Foundations of Wavelets, Filter Banks and time Frequency Analysis.
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Week-6.
Lecture -15.1.
Fourier transform Analysis of Haar Scaling and Wavelet Functions.

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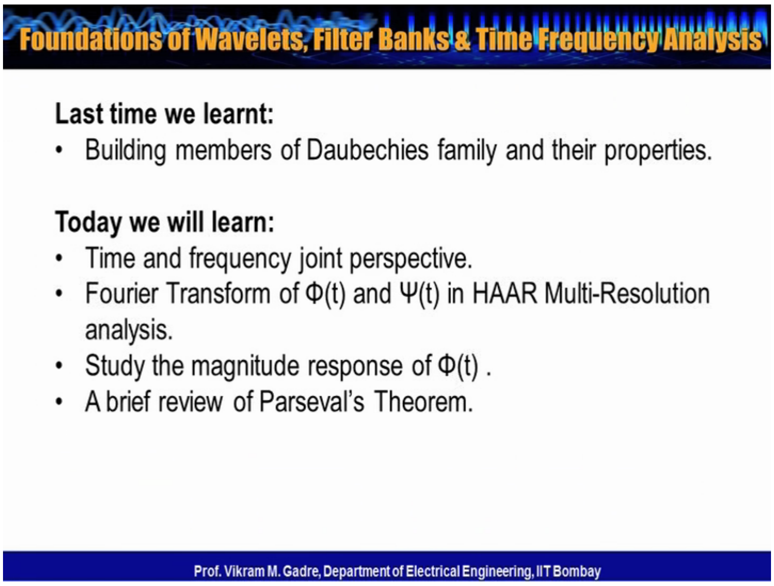

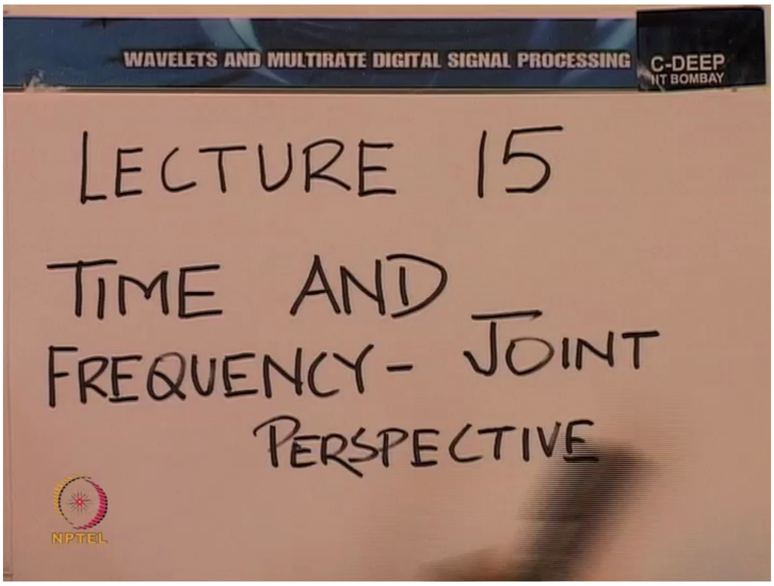

A warm welcome to this lecture on the subject of wavelets and multirate digital signal processing. We had raised a few questions in the previous lecture which we set out to answer in this lecture. the questions pertained to the fact that we were building more than 1 multiresolution analysis. And specifically the question was, what are we looking for when we go from one multiresolution analysis to the other. Why cannot we be content with the Haar multiresolution analysis? We have been singing the praises of the Haar multiresolution analysis so frequently in some of the previous lecture.

We have been saying that it tells us most of the things that a multiresolution analysis does, why then should we look for others, what is inadequate in the Haar? Well, if you remember, when we began this course, we set out to do something which a basic course on signal theory or signals and Systems does not. namely, look at 2 domains simultaneously. We have been trying to do this for a while. So when we took the Haar, we did at some point talk about the filtering aspects of it. We also talked about the ideal to which we strive in the filter bank and if you recall that ideal had to do with the frequency domain, not the time domain.

So somewhere we have not been giving as much importance to the other domains so to speak as we should. We have been working in the natural domain of time but we have not quite been doing justice to the frequency domain. When we take a Haar multiresolution analysis, what does it do in the frequency domain, we have not quite understood this as yet. And the 1st thing that we would like to do is to answer this question. What does the Haar multiresolution analysis do in the frequency domain and then how do these filter bank that comprise the Haar buildup to their frequency domain behaviour?

