Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-5.
Lecture-13.1.
Power Complementarity of lowpass filter.

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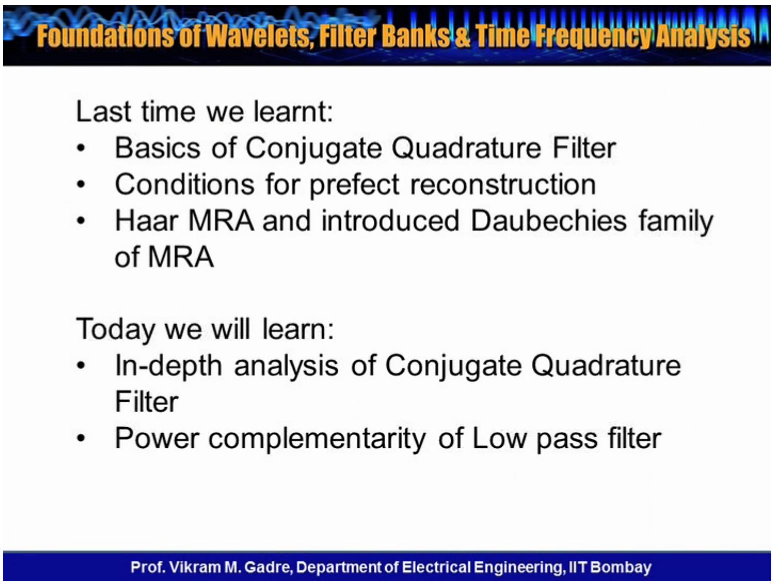

A warm welcome to the lecture on the subject of wavelets and multirate data signal processing. We continue in this lecture to build upon the particular class of filter banks which we had introduced in the previous lecture, namely the conjugate quadrature filter bank. A number of issues related to that filter bank were left unanswered in the previous lecture. To some extent our introduction of the filter banks seemed ad hoc at points.

What I mean by that is, it suddenly made little twists in the nature of the filters where a proper justification had not been given, simply because there was bit of a chicken and egg problem. The justification was best seen after we went through the discussion and that is what I had promised that after we complete an understanding of this filter bank, many things will be a little more clear. So let us embark then upon that filter bank once again, let us look at that conjugate quadrature structures once again, 1st in toto and then in specifics.

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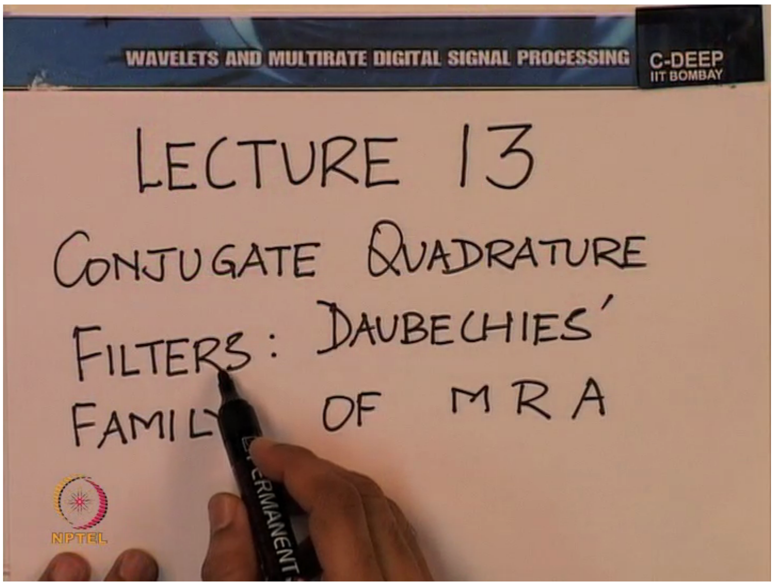

So, in today's theme we shall look at conjugate quadrature filters in depth and then we shall again consider one specific class of those conjugate quadrature filter banks, namely the family of filter banks and family of multiresolution analysis that emerged from Dobash filters. Incidentally as I mentioned, Dobash or you know sometimes it is pronounced as Daubechies, has been a mathematician, scientist, engineer, whatever you want to call her of repute.

Her important contribution in this field has been to propose a family of compactly supported wavelets which also have some other interesting properties. It turns out that the Haar wavelet is a baby of the Dobash family, the simplest of the Dobash wavelets and there are further and further ones of which we shall give an introduction today. In fact the central idea in the Dobash family is to build upon what we had briefly mentioned in the previous lecture, namely the idea of keeping and annihilating polynomials of higher and higher degree on one of the 2 branches of a filter bank.

