Foundations of Wavelets, Filter Banks and time Frequency Analysis.
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Week-6.
Lecture -16.1.
nature of Haar Scaling and Wavelet Functions in Frequency Domain.

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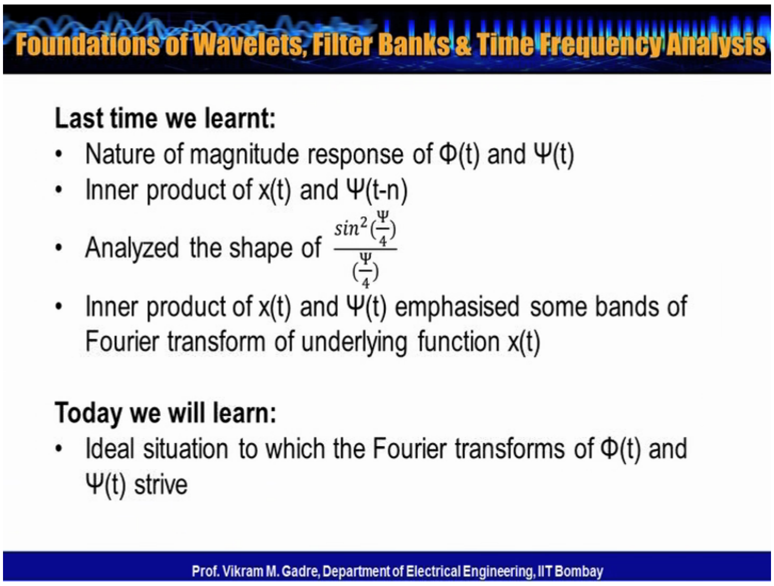

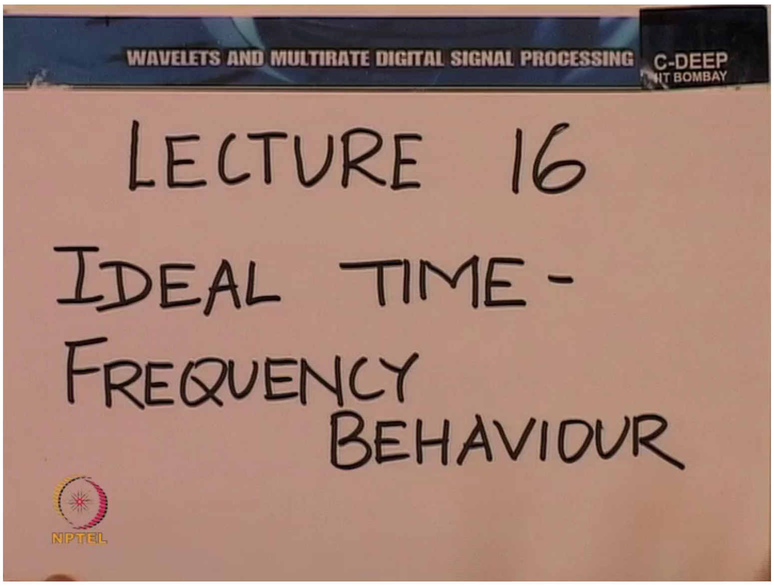

A very warm welcome to this lecture on the subject of wavelets and multirate digital signal processing. Let us put in perspective what we are going to do in today's lecture building up from what we did in the previous one. In the previous lecture we had looked at the Fourier transform of the scaling function or the so-called father wavelet Phi t and the wavelet function or the so-called mother wavelet Psi t in the Haar multiresolution analysis. What I shall do is to begin with a description of where we wish to go from here. You see, we have made some observations about the nature of the magnitude of the Fourier transform of Phi t and Psi t.

We also noted that when we multiply a given function xt by a translate of Phi t, the magnitudes of the Fourier transforms of x and Phi are getting multiplied and we saw that the nature of the Fourier transform of Phi or for that matter even that of Psi was such that it emphasised some band of the Fourier transform of the underlying function x which was being studied. Now what we intend to do today is to idealise from there, what is the ideal situation to which we strive. And therefore I have put down the central theme in the lecture today to be ideal time frequency behaviour, what is the ideal towards which we are trying to move?