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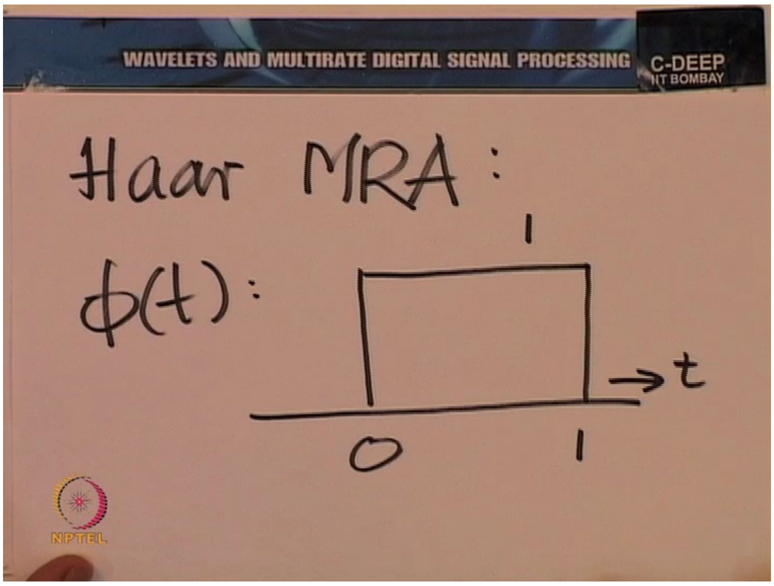

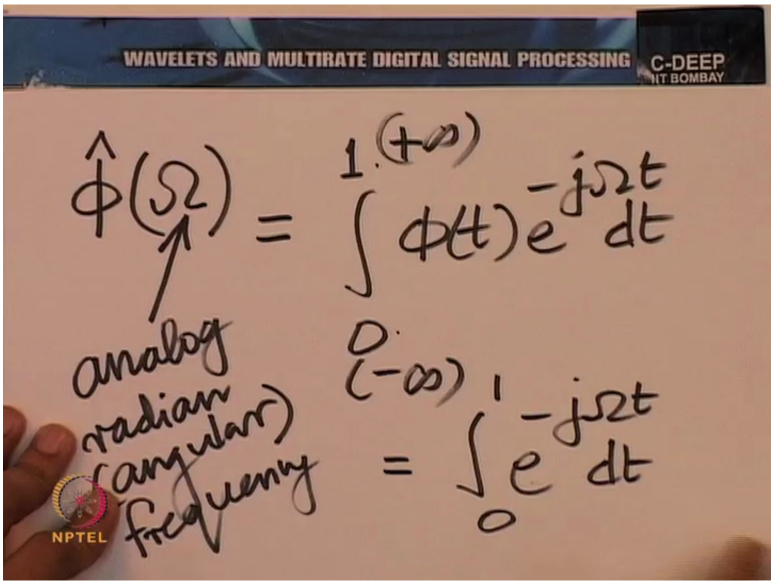

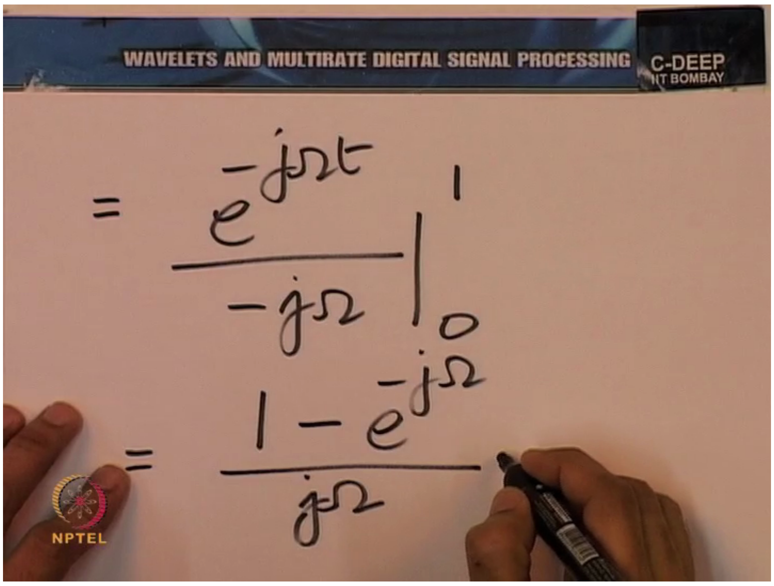

So let us set out 1st to answer a more basic question. What is the Fourier transform of the scaling function and the wavelet function in the Haar multiresolution analysis? So we shall call today's lecture a joint perspective, time and frequency. And we shall set out 1st to look at the Fourier transform of Phi t and psi t in Haar, the Haar mRA. Let us begin with the Phi t and let us look at its Fourier transform. Remember how Phi t looked, this was Phi t, let us obtain its Fourier transform.

