Foundations of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-1.
Lecture -2.2.
Dilates and Translates of Haar Wavelets.

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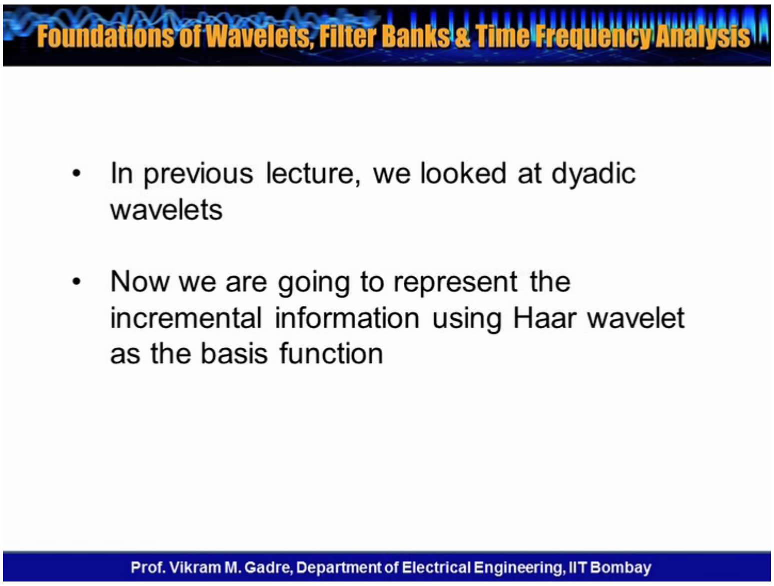

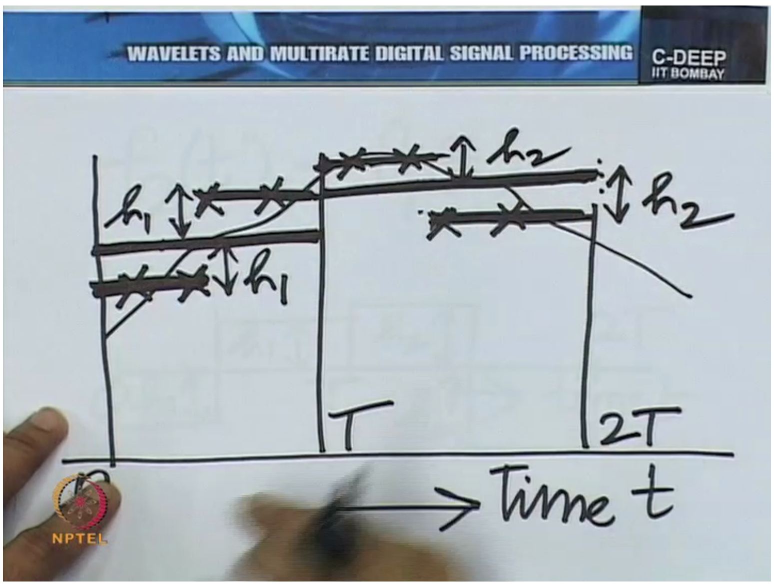

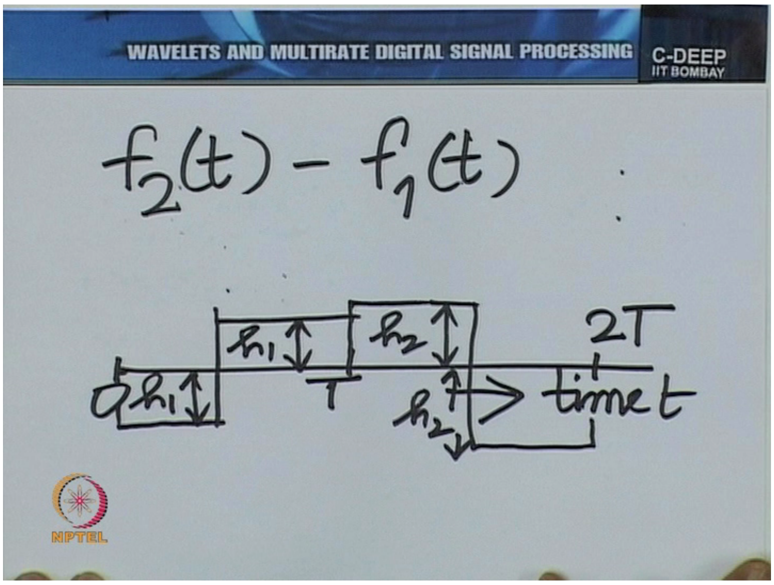

So now let us see how f2 t - f1 t would look. It is very easy to see that f2 t - f1 t has an appearance like this. Let me flash them before you f2 t and f1 t, just for a second here so that you get a feel. This is f2 and this is f1 and visualize subtracting this from this, what would you get, a function which looks something like this. So I have the time axis here, so if I mark the intervals of size T there, something like this, maybe this has height H1 and this has height H2.

