Foundations of Wavelets, Filter Banks and time Frequency Analysis.
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Week-6.
Lecture -16.2.
The Idea of Time-Frequency Resolution.

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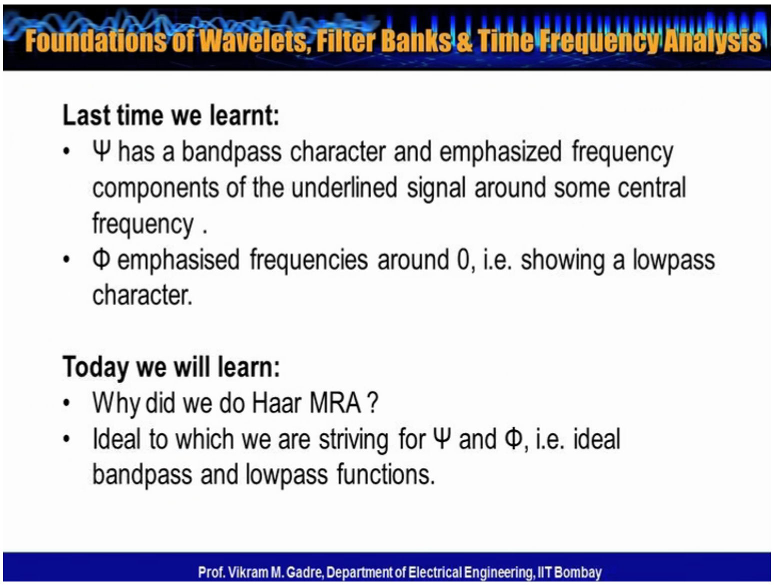

Now let us bring in the idea of time resolution and frequency resolution. If we use bandwidth as a measure, please note as a measure of the range of frequencies that are emphasised by the function psi. now why am I saying once again that these frequencies are emphasised, let me just recapitulate. I am saying this again and again because one must firmly understand this. I am saying that those frequencies are emphasised because in finding the dot product of a function xt with any translate of the function psi t or one of the stretched or compressed versions of psi t.

Parseval's theorem tells us that they also multiplying the Fourier transform of x with the Fourier transform of that particular translate and dilate of psi or for that matter phi, whatever it be. Now we also understood that translation has no effect on the magnitude, dilation does and when they multiply that Fourier transform of x by the Fourier transform of Phi or psi as the case may be, appropriately diluted, one is automatically emphasising multiplying that part of the band which lies in the region of large magnitude of Fourier transform of Phi or Psi by a larger number and the other parts are being multiplied by a tapering number.

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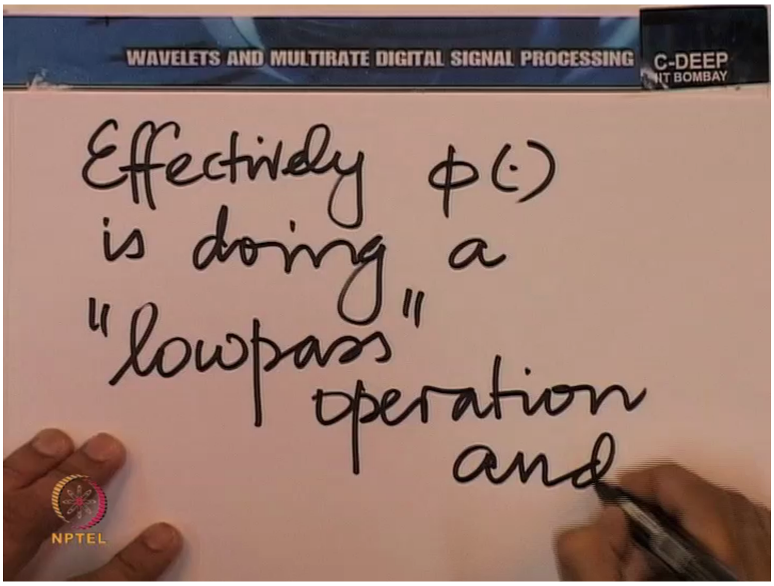