We shall use this quotation for the Fourier transform, a cap on top and capital Omega to denote the angular frequency, analog angular frequency, radian or angular frequency. And this is easy to calculate, well I am saying 0 to 1, actually I should write -, I can write - infinity to + infinity but the nonzero part is only between 0 and 1, so it is also all right to write between 0 and 1, only for this specific function. And this is of course equal to integral 0 to 1e raised to the power - J Omega t dt which evaluates to e raised to the power - J Omega t by - J Omega from 0 to 1 and that is 1 - e raised to the power - J Omega by J Omega.

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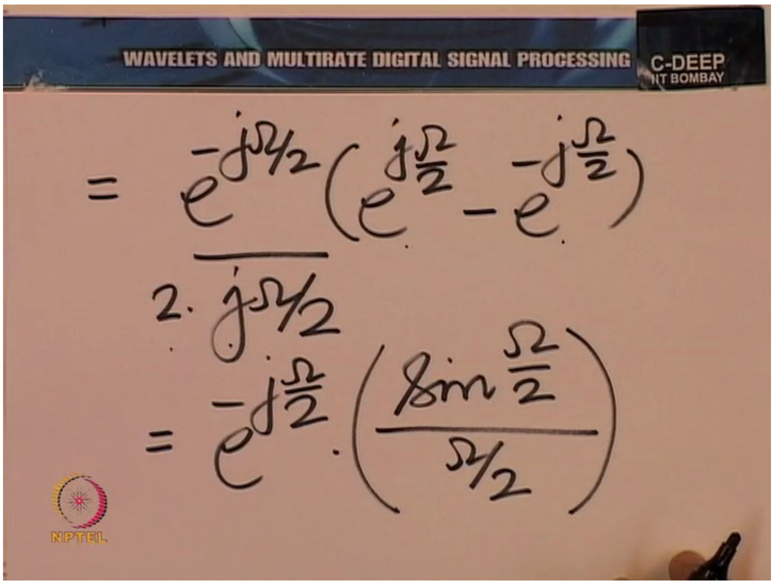

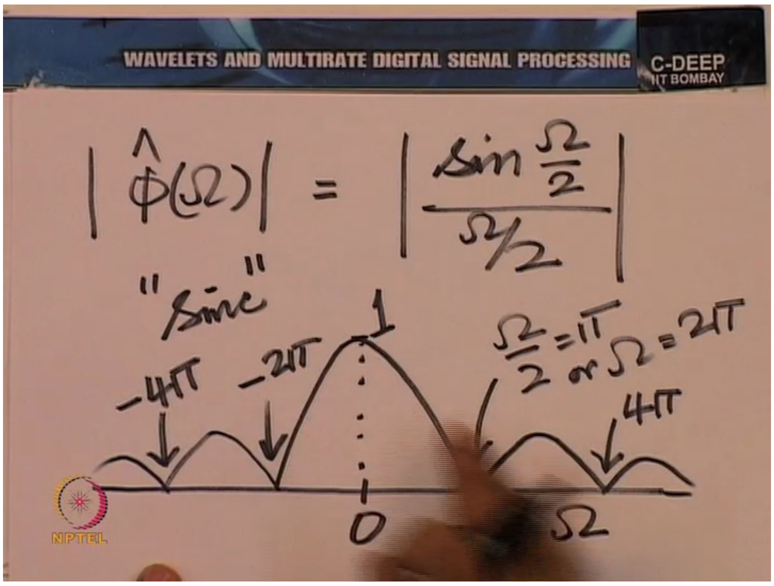

And we can simplify this, we can take e raised to the power - J Omega by common in the numerator. And that is easy to interpret, in fact if you wish you can make, even make this by J Omega, all right, 2 and then this becomes e raised to the power - J Omega by2, if I take the 2J and this together, I get sin Omega by2 there and Omega by2 here. So this is the Fourier transform. Now let us look at the magnitude of this Fourier transform. So in fact I could straightaway sketch this, it is essentially the magnitude of sin Omega by2 by Omega by2.

The sketch would look like this, this is a very familiar function to most Electrical engineers, we call it the so-called Sinc function, the Sinc pattern. People have different names for it, they call it the sampling function, the Sinc function and whatever other names. Anyway, this is a point where Omega by2 is equal to pie for Omega is 2 pie and this of course is the point where Omega is 4 pie and this is where it is - 2 pie and this is where it is - 4 pie and so on so forth. At this point the magnitude takes the value 1, in fact recall that the magnitude of the Fourier transform, in fact the value of the Fourier transform at omega equal to 0 is indicative of the area under Phi t, so the integral of Phi t over all t.