Let me mark H1 and H2 on this diagram too, so this is H1 and this is H2, of course, so is this. Simple enough, now if we look carefully, we can construct all of this by using just one function and what is that function?

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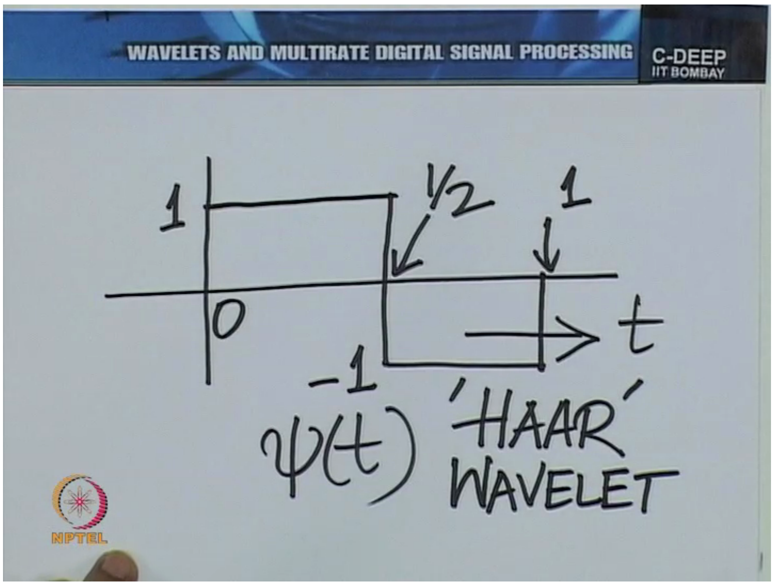

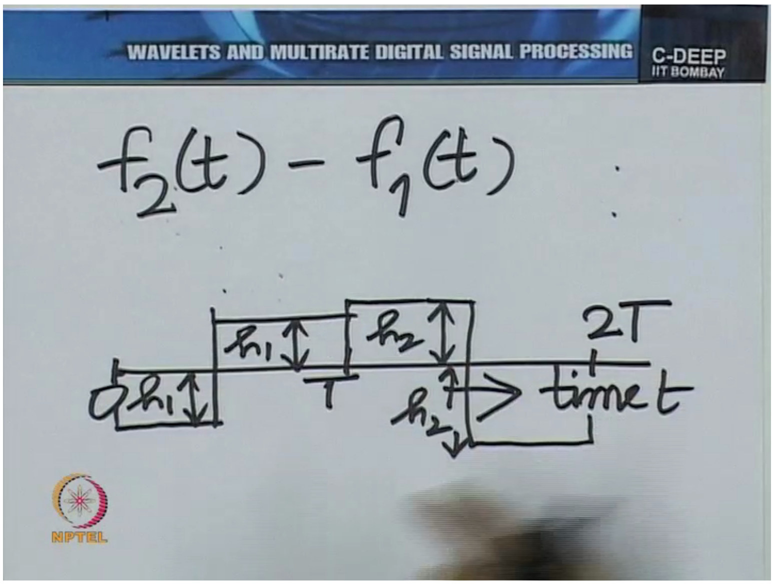

Suppose I were to visualize a function like this 1 over the interval of 0 to one half and -1 over the next half interval, this is a point half, this is a point 1, point 0, 1 here and -1. And let us give this function a name, let us call it psi of t. In fact this is indeed what is called the Haar wavelet, Haar again the name of the mathematician. It is very easy to see that using this function I can construct any such f2 t - f1 t. Indeed, if I were to take this function, stretch it or compress, whatever might be the case depending on the value of capital T, dilate, dilate is the more general word.

So if I were to dilate this function to occupy an interval of T and bring it to this particular interval of T, so I dilate that function psi t and bring it to this interval of T. And then I multiply psi t so dilated by the constant H1, of course H1 should be, is an algebraic constant, it should be given a sign. Here for example H1 should be given a negative value because we started psi t with a positive -1 here. Similarly H2 has a positive value here. So in other words, this segment of f2 t - f1 t is of the following form.