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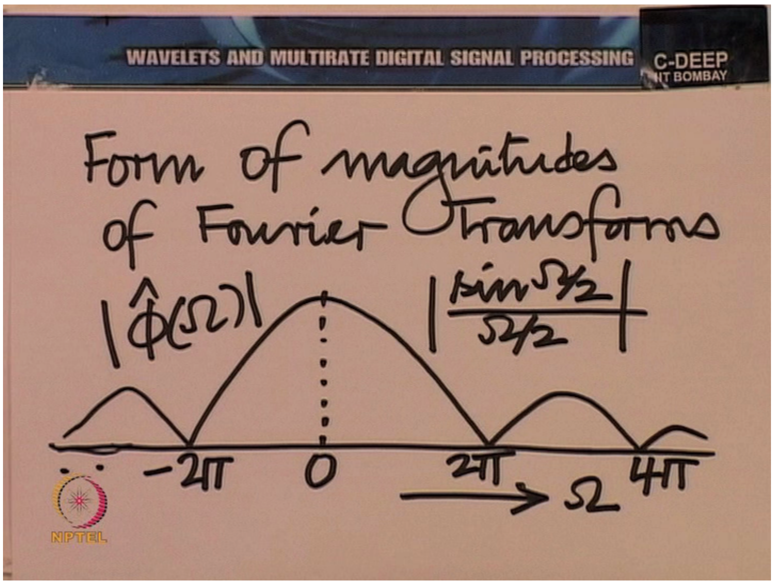

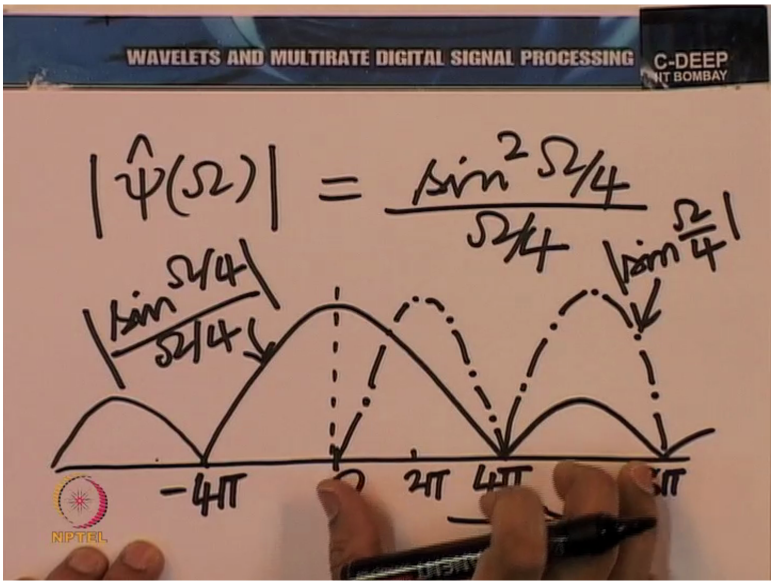

Now let me put before you once again the nature of the Fourier transforms of Phi and Psi. By nature I am essentially going to refer to the magnitude. The phase though important in general is not of prime importance at the moment because it is the magnitude which makes the selection of bands. So let us put down the nature of the magnitude. So form of magnitudes Fourier transform, for Phi it had an appearance like this, so this was 0 Frequency and this was to 2 pie here and all multiples of 2 pie subsequently and so on, this is mod phi cap Omega.

You see when I say form, what I imply that I am not going to consider any constants or phase. Constants would only scale this up or down and the phase would not affect the magnitude of course. So let us also look at that of Psi. The form of the magnitude of the 4 written form of Psi would look something like this. Sin squared Omega by 4 divided by omega by 4 and we have made an attempt to sketch this last time. We 1st sketched sin Omega by 4 by omega by 4, so we said, this was a form of mod sin Omega by 4 divided by omega by 4, the solid line here.