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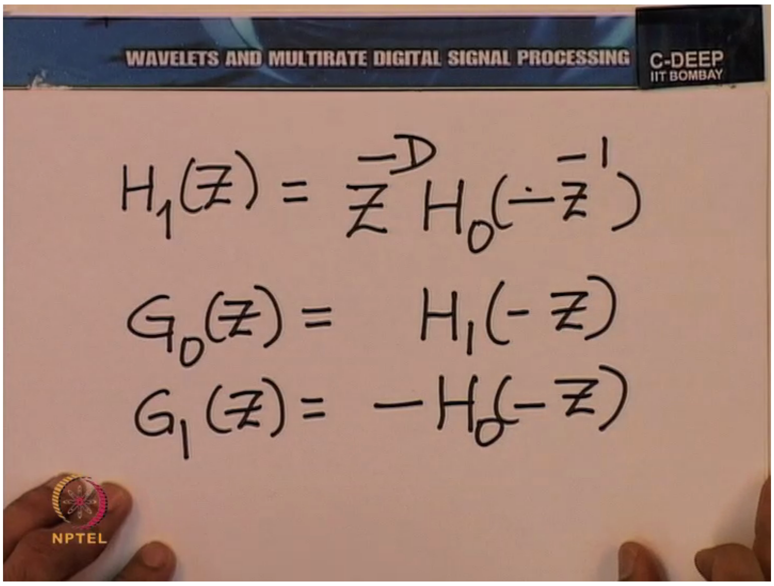

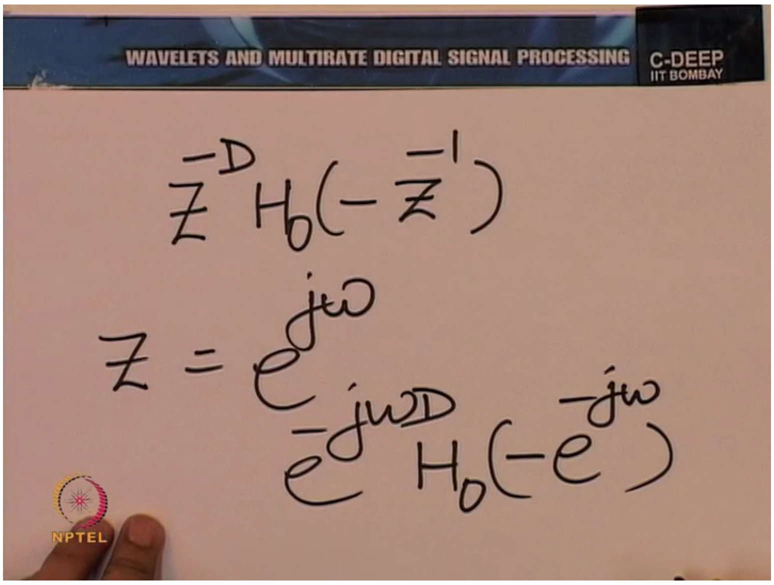

Anyway, we shall look at specifics as we go along but this is to put the lecture in perspective. So we shall talk today about the conjugate quadrature interbank and we shall look specifically at the Dobash family of MRA. Now you see the conjugate quadrature filters structure as we understood it had the following relationship between the filters. So we had the analysis high pass filter was related to the analysis lowpass filter by the following relationships. And we had promised that we shall understand this a little better today.

Of course the synthesis filter was related very easily to the analysis filters, so you had G0 Z being H1 of - Z and G1 Z being - H0 of - Z, this of course was essentially alias cancellation for you, these 2 conditions. But now let us focus on the relationship of H1 to H0. So, 1st let justify why did the hyper filter. Let us consider this expression. Z raised to the power - D H0

- Z inverse. And let us put Z equal to e raised to the power J omega as we do obtain a frequency response. Whereupon we will have e raised to the power - J omega D H0 - e raised to the power - J omega.

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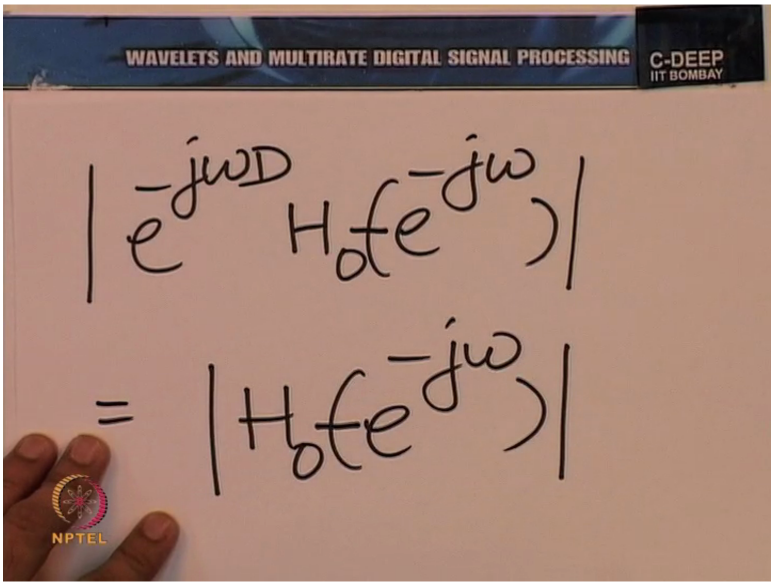