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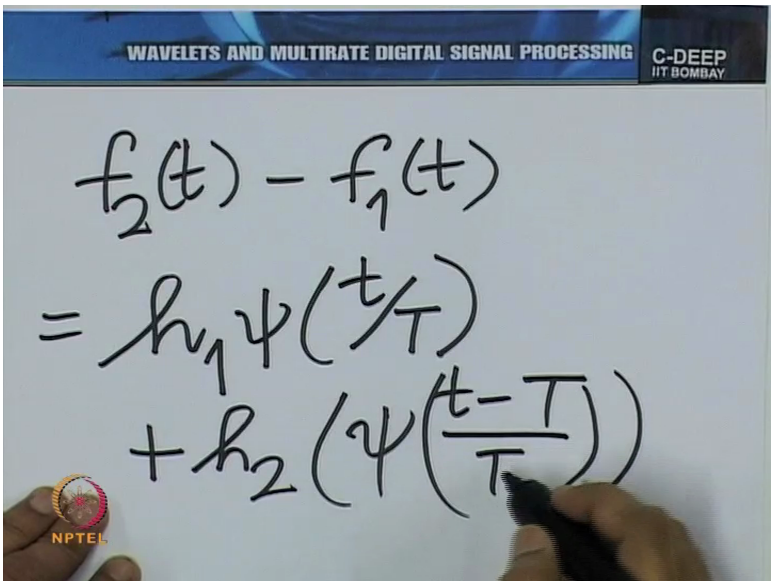

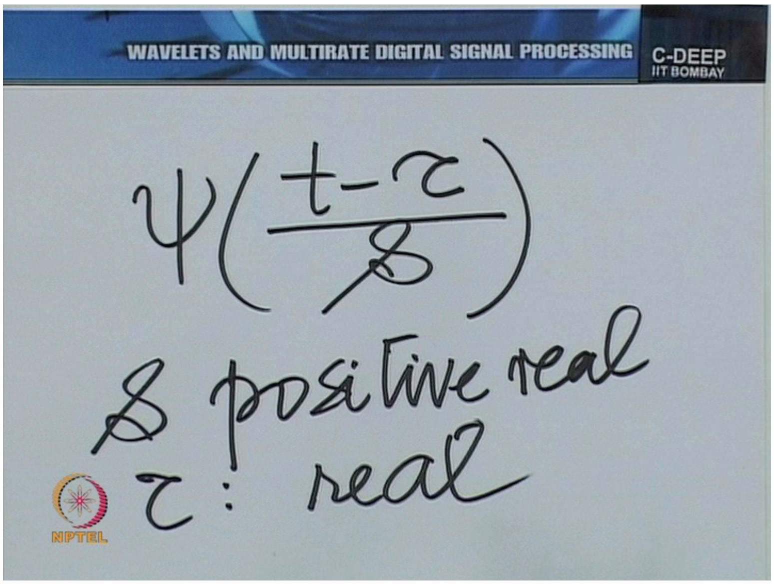

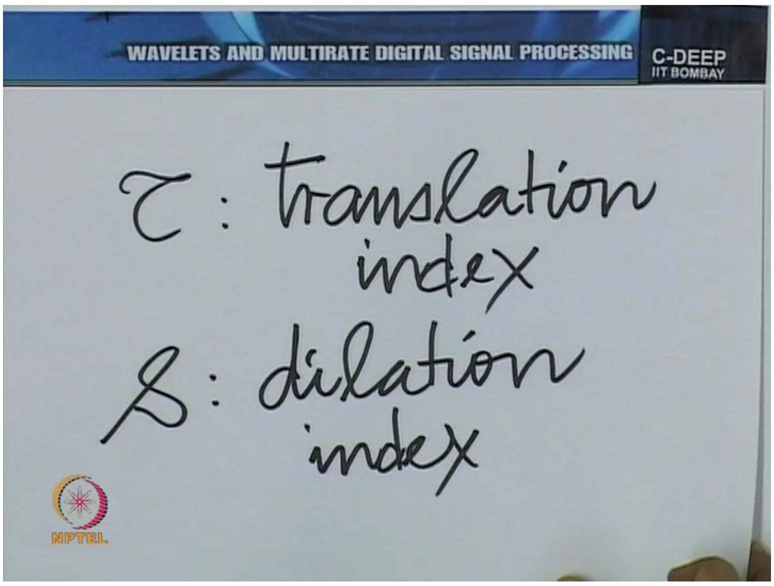

Some H1 times psi of t dilated by T + H2 times psi of t - T dilated by T. So here this is both dilated and translated, all right. In other words, in general, when we start from the function psi of t, we are constructing functions of the form psi t - tao by S, where of course S is a positive real number and tao is real. This is a general function that we are using as a building block, different values of tao and different values of S. Of course here, at a particular resolution, at a particular level of detail, the value of S is only one.