And I also drew a dotted line to indicate, this time let us use a dot dash line to make a distinction from this margin or this axis. So let us use a dot dash line to denote the magnitude of the other term, sin Omega by 4, so that would have a peak at 2 pie, this is the form. Now this solid line is multiplied by this dot dash line here. So of course you must visualize this dot dash line being replicated on the negative side and as you know for a real function the Fourier transform is magnitude symmetric. So it is enough for me to study the positive side of omega and the negative side would be a mirror image. So let me expand this part here.

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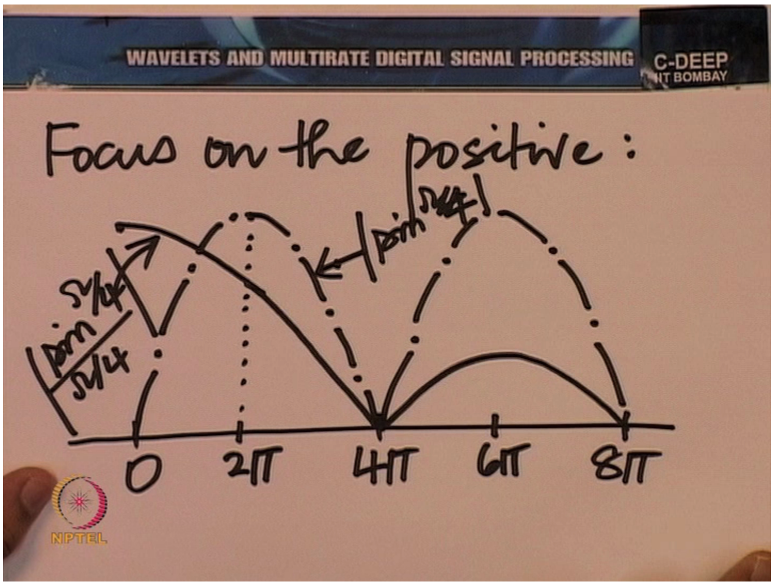

Focus on the positive, a good motto in general. You see, sin Omega by 4 by omega by 4, this one has a monotonically decreasing character from 0 to 4 pie, this one has a monotonically increasing character between 0 to pie and then monotonically decreasing. Now it is very clear that from this point onwards, the product of these 2 is only going to decrease, so you cannot possibly have a value of this product, this of course being mod sin Omega by 4, you cannot have a magnitude of the product of this dot dash line with the solid line greater in the segment between 2 pie and 4 pie than it is at 2 pie.

So in other words after 2 pie, between 2 pie and 4 pie this product is only going to decrease and therefore I can get a feel, you see it is clear the product is 0 at Omega equal to 0, whatever is after 2 pie is going to be less than what is that 2 pie. So somewhere in between it is going to achieve the maximum and then continue to drop. So we get a feel of this, so we must get a final feel of this than we had the last time. In fact let me also make one more remark. You see if you look at the region between 4 pie and 8 pie, the situation is a little simple.

There is a kind of tendency to maximum somewhere in between in both of these functions and then a drop. So that similar pattern would be replicated in the product, a maximum somewhere in between, well, not quite at 6 pie, please remember, this is not quite symmetric, you must remember that, although this is, this is not quite symmetric. So the maximum will be somewhere other than 6 pie but that is not of so much of concern, we will have maximum somewhere near the 6 pie and it would drop off on both sides.