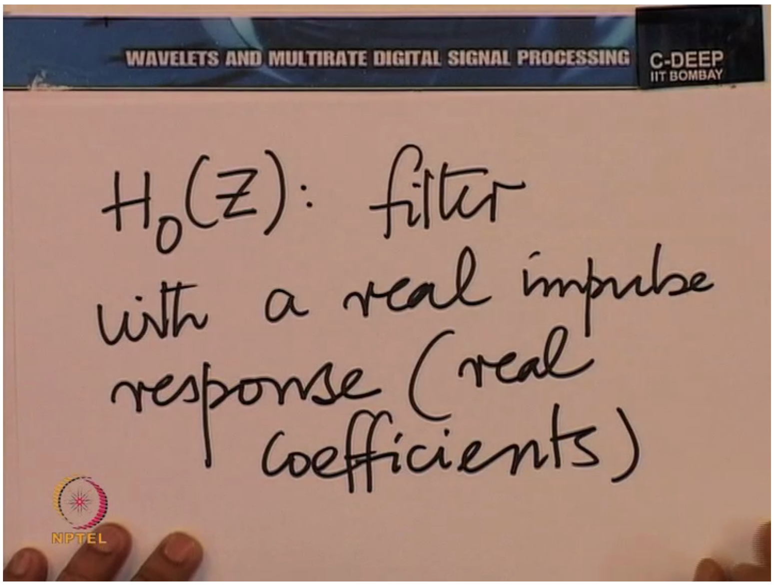

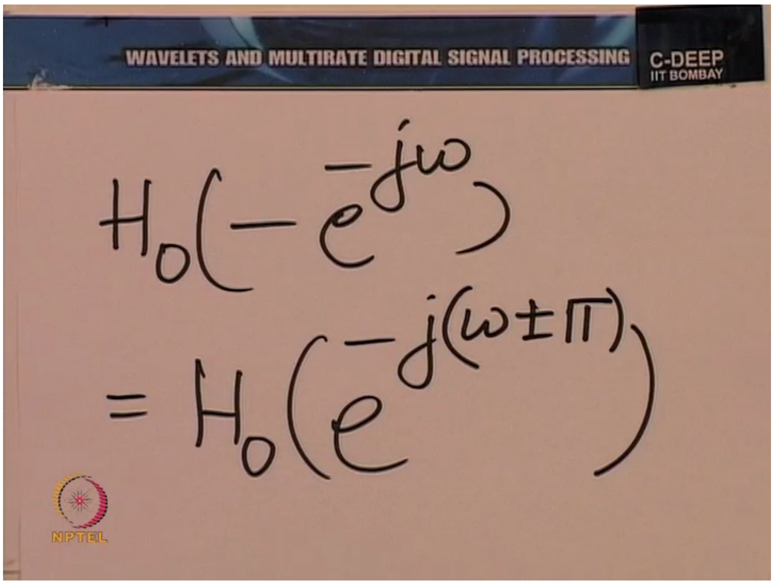

Now if we take the magnitude of this, as is normally what we are interested in, we have the magnitude of e raised to the power - J Omega D H0 e raised to the power - J omega is the same as the magnitude of H0 e raised to the power - J Omega. Well, -, I am sorry, - of this, that is because of the magnitude of this is one. And now let us look at this quantity, the magnitude of a H0 - e raised to the power - J omega.

You see, if H0 is a filter with real coefficients, so if H0 Z corresponds to a filter with a real impulse response and that is the class in which we are most interested. In that case, H0 e raised to the power - J Omega is going to be H0 e raised to the power J Omega complex conjugate. That follows in a straightforward way from some basic properties of the discrete time Fourier transform. What we are saying essentially is the magnitude response of a filter with real impulse response is symmetric in omega and the phase is anti-symmetric.

Now, as we now replace e raised to the power - J Omega - pie - e raised to the power - J omega, what are we really doing? So, H0 - e raised to the power - J omega is essentially H0 e raised to the power - J omega + - pie. We have done this before, we have noted that -1 is essentially e raised to the power + - J pie. And therefore what we have done here is essentially to shift by pie, either forward or backward, it does not make any difference because there is periodicity with a period of 2 pie.