For example when we are representing the function on intervals of size T, we take S equal to T. If we were to represent the function on intervals of size T by 2, then S would become T by 2 and so on. Then, what we are doing in effect is dilating and translating, now we introduce those terms. Tao is called a translation index or translation variant and S is called the dilation index or dilation variant and we are dilating and translating or we are constructing dilates and translates of a basic function.

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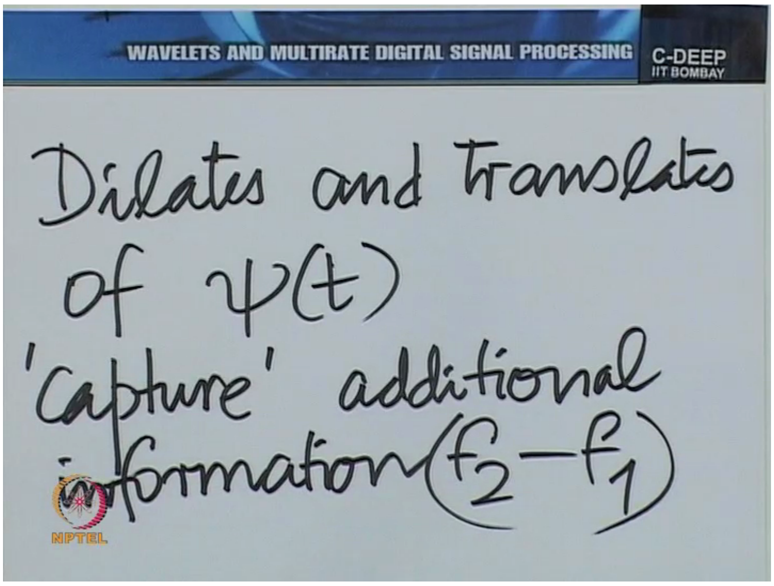

Dilates and translates of psi t capture the additional information in f2 t - f1 t. now let us spend a minute in reflecting about why this is so important. What have we done so far, just looks like very simple function analysis or just a very simple transformation or algebra of functions? What is so striking in what we have just said? What is striking is that what we have done to go from T to T by 2 can also be done to go from T by 2 to T by 4.

Not only that, what we have done to go from T to T by 2, in other words for intervals of length T to intervals of length T by 2, all over the time axis can be done all over the time axis to go from intervals of size T by 2 to intervals of size T by 4. And then you could go from intervals of size T by 4 to intervals of size T by 8, T by 16, T by 32, T by 64 and what have you to as small an interval as you desire. each time what you add in terms of information is going to get captured by these dilates and translates of the single function psi t.

A very serious statement if we think about it deeply enough. That one single function psi t allows you to bring in resolutions step-by-step to any level of detail. In fact, in formal language in functional analysis, we would put it something like this. You know in mathematics, in these arguments of limits on continuity and so on or in some of these proofs related to convergence, there is this notion of this adversary and the defendant. So here, the defendant is trying to show, the one makes the proposition is trying to show that by this process you can go arbitrary close to a continuous function, as close as you desire.

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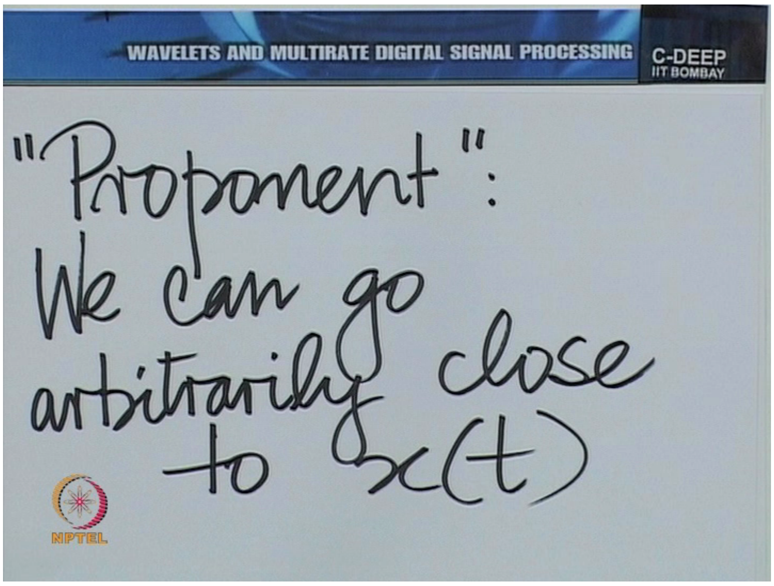