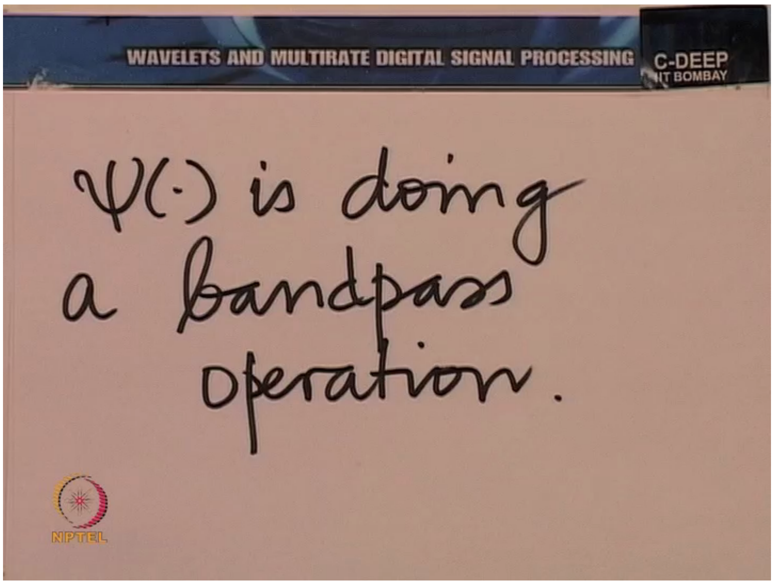

So in effect there is a filtering operation also being done by Phi and psi. Effectively Phi is doing a lowpass filtering operation and psi is doing a bandpass filtering operation, let us make a note of this, this is very important. So effectively Phi is doing a lowpass operation and psi is doing a bandpass operation. Now then it almost seems trivial, what is so great, we could have built a bandpass or lowpass filter otherwise, why did we have to do all this Haar business? Well, you see the beauty is in the 2 domains together and this is where the whole catch lies and this is where the whole struggle lies.

You are able to do some kind of a crude lowpass operation, I say crude because nobody would agree if you look at the frequency response, the Fourier transform of Phi that it is really very close to a good lowpass filter, crude in that sense. You are doing a crude lowpass filter operation but with the proviso that you are also confining yourself in time. So you are saying, you are able to say with some confidence and that confidence depends on how well localised that Fourier transform is around 0 frequency. So you are able to say with some confidence that when I multiply xt by a certain dilate and translate of Phi, I am emphasising that band of frequencies around 0 which is covered by the appropriate dilate of Phi.

So if you take Phi itself and if you focus your attention on the main lobe of the Fourier transform, you may say in a crude sense that you are emphasising the frequency around 0 up to the extent of 2 pie, the main lobe goes up to 2 pie. And you are doing this in a time region in which Phi lies, in fact that can be said non-crudely. So Phi is indeed very very localised in time, I think nobody will disagree with that, so is psi. So when you multiply by a certain dilate and a translate of psi, you are in effect doing a kind of localisation in frequency around that point of maximum as you saw it lays somewhere near 2 pie, before 2 pie actually and as you take different dilates of psi, you are taking different bands and this is being done in the time zone covered by that particular translate.

This is a serious statement we are making, we are making a statement about localisation in 2 domains simultaneously, in time and in frequency. And if you recall, in the very 1st lecture when I introduced the subject of wavelets and time frequency methods, this is one of the things are mentioned as a fundamental challenge in signal processing. In fact I went to the extent of saying the same challenge appears in different manifestations in different subjects. In signal processing we see there is a conflict between time and frequency. Where is the conflict?

The conflict is partly seen now, partly I say. You see as you notice, in time we are very correct in saying that we have localised, after all Phi t and psi t and their translates and dilates are nonzero only over a finite region of time. So localisation in time in this case is not under question at all, it is localisation in frequency which is somewhat suspect. We can crudely say that because if you focus your attention on the main lobe, then in some sense it is localised, but there are the side lobes in the Fourier transform, both of Phi and of psi.