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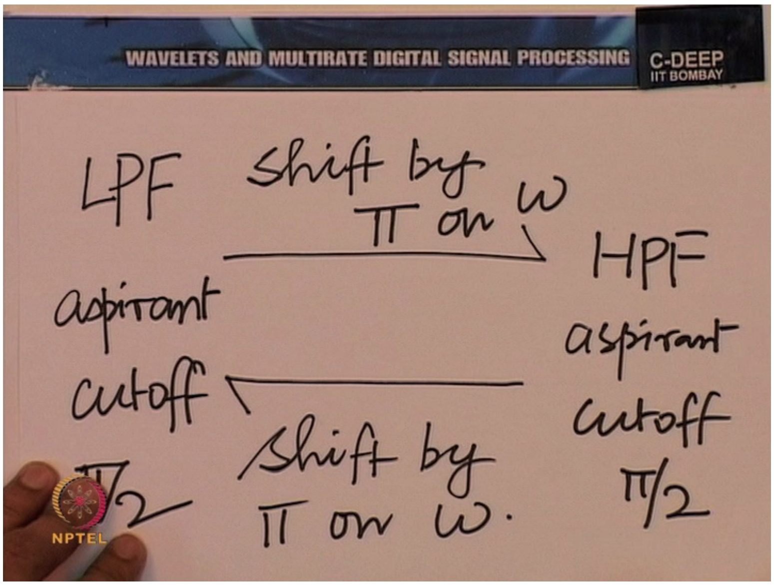

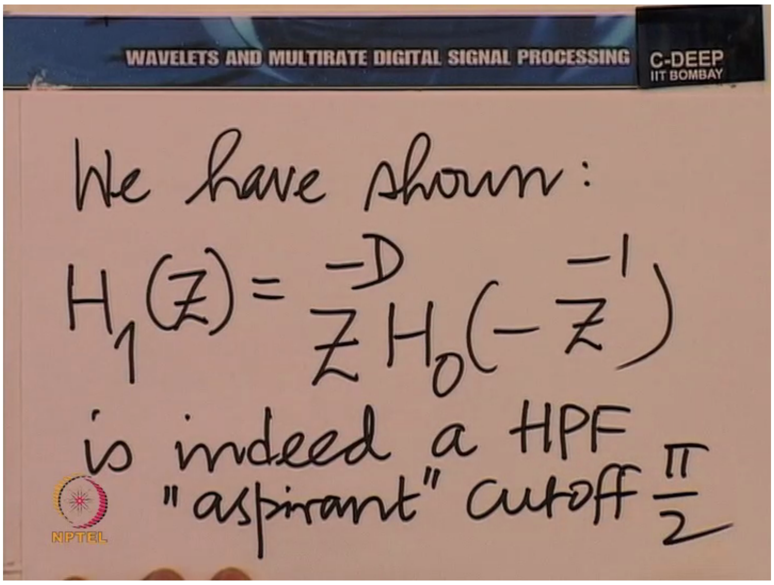

Anyway, what we do know is that a lowpass filter, I think we have seen this quite frequently now, a lowpass filter when shifted by pie on the frequency axis becomes a high pass filter, of course a lowpass filter aspiring to be a lowpass filter with a cut-off of pie by 2. It becomes an aspirant for a high pass filter with a cut-off of pie by 2 again. And similarly when a high pass filters is shifted by pie on the omega axis, it becomes a lowpass aspirant with a cut-off of pie by 2, we have seen this pretty much before.

Anyway, recognising this then we have an interpretation for what we just did. So we said this - essentially shifts by pie and therefore H0 e raised to the power - J omega without the - sign would have been a lowpass filter because of this conjugate symmetry that we have here and now with the introduction of a - sign it becomes a high pass filter. So, we have a convincing argument now that H1 Z, the way we have constructed it, so we have convinced ourselves, we have shown H1 Z, in the way we have constructed it, namely Z raised to the power - D H0 - Z inverse is indeed high pass or a high pass aspirant, aspires to be an ideal high pass filter with cut-off of pie by 2.

What I did of course H0 Z is an aspirant to be a lowpass filter with cut-off pie by 2. So now things are following into place, the only issue is why have we taken this peculiar expression? Not so peculiar really, now we do not see it so peculiar but why the Z inverse and so on? So we will understand that in a minute. You see, I will just give you a trailer for the reason, the trailer is that this automatically brings a condition of the magnitude, we will see that shortly.