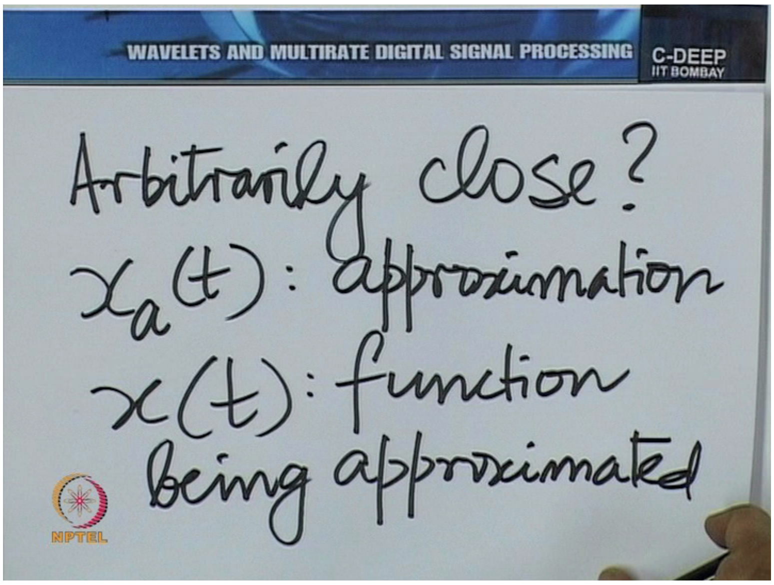

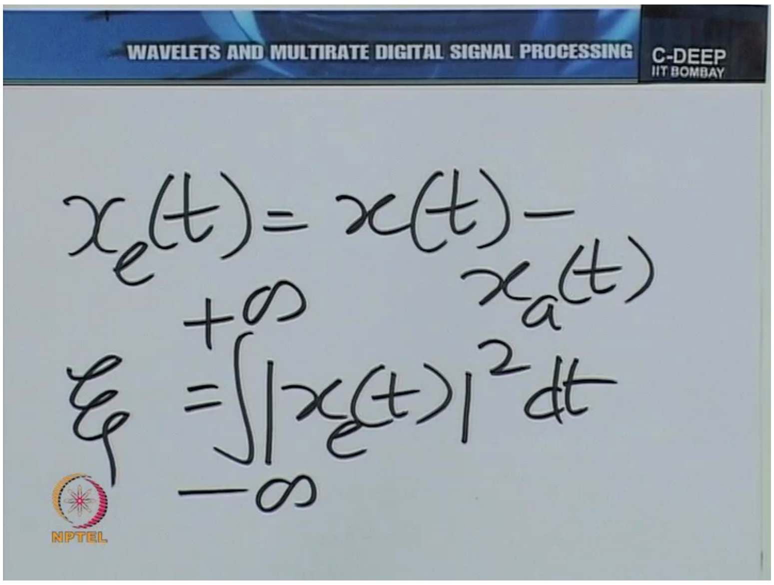

Now as close in what sense? Well it could be in terms of what is called the mean square error or the square error. So let us formulate that adversary-proponent kind of argument here. So what we are saying is, the proponent says we can go arbitrarily close to xt, to a continuous function xt by this mechanism. Arbitrarily close in what sense? In the sense if xa is the approximation, approximation at a particular resolution and if xt is the original function, then if we take what is called the square error, so we look at xe t, that is xt - x a t and integrate x e t the whole square, the modulus whole square actually over all t, we call this the squared error, script e.