So now we want to ask the question, what ideal would like to strive towards? If I were to have my way, how should I make the Fourier transform of Phi and psi look? We know how they should be in time, we should be packed into a finite region of time, we are able to do that. I would also like to pack them into a finite region of frequencies simultaneously. Now what would that region of frequency be? Let us use our understanding of signals and sampling a little bit here. You see, let us write down the dot product of xt with a particular integer translate of Phi as a sampling problem now.

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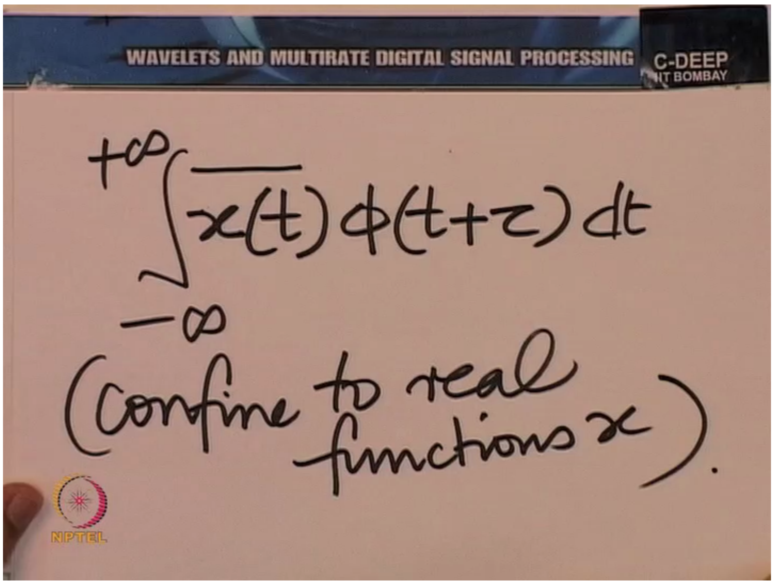