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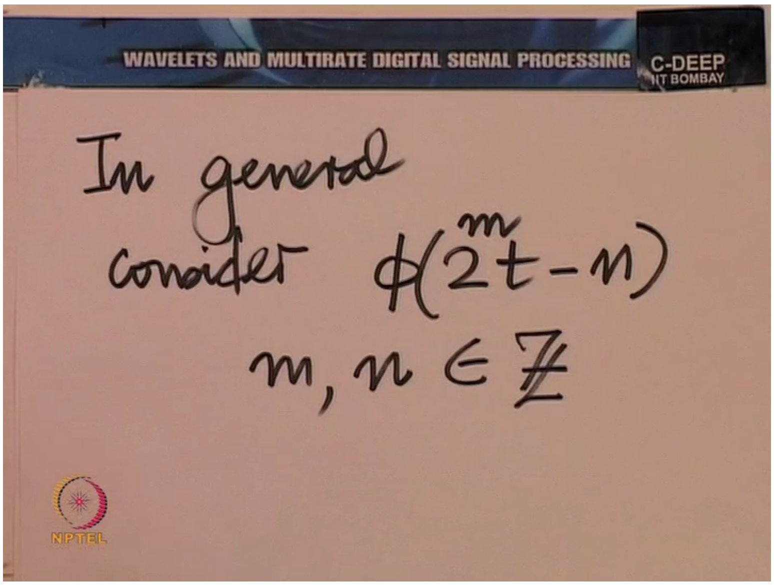

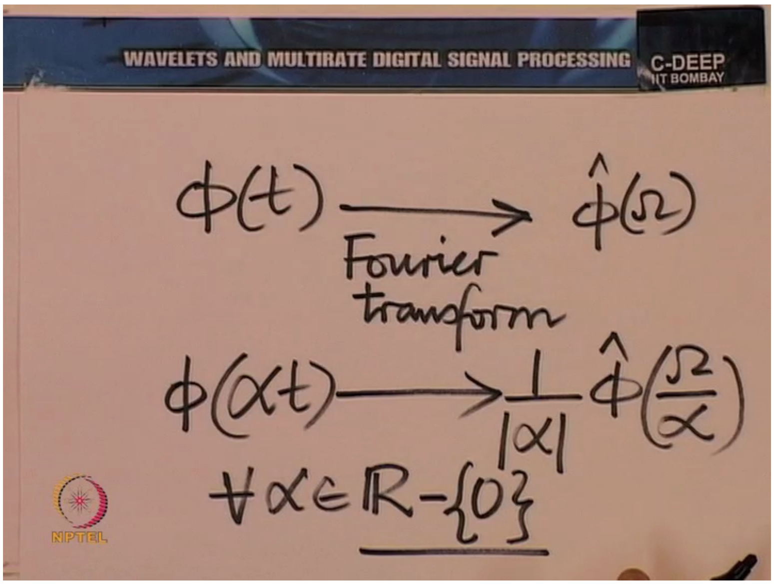

Anyways, so much so for the magnitude and now we know what to do when we compress and expand. Let us look in general had the Fourier transform of Phi 2t - n, so we should interpret that. You know, we are talking about dyadic dilates and translates, so let us consider phi, in fact symbol 2 raised to the power m t - n, for m and n belonging to the set of integers.

Now, you will recall that there is a very simple result in the Fourier transform which says that if Phi t has the Fourier transform phi cap Omega as we do in general, then Phi Alpha t has the Fourier transform 1 by mod Alpha phi cap Omega by Alpha for all Alpha belonging to the real numbers other than 0.

This notation says all real numbers except 0, of course Alpha cannot be 0. So for example even if Alpha is negative, we can use this, in particular for example if Alpha is -1, we have a reflection of the Fourier transform as well. So using this week will not take care of the 2 so-called distortions or modification that we have made in phi, the translation and dilation. In fact the translation does not affect the magnitude, the translation only affects the phase or the angle of the Fourier transform. So I can even forget about the translation, I need only look at the 2 raised to the power m term there.