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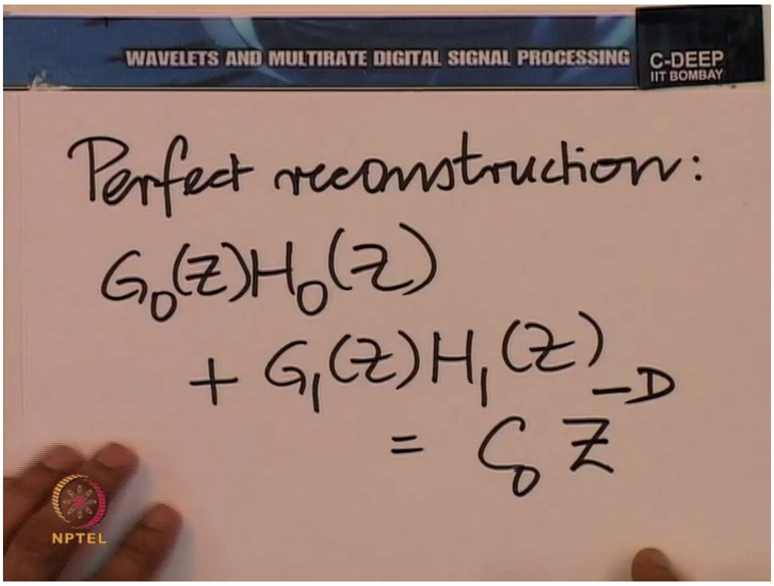

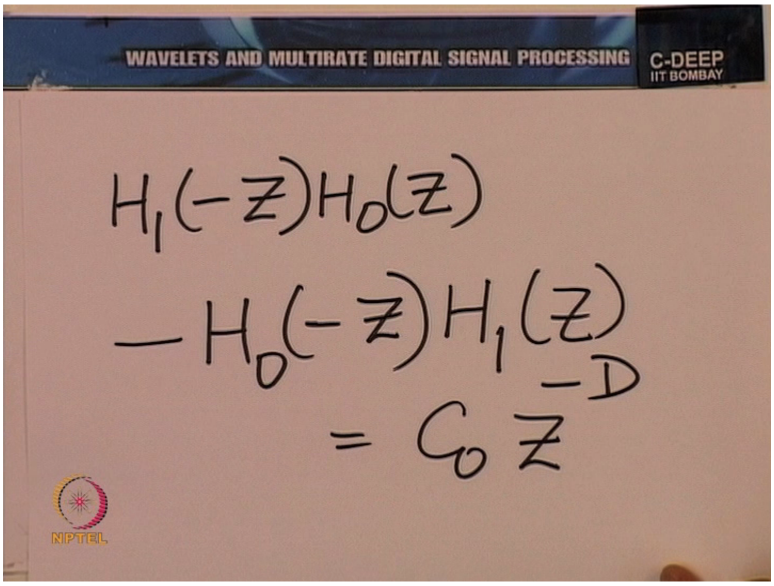

Now let us put down, the alias cancellation condition is anyway put down, we need to put down the perfect reconstruction condition. So let us put down the perfect reconstruction condition, we did that yesterday, we will do it little more carefully. The perfect reconstruction condition is essentially says that you would have G0 Z H0 Z + G1 Z H1 Z must be some constant, we call it C0 times Z raised to the power - D. And what are G0, G1, H0, H1 here?

Now we have agreed that G0Z is essentially H1 of - Z, so we have H1 - Z H0 Z + now G1 Z we had agreed to make - H0 - Z. And H1 Z of course we have agreed to make it Z raised to the power - D and so on, but let me write H1 Z for the mind. And we want this whole thing to be C0 Z raised to the power - D, now we will substitute H1 Z in this equation.

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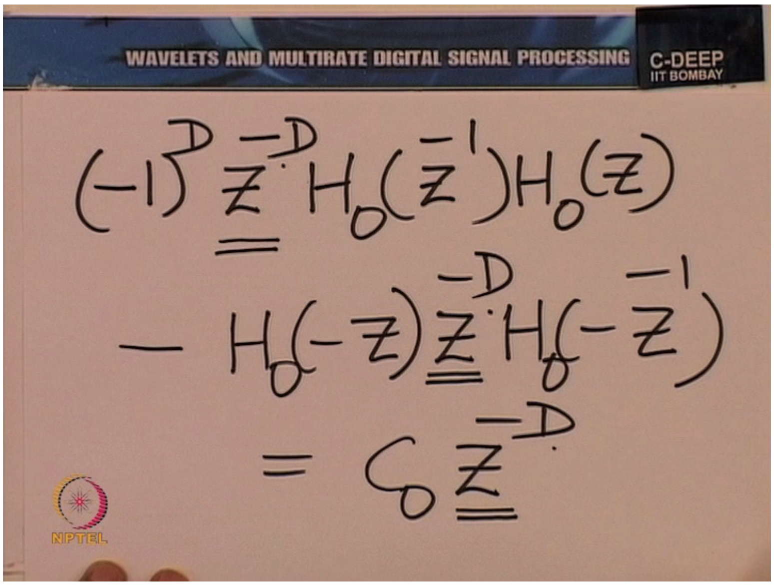