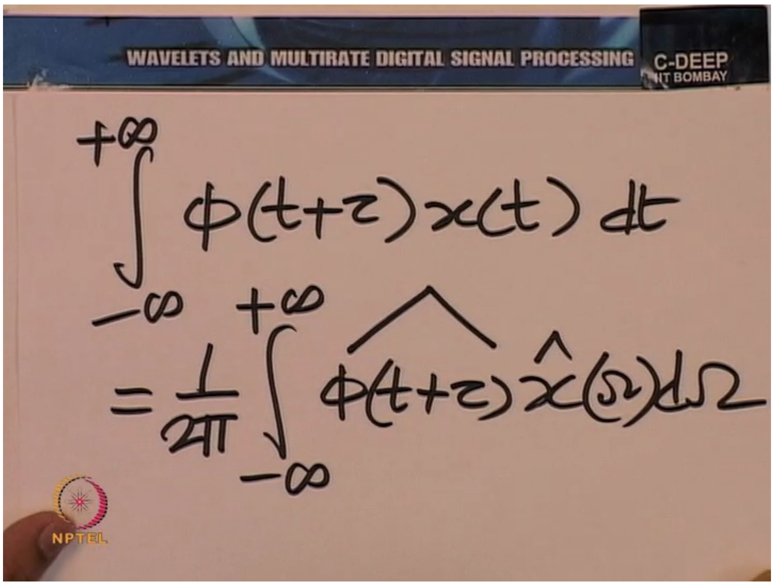

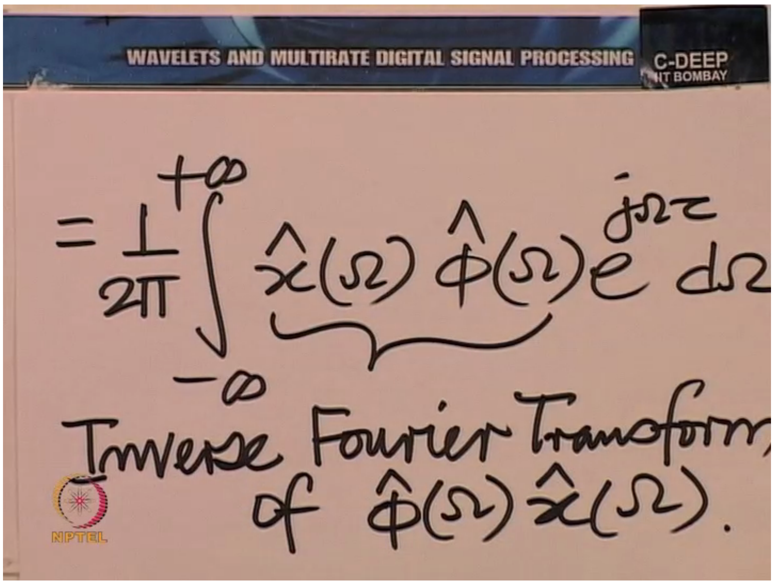

So if you take this product, if you wish I can put complex conjugate, maybe I should put a complex conjugate there, it would be a dot product in the strict sense. But even if I do not would be complex conjugate and confine myself to real function x, I am doing rather well. In fact we will do that for the moment because we do not want to mix too many issues. Let us confine to real functions. And then I have, this is of course equal from Parseval's theorem too, the Fourier transform of Phi t + tao times the Fourier transform of x integrated over all omega and this is easy to evaluate.

So essentially we have a product of Fourier transforms x and Phi, x Cap and phi Cap multiplied together and then and inverse Fourier transform is being computed at the point tao. So this is like, you know even if you were to use a complex function, the only thing would be here, there would, we would need to put a complex conjugate there. That is why I said that that is not such a serious issue, at the moment we just focus on real functions and interpret. So here when you multiply by Phi Cap omega, you are in effect doing some kind of a lowpass filtering.

And when you take the inverse Fourier transform, you are calculating what comes out of that crude lowpass filter whose impulse response is essentially phi, essentially phi I mean, do not worry about inversions or you know time in version, it relates to phi, very closely phi. Now what we are saying is, when you sample this, when you put tao equal to all the integers.

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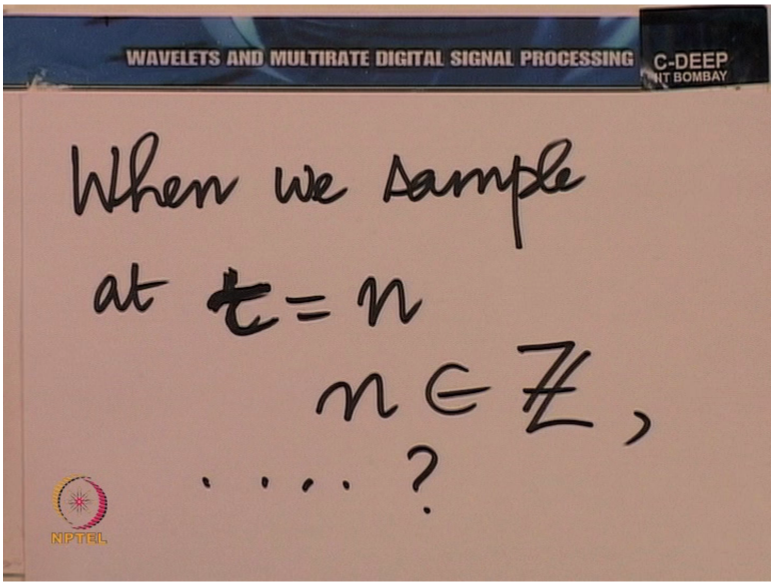