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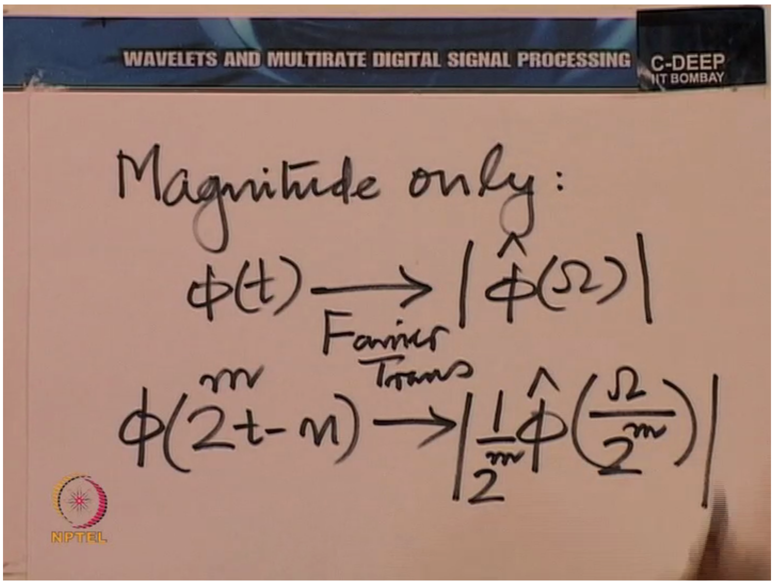

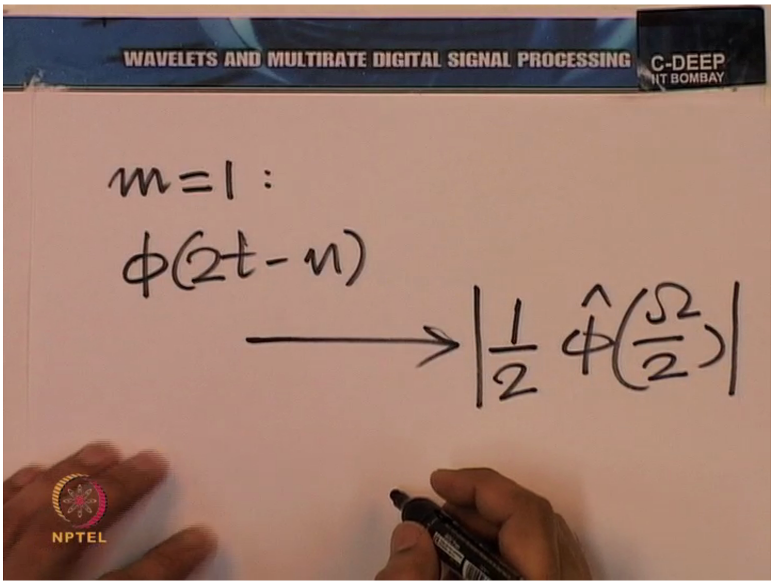

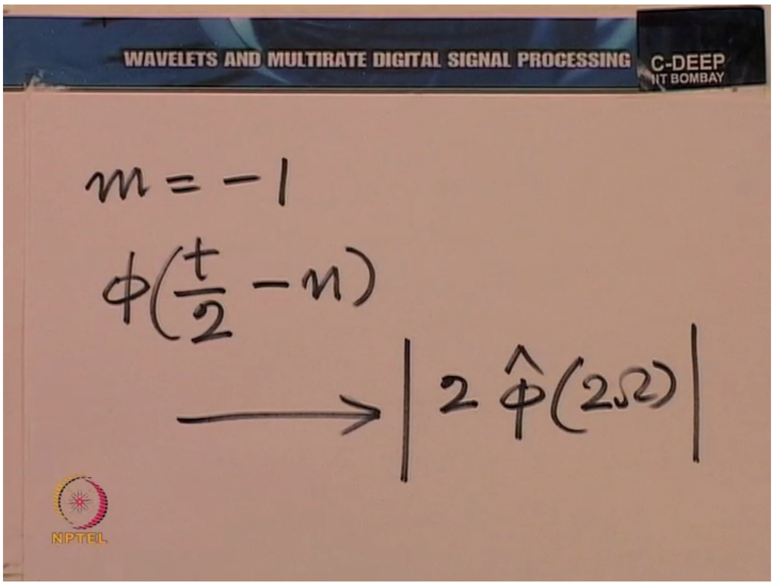

So here if I restrict myself to magnitude only, then Phi t has the Fourier transform Phi cap Omega with the magnitude of mod of this, Phi 2 raised to the power of m t - n would then have a Fourier transform mod phi cap Omega divided by 2 raised to the power of m, of course with the constant, the same 2 raised to the power of m but in the denominator. So of course let us take an example, suppose m is equal to 1 and - 1 to fix our ideas. And m equal to -1. Now notice that the n is entirely absent, here or here, the n is irrelevant as far as the magnitude goes, the n only contributes to the phase, let us sketch both of these, how would be look.