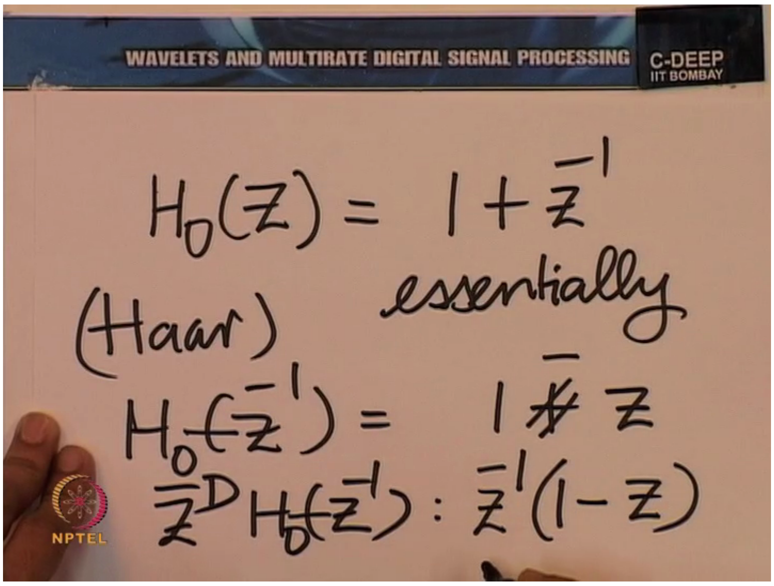

And we have -1 to the power D Z raised the power - D H0 Z inverse times H0 Z -, now again you have H0 - Z here and H1 Z becomes Z raised to the power - D H0 - Z inverse. This you desire should be C0 Z raised to the power - D. Now you know this Z raised to the power - D that we have here, in fact we should not quite have written it like this, so what we have written now happens to be correct.

We should have started by giving a different value for the delay here and the delay on this side. But now again through serendipity or through convenience we can actually make them the same, the purpose of putting this Z raised to the power - D here was actually to take care of this term here. So it is not coincidental that we have written the same D on both sides, that should not have been done initially, we are doing it right away to emphasise that this Z raised to the power - D term that we introduced in H1 was meant to take care of this.

So what we are saying in effect is that we want the rest of it match as well. So what we desire for perfect reconstruction is essentially this -1 raised to D H0 Z, H0 Z inverse - H0 - Z H0 - Z inverse is a constant. Now again we have the freedom to choose the value of capital D here. Again the main issue is whether capital D is odd or even. If capital D is odd, then we have - in both places, for both the terms, if it is even, then this is a + and this is a -.

Let us choose capital D odd and in fact again there is a reason for that, it is not arbitrary. You know, we have just looked at the Haar filter bank where we have a filter of even length, length 2 filter actually, all of them, 1 + Z inverse, 1 - Z inverse on both sides are of length 2. When we replace Z by Z inverse, so let us take the Haar case once again, you have H0 Z of the form 1+ Z inverse, whatever, forget the by 2 here, in the Haar case H0 Z inverse would have been 1+ Z and the Z to the power - D H0 Z inverse, or if you choose, you can write - Z inverse as we do and this would then become - here.

Z raised to the power - D H0 - Z inverse is actually intended to make this causal, this filter is noncausal. So you need to introduce a Z raised to the power - 1 here to make this causal and therefore D becomes one in this case. You see the role of D, I have hinted at this yesterday. You see, we said that the reason why we cannot avoid the delay is because you want the filters to be causal, now you see what we mean.

This Z raised a power - D term has been put there to retain causality. And you just put as much of a D as is needed to allow for causality and so here the D required is 1. Now in the Dobash family, we keep augmenting the filter length by 2 in every rung of the family ladder.

So when we go from the baby of the family, namely the Haar MRA to the next member of the family, we augment the length by 2, so we have a length of 4. When we go to a length of 6, it gives us the 3rd member and so on, length in the 4th number and so on and so forth.

So, successive even lengths of filters give us successive members of the family in the Dobash family. Now what we are going to do is slowly move towards building the 2nd member of the Dobash family. And therefore the next case would be capital D equal to 3, so you would have length of 4 and they would have a maximum power of Z equal to 3, Z cube, when you write this H0 - Z inverse. That is the role of Z raised to the power - D here, all right.