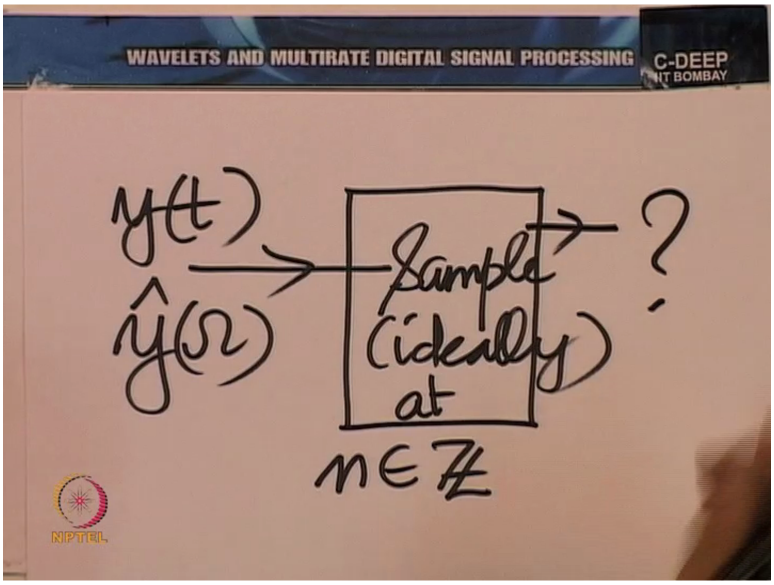

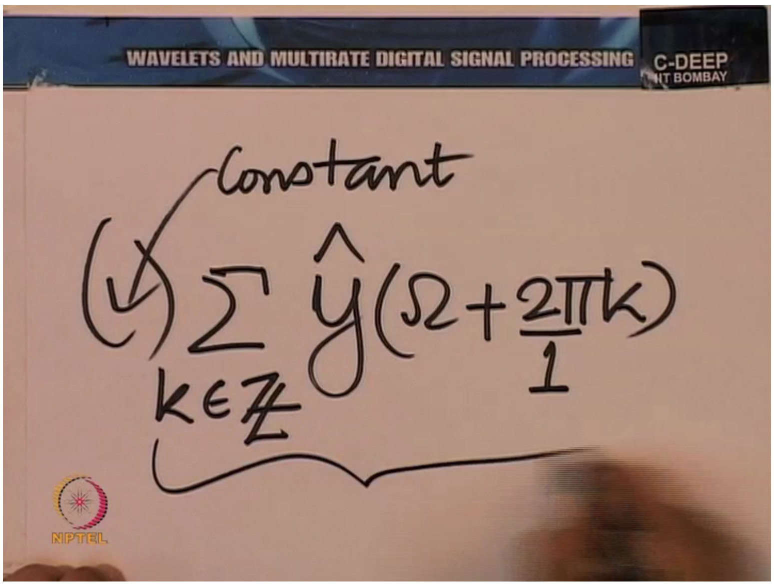

So if you take this and substitute tao by different integer values, when we sample at tao equal to n, n all integers, what is going to happen? We are going to take the original Fourier transform, you see when we sample, if you take a function let us say Yt with Fourier transform Y Cap omega and you sample this, sample ideally if you like at all integers. That essentially means that you sampling and sampling rate of 1. So that amounts to taking the original Fourier transform, translating it by every multiple of 2 pie divided by 1 which is 2 pie on the angular frequency axis and adding up these translates.

So let me write that down in terms of an algebraic expression. What we are doing essentially is we are taking the original Fourier transform, translating it by every multiple of 2 pie divided by 1 if you please, every multiple of that and summing up these translates, some constant possibly, that constant relates to the sampling process. Let me ignore that constant for the moment, our attention is here. So in to reconstruct Y from its samples, what should we have desired? We should have desired that these translates do not interfere with the original.

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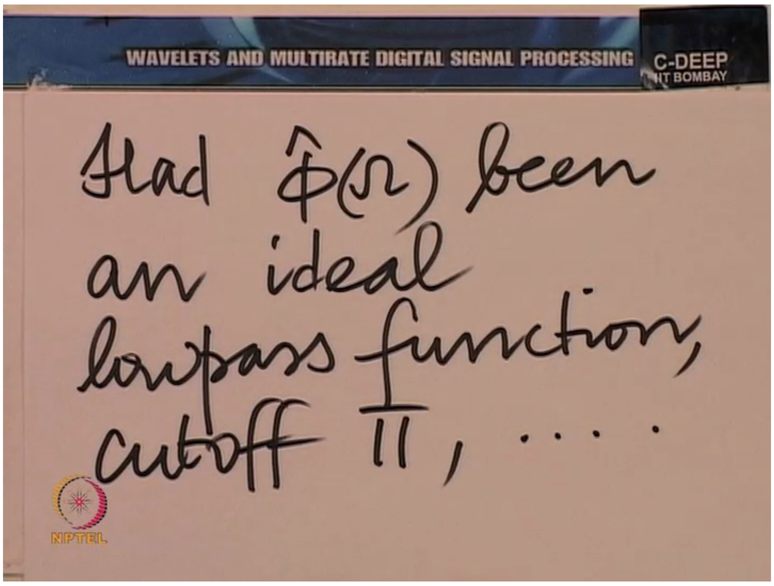