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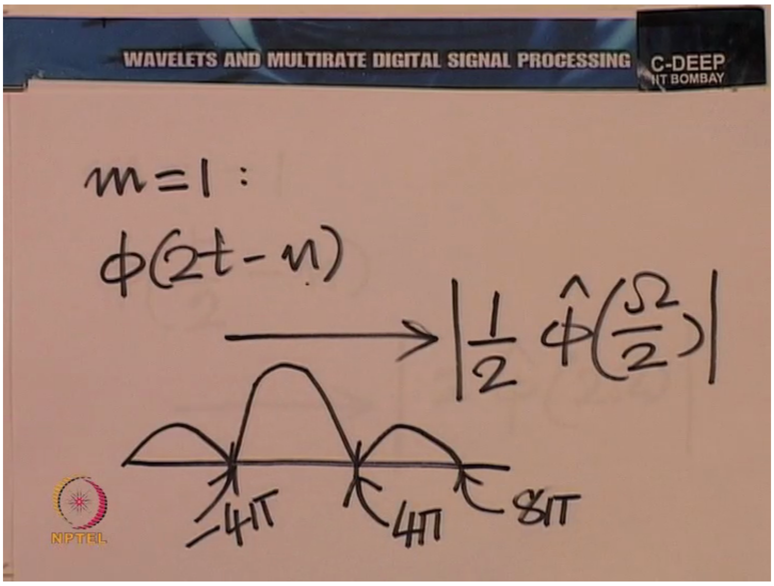

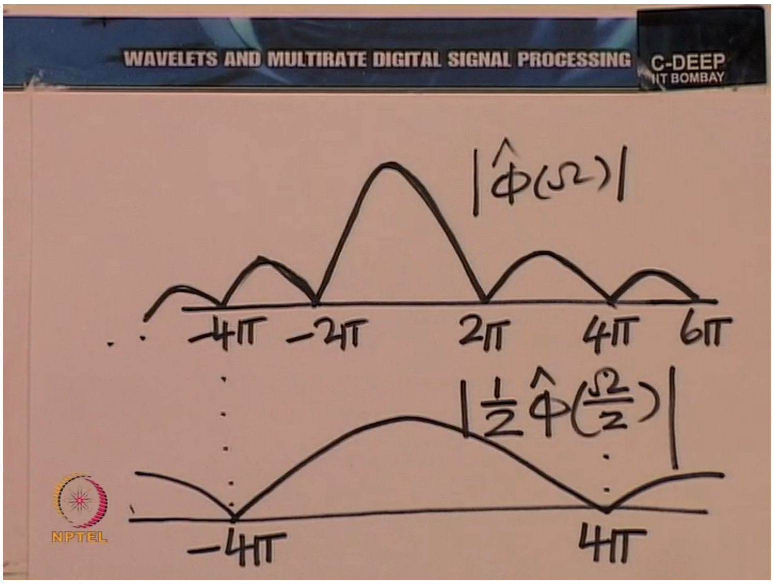

So let me take the m equal to1 case, let me sketch this. So Omega by2, so if I focus my attention on the main lobe and the principal side lobes that I have here. This was originally 2 pie, now it has become 4 pie, this was originally 4 pie, now it has become 8 pie and so too on this case, this is - 4 pie and so on. So you have stretched by factor of 2, let me put them together, the original Fourier transform and this Fourier transform with m equal to1 and so on. I am just sketching mod phi cap Omega and here I sketch mod phi cap omega by2 multiplied by half, of course this should be smooth here, all these are smooth.

So, as expected when we squeeze in time, we have stretched in frequency. And now let us interpret what we are doing when we take the dot product to calculate the coefficients in a Vm. To calculate coefficients in a subspace Vm, what are we effectively doing? We are taking an inner product, namely the inner product of x with phi 2 raised to power m t - n, remember. Now you know this of course normalisation here. So if you want to work with an orthonormal basis, then it should not quite be phi 2 which is the power of m t - n, one must normalise it to make it unit normal.