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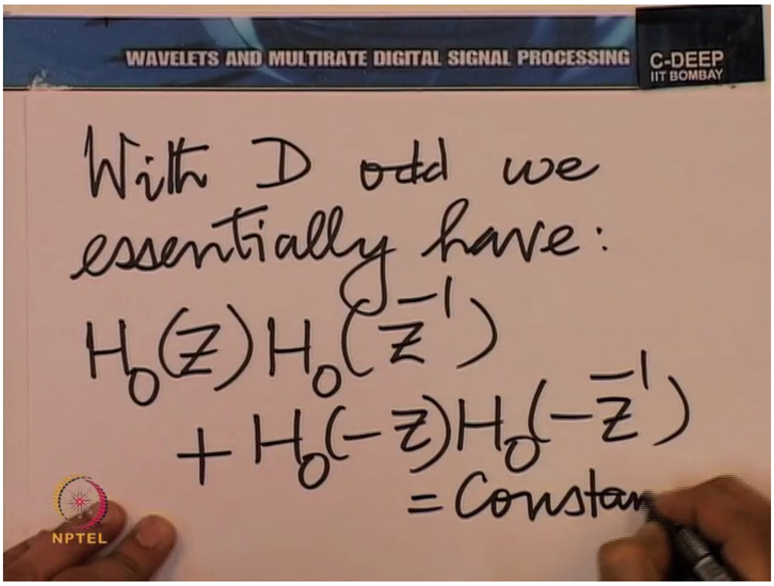

Therefore it is justified for us to begin by assuming that C is odd. So let me put that down once again for you. In this relationship that we have here, we shall now assume D to be odd. With D odd, we essentially have for perfect reconstruction H0 Z H0 Z inverse + H0 - Z H0 - Z inverse is a constant. Let me explain, you see when D is odd, then both of these are - sign so you can take away the - sign from the left-hand side and put it on the right and this is anyway a constant, so negative of a constant is also a constant, so there we are.

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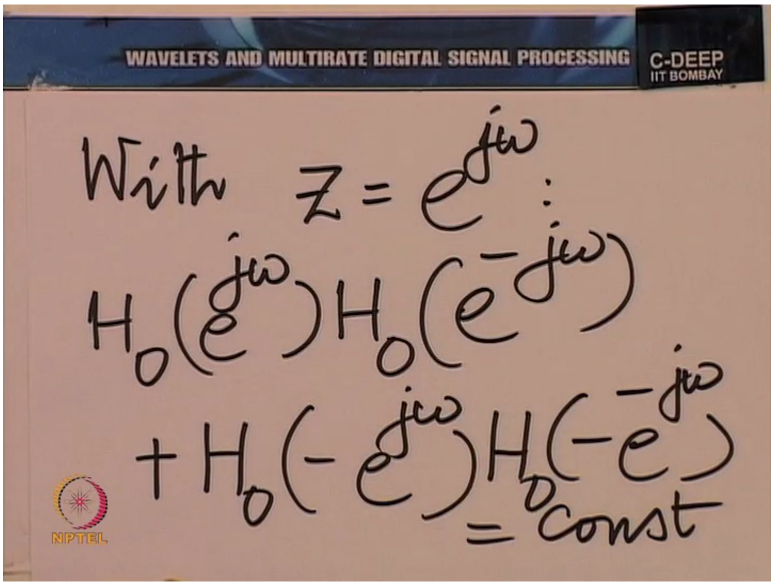

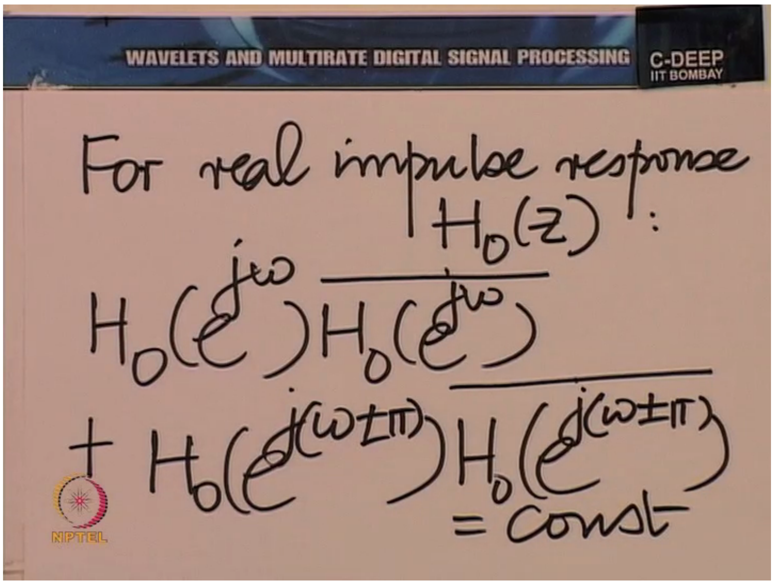

Now what does this mean, we need to reflect on this little. We 1st reflect on it in the frequency domain. So when we put Z equal to e raised to the power - J Omega, what do we have here? H0 Z, or rather H0 e raised to the power - J Omega times H0 e raised to the power - J Omega + H0 - e raised power J Omega H0 - e raised to the power - J Omega is a constant. Now, once again we shall remove the - sign here and shift omega by pie.