then this 'aliasing'

$$\sum_{k \in \mathbb{Z}} \hat{y}(\Omega + 2\pi k)$$

would leave $\hat{y}(\Omega)$ unaffected.

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For V_1 (MRA ladder Haar)

we expand by 2 in frequency.

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That means we are asking for a lowpass filter with cutoff 2π (instead of π)!

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So it would have really been nice if we had been able to ensure that these carbon copies created by Y Cap omega + 2 pie K nonoverlapping with the original and that is ensured by ensuring that the lowpass filter cuts of at capital Omega equal to pie. Let me sketch that for you. Had Phi Cap omega been ideal lowpass function with a cut-off of pie, then, then this aliasing process would leave Y Cap omega unaffected. So that is the ideal towards which we are striving as far as Phi goes. Now what is the ideal towards which we are striving as far as psi goes, let us see.

You see when you go from V0, which is what brought us to phi to V1. What is V1? Just essentially V0 but compressed by a factor of 2 in time and therefore expanded by a factor of 2 in frequency. So for V1, I am talking about the ladder, MRA Ladder, Haar Ladder, we expand by 2 in frequency, we are talking about frequency domain behaviour. So we expand by 2 in frequency. That means we are asking for a lowpass filter with cut-off 2 pie instead of pie. Now we also have an interpretation for the incremental subspace. Obviously, if V0 is going to contain information between 0 and pie and V1 is going to contain information between 0 and 2 pie, then the difference subspace W0 should contain the information between pie and 2 pie, simple.

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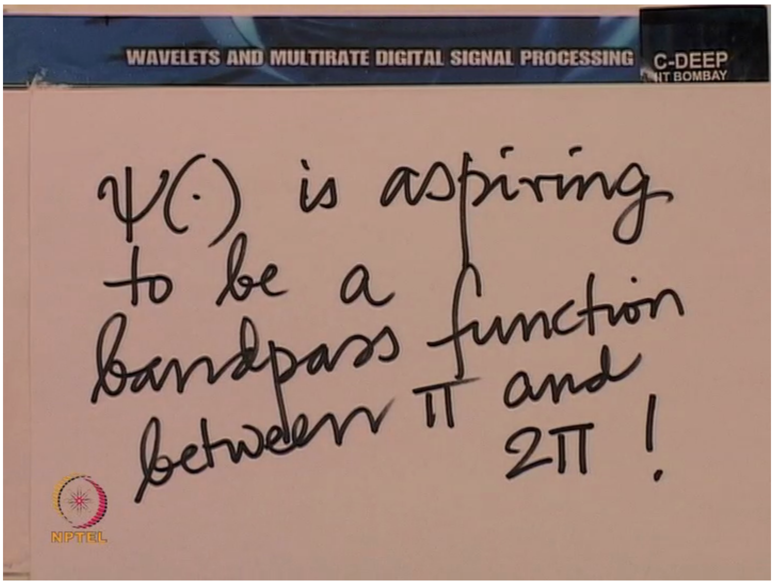

So what we are saying in effect is psi is aspiring to be a bandpass function between pie and 2 pie. And of course this is for going from V1, from V0 to V1, when you go from V -1 to V0, you use a corresponding dilate of psi which is aspiring to be bandpass function between I buy 2 and pie. When you go from V1 to V2, then you bring in a dilate of psi which aspires to be a bandpass function between 2 pie and 4 pie and so on.