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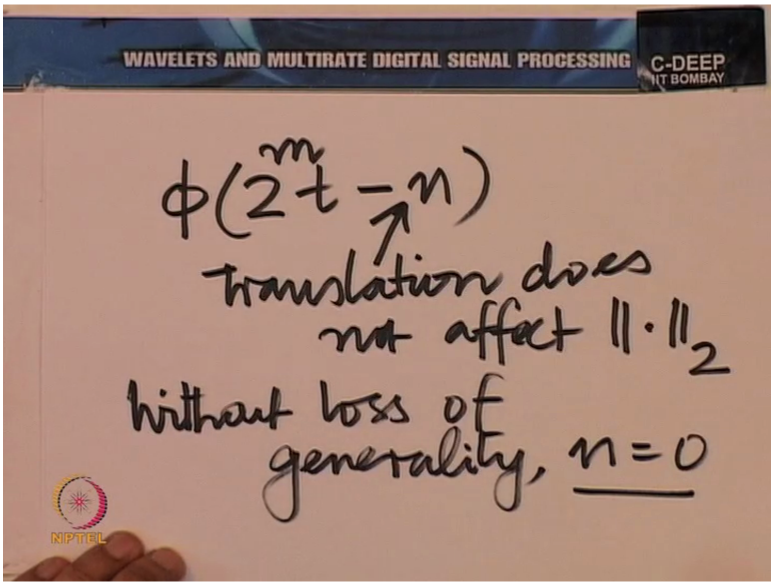

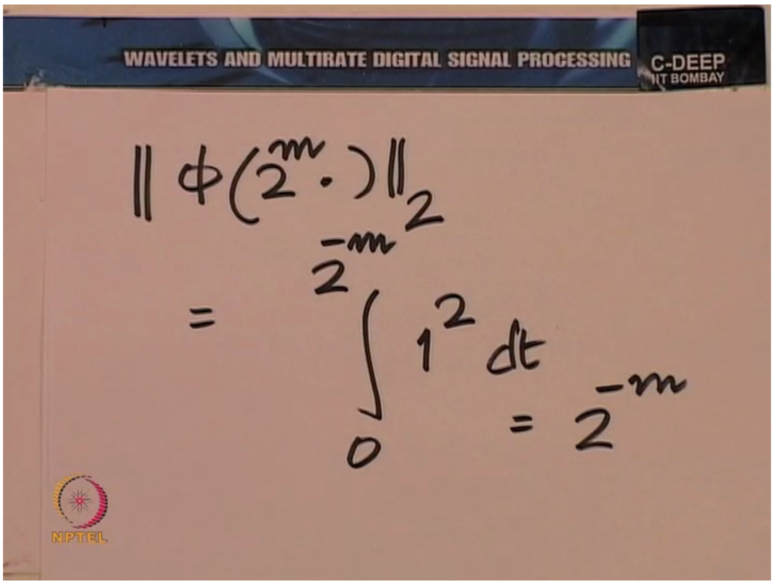

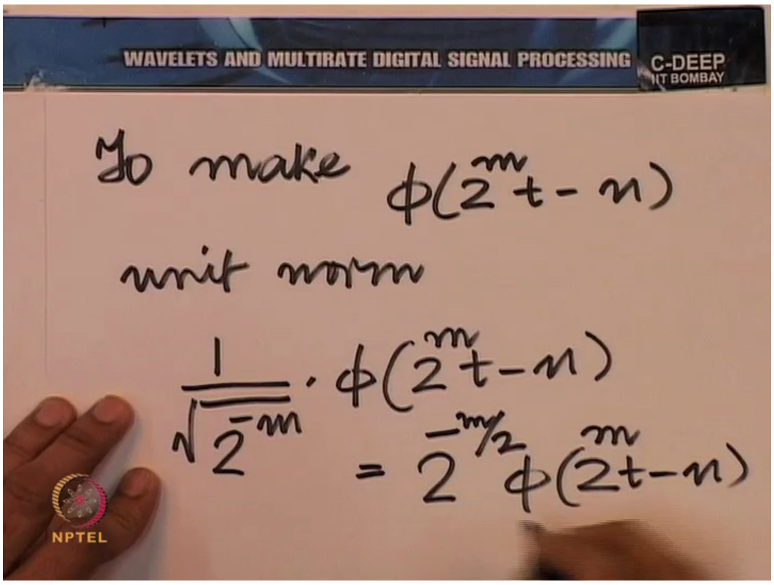

So I will take an instance, let us take any arbitrary m and let us look at the norm. So if we consider phi 2 raised to the power of m t and again the - n does not affect the norm, I am talking about the L2 norm. And therefore we can as well take without loss of generality, we could take n equal to 0. Let us find out the norm therefore, so I am putting 2 raised to the power of m and then a dot, dot denotes the argument of the function but we are treating the function as an entity, so I do not use the explicit argument here. The norm of this is essentially integral, now you know when you go over phi 2 raised to the power of m dot, you are talking about 0 to 2 raised to the power of - m here, 1 square dt and that is obviously 2 raised to the power - m.