And we shall also note that if you have a filter with a real impulse response, then H0 e raised to the power - J Omega is essentially the complex conjugate of H0 e raised power J Omega. The same goes here, when you have omega replaced by - omega here, you again get a complex conjugate of this. So all in all for real filters, we have H0 e raised to the power J omega H0 e raised to the power J Omega complex conjugate + H0 e raised to the power J

Omega + pie, + - if you please, H0 e raised to the power J Omega + - pie complex conjugate is a constant.

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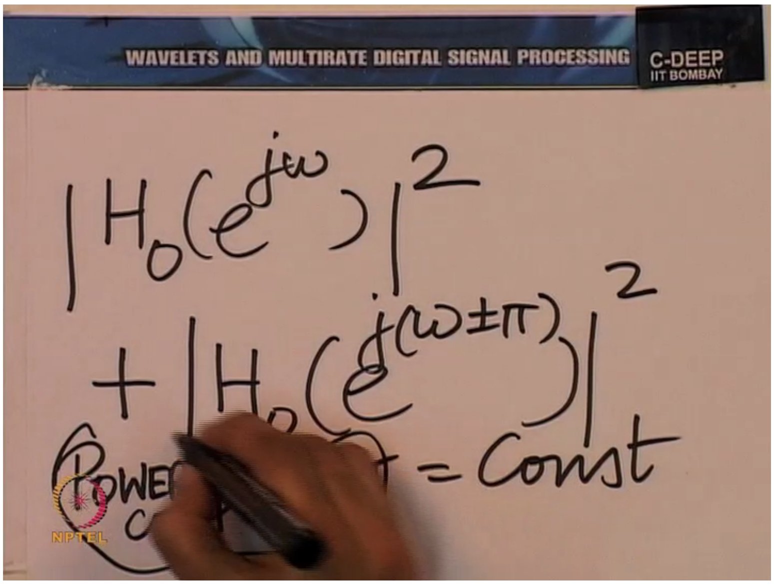

Now we have a very beautiful conclusion here. You see this is the magnitude square, and this is again the magnitude square, so there we are. What we are saying in effect is mod H0 e raised to the power J Omega square + models H0 e raised to the power J Omega + - pie the whole square is a constant. Now, this is very interesting, this is exactly one of the properties that we had introduced in the context of the Haar system. Namely the property of what is called power complementarity.

Here it is clear now that by this construction we have achieved power complementarity in the high pass and lowpass filters of the analysis side and in fact it is a simple consequence that if we look at the synthesis side, they are also power complimentary. In fact I leave it to you as an exercise by using the relationship between G0, G1 and H0 to show that the synthesis side is also power complimentary.

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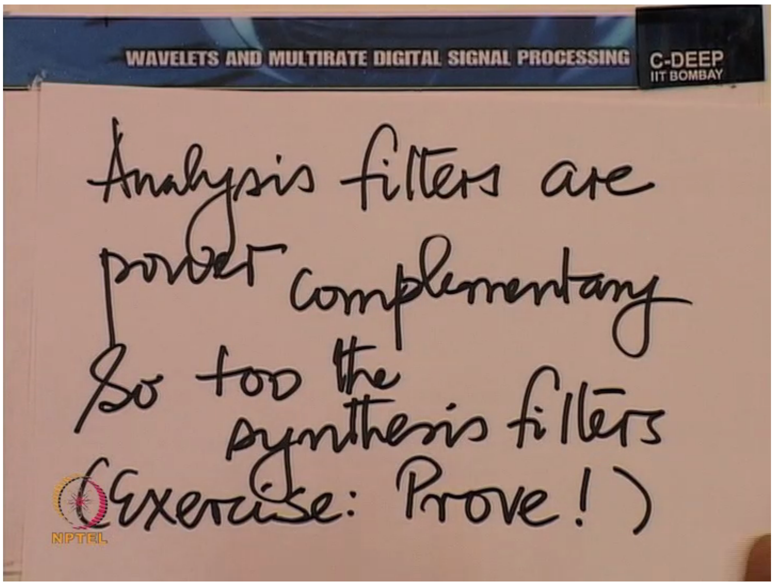

So what do we have here, it is very interesting, the analysis filters are power complimentary and so too the synthesis filters. So as I said, exercise, show this, we have already proved it more or less, it is just a little bit of as they say totting your Is and crossing your Ts you need to write down neat proof but I think that is a good thing to do, we must leave a couple of exercises for the class to do. And this is a very simple exercise with which we begin. Use the discussion that we just had over the last couple of minutes to work out the details to show that the analysis filters and the synthesis filters are both apower complimentary pair.