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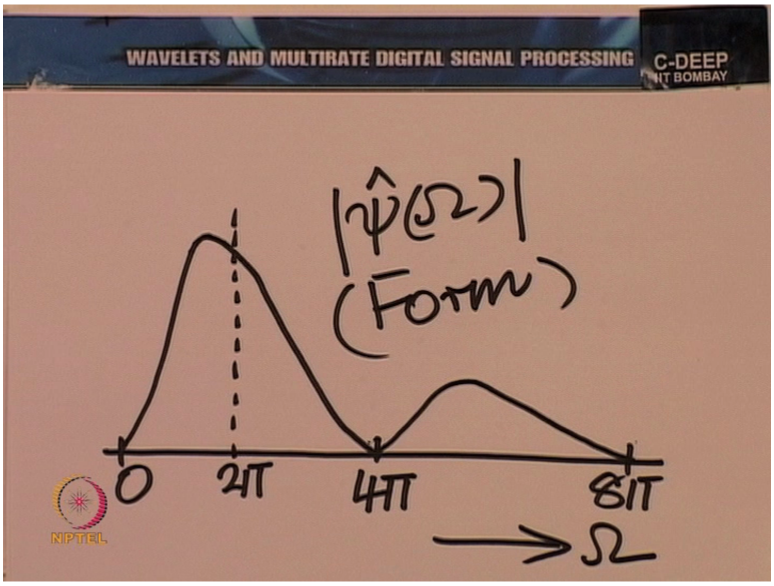

So in toto this is what the product would look like, I will not mark this maximum, it is a little difficult to calculate. But this is the nature, this is the form. Now let us take the trouble to draw them together, again only on the positive side of the frequency axis. The form of the Fourier terms of Phi and of Psi, let us focus only between 0 and 4 pie. So Phi looks something like this and Psi looks something like this and we had made a remark on what Phi does and what Psi does. Phi in effect emphasises those frequencies lying around 0 frequency and Psi emphasises those frequencies lying around its maximum in the band between 0 and 4 pie and deemphasises frequencies on either side.

So in fact if you look at Psi, it deemphasises frequencies around 0 and then after that band, so it emphasises a band of frequencies. It is clear that Psi has a bandpass character, it is a bandpass function. A bandpass function is one which emphasises frequencies around some centre, so-called Centre frequency by the response is the maximum and deemphasises frequency both around 0 and around infinity so to speak. So there is a finite band of frequencies, one band which is that function emphasises. Loosely speaking, this Psi Omega here emphasises those frequencies lying around its maximum here and of course Phi emphasises frequencies around 0.

We also made one more remark on the distinction between phi and Psi. You see we noted that when we contract expand, so when we go up or down the ladder, what are we doing in the Fourier domain? When we go up the ladder, we are expanding in frequency because we are contracting in time, when we go down the ladder, we are expanding in time and therefore we are contracting in frequency. So let us look at this figure once again. As we go down the ladder we are contracting in frequency, so we are emphasising smaller and smaller bands around 0 and again, since they are contracting this as well, we are emphasising frequencies around a smaller and smaller Centre frequency.

In fact it is very easy to see that this Centre frequency, the point where there is a maximum in the magnitude of Psi decreases geometrically or logarithmically as we go down the ladder in the Haar multiresolution analysis. And the width of this band also decreases geometrically or logarithmically, this is something very interesting. The band decreases geometrically, the Centre frequency also decreases geometrically. So we have a situation where the ratio of the band to the Centre frequency is a constant. So we have a name for that kind of analysis in the literature on wavelets for time frequency methods, we call it constant quality factor analysis or constant Q analysis, let us write that down.

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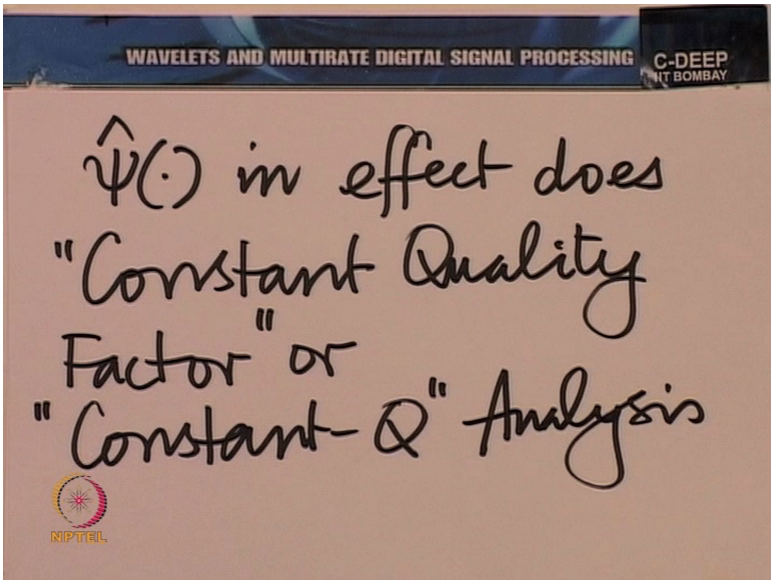