And therefore if you want to make this unit norm, then you must divide by, this is of course the square of the norm. So if you want to make this unit norm, you must divide the function by the square root of this. We must consider 1 by square root of 2 raised to the power - m times Phi 2 raised to the power m t - n and that is easily seen to be 2 raised to the power - m by2 phi 2 raised to the power m t - n. So this is the unit norm, this is now an orthonormal basis. So let us make that note.

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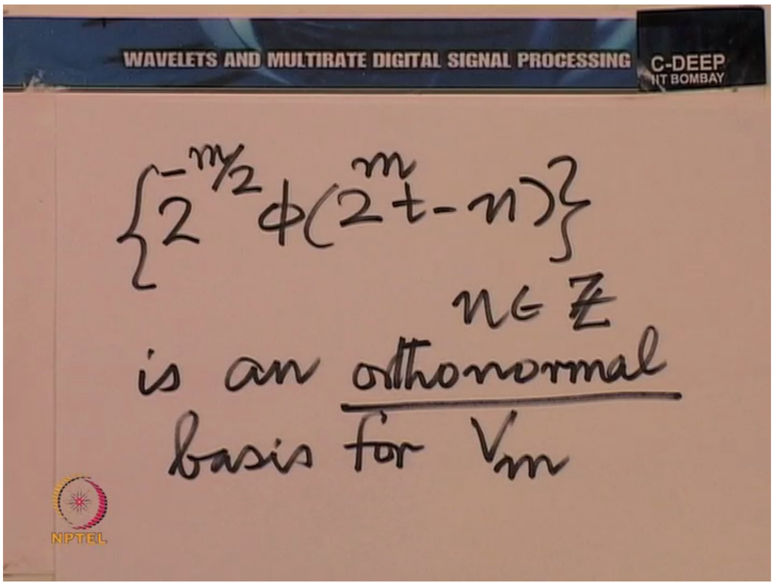

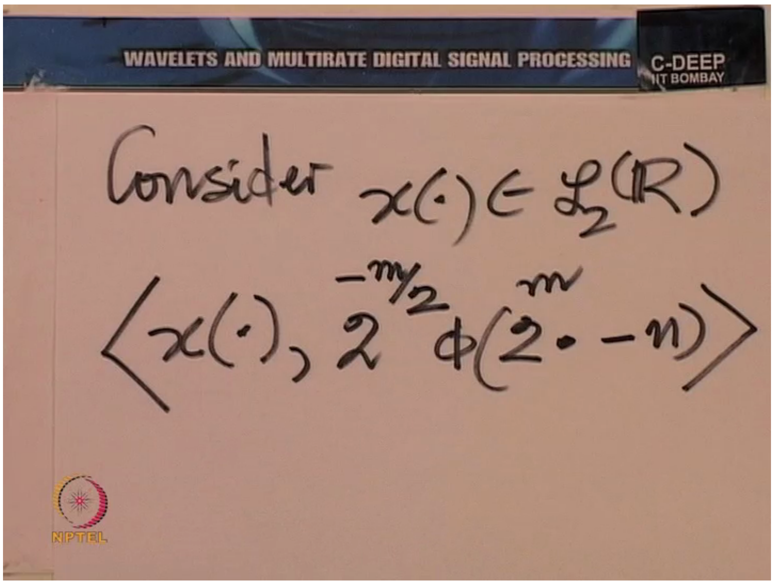

2 raised to the power - m by2 phi 2 raised to the power m t - n for all integer n is an orthonormal basis for Vm, you know what Vm means, Vm is the mth subspace in the ladder of subspaces that leads to L2R as you go rightwards and to the trivial subspace with only the 0 element as you would left words. So it is that mth subspace in the ladder and now we have an orthonormal basis for it. Anyway now let us interpret what happens when we take the dot product of any function in L2R with an element of this orthonormal basis.

So, consider Xt belonging to L2R or if you please X with the argument belonging to L2R, as is the correct way to write it. And then consider the inner product of this X with this orthonormal basis elements, 2 raised to the power - m by2 phi 2 raised to the power m dot - n. Now we are going to invoke the Parseval's theorem. You will remember that we have discussed the Parseval's theorem awhile ago in one of the earlier lectures. When I talked about the relationship of functions and vectors, I had mentioned the significance of Parseval's theorem.

There are different ways of stating it, Parseval's theorem on one hand says that the inner product is preserved as we go from time to frequency. Now here if we use angular frequency, a factor of 2 pie is needed, if we use hertz frequency, that 2 pie factor is not required. But since we are working with angular frequency, it would be safer to retain that factor of 2 pie. But that factor of 2 pie apart, what Parseval's theorem says is that after all when you go from the function to its Fourier transform, in effect you are representing the same function in a different basis.