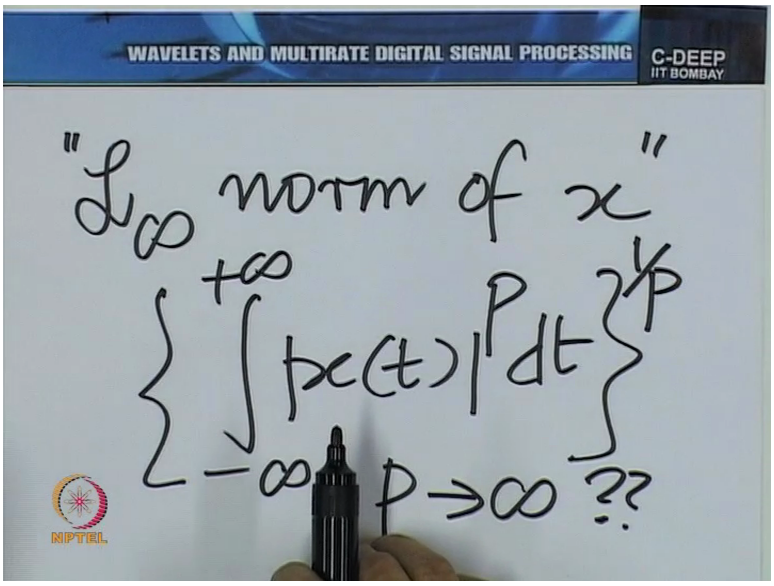
WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING
C-DEEP IIT BOMBAY
"$\mathcal{L}_\infty$ norm of x"
$\left\{ \int_{-\infty}^{+\infty} |x(t)|^p dt \right\}^{1/p}$
$p \to \infty$??
NPTEL

The energy as we call it in signal processing is called the L2 norm by mathematicians. And you know it helps to introduce the terminology little by little from the beginning because if one happens to pick up literature on wavelets, these terms could be used. So let us introduce that notation slowly. So we say the L2 norm of x is essentially mod xt square dt integrated over all t and to be very precise this needs to be raised to the power half. Similarly one can talk about LP norm, you could talk about of x.

And that would correspondingly be mod xt to the power P dt integrated on all time and raised to the power 1 by P. And of course P here is a real number. So for any real, in fact real and positive. You could talk about in L1 norm, you could talk about L2 norm, you could talk about an L infinity norm, what would L infinity norm be, let us take some examples. What would an L1 norm be? It would essentially be integral mod xt, the L2 norm we already know. What would the L infinity norm be?

That is interesting, so you see in principle it would be something like this, what now is this? What do we mean by this? You see as P becomes larger and larger, what are we doing, we are emphasising those values of xt which are larger. So for a larger value of P, we are emphasising those values of mod xt which are larger. And as P tends to a larger and larger and larger value, as P tends to infinity, we are in some sense highlighting that part of xt which is the largest.

So in other words, the L infinity norm of x essentially would correspond to the maximum or the supremum, you know the very largest value that xt can attain all over the real axis. So it has a meaning, even as P tends to infinity. Anyway, this was just to introduce some notations which we are going to find useful. And what we are saying in this language is that we are going to focus on functions which belong, now here you know we are going to start talking about functions that belong to a space. We say, you know, we say the space L2, what is the space L2?

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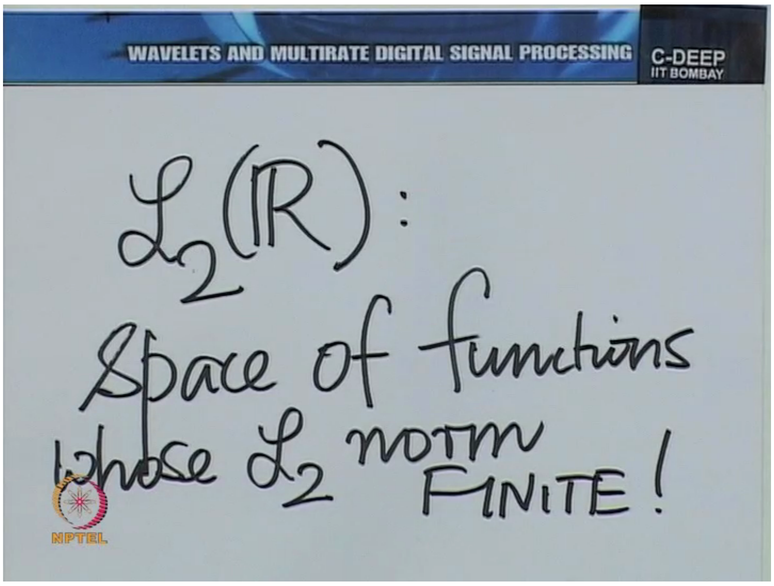

L2 is over the real axis, it is a space of functions and it is a space of functions whose L2 norm is finite, simple. Similarly you could have the space LP. The space LP is a set of all functions whose LP norm is finite. Now the word space is used with an intent. You see space really mean that if I take a linear combination of functions that set, it gets back to a function in that set. So if I take any finite linear combination of functions in a space LP, the resultant is also in that space, in that set LP and that is why we call it a space.

So LP, only LPs for any particular P are spaces, linear spaces, they are closed under the operation of linear combination. So in other words we are saying let us focus our attention on the space L2. Now what we have said in the Haar analysis that we talked about a few minutes ago is that if you take any function in the space L2, I am in if you are adversary, picks up any function in the space L2 and puts before you a value e0, saying please give me an m, so that when I make the piecewise constant approximation on intervals of size T by 2 raised to the power m, my error, square error is less than e0.