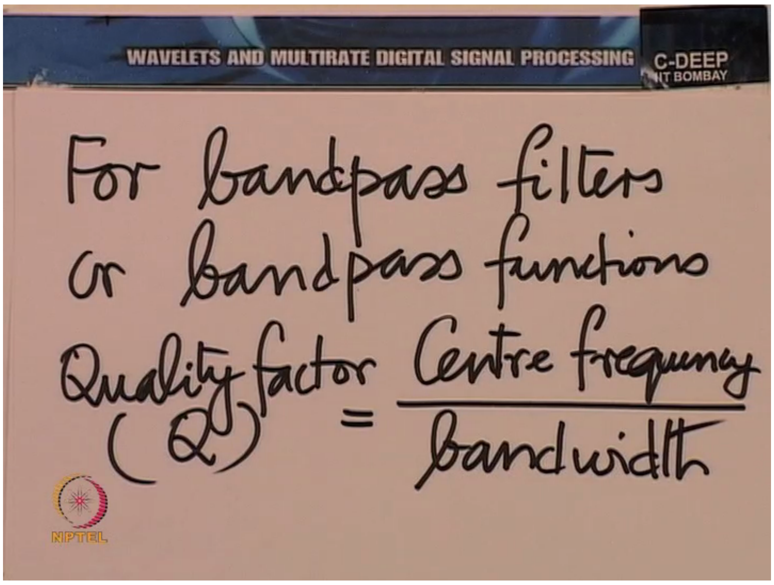

Psi in effect does constant quality factor analysis or constant Q analysis. And this word quality factor comes from a term used in the context of bandpass filter. For bandpass filters or bandpass functions, the quality factor or Q as it is often denoted in brief is a ratio of the Centre frequency to the band or the bandwidth. You know the word bandwidth of course has to be taken with a pinch of salt. What does bandwidth mean, there are different definitions, particularly when you do not have a clear brick wall situation, you have a smooth variation of magnitude with frequency as you do here.

So there is the maximum and the frequency falls off on either sides. Typically we denote the word bandwidth to denote that range of frequencies within which the magnitude remains within a certain percentage of the maximum magnitude. So for example where the magnitude remains between 70 and 100 percent say of the maximum magnitude. Or where it remains even more specifically for most situations we talk about what is called the half power bandwidth where the amplitude of the magnitude response falls to the square root of half from the maximum.

And the square root of half has a significance that at that point where it falls to the square root of half, the power of a sine wave is half of what it would be in proportion to the original as compared to the maximum point. So if at the Centre frequency the point where the magnitude response is maximum, the power ratio of input to output is say 100 units then at the point where the power, you know the magnitude falls to 1 by square root of 2, the power would be only 50 units, the ratio, the power ratio, output divided by input, so it is called the half power point.

Very often we talk about the half power bandwidth. In any case it does not matter what percentage we use, 70 percent so be it, 60 percent so be it, whatever it be. With this notion of bandwidth the ratio of the Centre frequency to the bandwidth in this sense is a constant as we stretch or compress the Fourier transform of Psi and then also, of course, you know whatever you do in terms of stretching or compressing in the time domain, you are doing exactly the opposite in the frequency domain. So as you go up the ladder, you are going towards have frequencies and you are also spanning a larger bandwidth.

As you come to lower steps, as you go descend in the ladder, you go to lower rungs of the ladder, you are essentially going to smaller Centre frequencies and using a smaller bandwidth.