The proponent is able to come up with an m which gives the answer. And this could be done no matter how small the e0 is, the proponent will always come out with a suitable m. That is the idea of what is called closure. You know, so what we are seeing is when we do an analysis using the Haar wavelet, in other words, when we start from a certain piecewise constant approximation on intervals of size say 1 for example and then bring it to intervals of size half, 1/4th, 1/8th, 1/16th, as small as you desire, you can in principle go as close in the sense of L2 norm, that means if I look at the L2 norm of the error between the function and its approximation, that L2 norm of the error can be brought down as much as you desire.

And in that sense whatever the Fourier series was doing, after all what does the Fourier series do, it allows you to bring the L2 norm of the error between the functions and Fourier series as small as you desire for the reasonable class of functions. For a wide class of functions, give me the Epsilon, give me the e0 and I will give you certain number of terms that you must including the Fourier series. So the adversary says, well here is an e0 for you, the proponent says okay, include so many times in the Fourier series and you can bring your error down as low as you desire.

The same kind of thing is happening here, the proponent-adversary principle. Now this is a deep tissue that one function psi T is able to take you as close as you desire to the function that you want to approximate. And by the way this is only one psi T which can do it. The whole subject of wavelets allows you to buildup many such psi Ts. Here we had a good physical, a very simple physical explanation.

We started from piecewise constant approximation, we said well, when you want to refine your piecewise constant approximation, you could do it by using the Haar wavelet. And this you could do to go from any resolution to the next resolution. Please remember here we are increasing the resolution or improving the amount of information contained by factors of 2 each time. And that is why we use the term dyadic, let me write down that term, dyadic.

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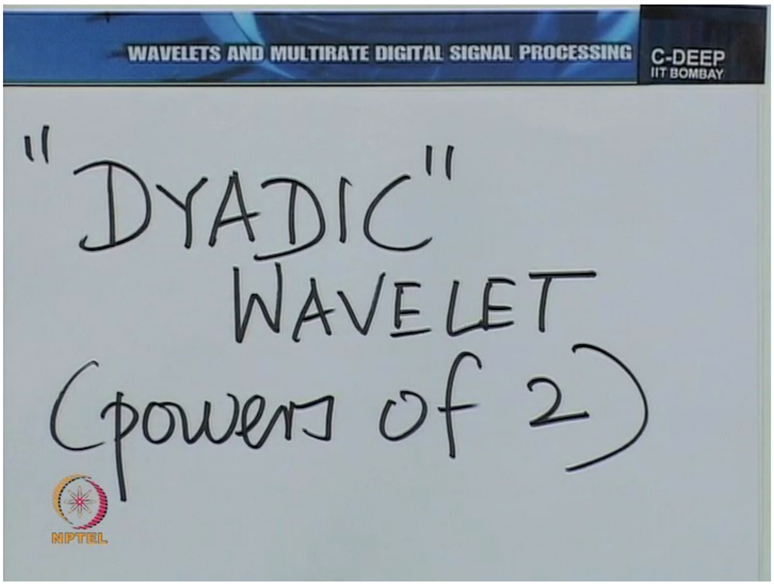

So what we have introduced in this lecture is the notion of a dyadic wavelet and dyadic refers to powers of 2, steps of 2 every time. The Haar wavelet is then example of a dyadic wavelet and in fact for quite some time in this course we are going to focus on dyadic wavelets.

Dyadic wavelets are the best studied, they are the best and most easily designed, they are the best and most easily implemented and I daresay the best understood. So for quite some time in this course, we shall be focusing on the dyadic wavelet, the Haar is the beginning.

I mentioned in the previous lecture that if one understands the Haar wavelet and if one understands the way in which the Haar multiresolution analysis is constructed, many concepts of multiresolution analysis would become clear. What we intend to do now after this in subsequent lectures is to bring this out explicitly. So let me give you a brief exposition of what we intend to do in subsequent lectures, and then we shall go down to doing it mathematically step-by-step.

You see, we brought out the idea of the Haar wavelet explicitly here, what is the Haar wavelet, we know, we know what function it is and we know that dilates and translates this function can capture information in going from one resolution to the next level of resolution in steps of 2 each time. Now, how is this expressed in the language of spaces, after all we talked about the space L2R, L2R is the space of square integrable functions. So how can we express this in terms of approximation of that whole space, so can we express this in terms of going from one subspace of L2R to the next subspace?

And in that case can be expressed this Haar wavelet or the functions constructed by the Haar wavelet and its translates and perhaps also dilates in terms of adding more and more to the subspaces to go from a course of surface all the way up to L2R on one side and all the way down to a trivial subspace on the other. So we are going to introduce this idea of formalising the notion of multiresolution analysis.

We need to think of what is called a ladder of the spaces in going from coarse subspace to finer and finer subspaces until you reach L2R at one end and coarser and coarser and coarser space until you reach the trivial subspace at the other end. Further, we are going to see that the Haar wavelet and its translates at a particular resolution, at a particular power of 2 so to speak, actually relates to the basis of these subspaces.

So we are going to bring out the idea of basis of these subspaces and how is the Haar wavelet captures what is called the difference subspace, in fact the orthogonal complement to be more formal and precise, simple but beautiful and what we do for the Haar will also apply to many other such kinds of wavelets. Let us then carry out this discussion in more detail in the next lecture where we shall formalise whatever we have studied today for the Haar wavelet by putting down the subspaces that lead us towards L2R at one end and towards the trivial sub space at the other, thank you.