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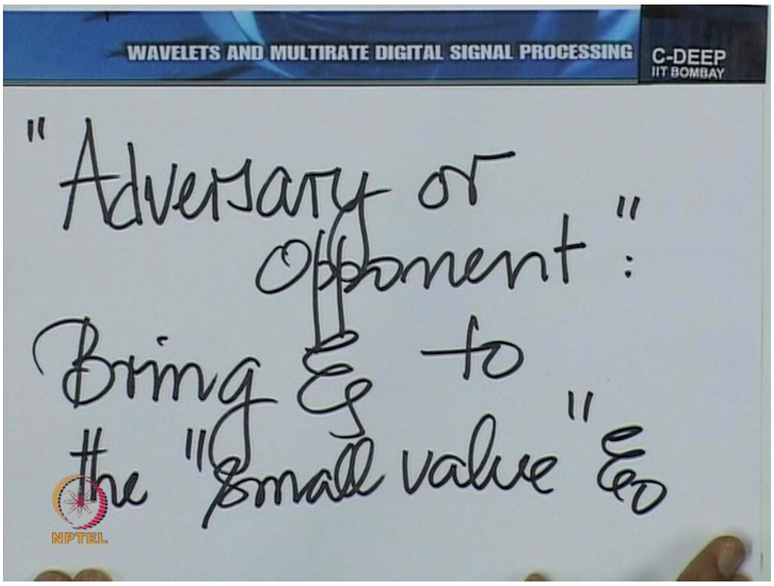

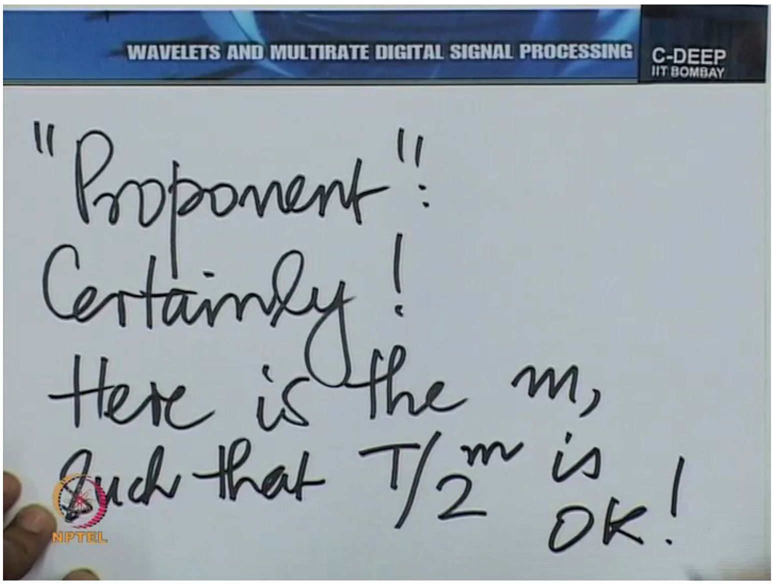

Then adversary or opponent says bring e to the small value, let us say e0 and the proponent says certainly here is the m such that T by 2 raised to the power of m is okay. So that is the idea of proponent and opponent here. The adversary or the opponent gives you a target. He says I want the square error to be less than this number e0 and the proponent says, well here you are, if you make that that interval of size T by 2 to the power m, lo and behold, your error is going to be less than or equal to e0.

And what is striking in this whole discussion is, no matter how small we make that e0, the proponent is always able to come out with an error, such that T by 2 raised to the power of m, I mean piecewise constant approximation for intervals of size T to the power of 30 by 2 raised to the power of n would give you an approximation close enough for that small e0. We need to spend a minute or 2 to reflect on this, it is a serious thing we are saying. In fact let us for a moment think on how this is dual to the idea of representation of a function in terms of its Fourier series for example.

In the Fourier series presentation, what do we do, we say give me a periodic function, or for that matter give me a function on certain interval of time, let us say an interval of T, size T. If I simply periodically extend that function, that means I take this basic function on the interval of T, I repeat it on every such interval of T translated from the original interval. So suppose that original interval is 0 to T, then repeat whatever is between 0 and T, between T and 2T, between - T and 0, between - 2T and - T, between 2T and 3T and go on doing this, so you have a periodic function.

Decompose that periodic function into its Fourier series presentation, so what am I doing in effect, I have a sum of sinusoids, sine waves, all of whose frequencies are multiples of the fundamental frequency. What is that fundamental frequency? In angular frequency terms, it is 2 pie by T some in hertz terms it is 1 by T. So in hertz terms, you have sine waves with frequencies which are all multiples of 1 by T.

And an appropriate set of amplitudes and phrases assigned to these different sinusoidal components with frequencies of multiples of 1 by T when added together would go arbitrarily close to the original function, of course the original periodic function on the entire real axis or for that matter specifically on the interval from 0 to T if you restrict yourself to the function from where you started.

So not only does the Fourier series allow you to represent by using the tool of continuous functions, analytic functions, remember we talked about sine waves in the previous lecture. Sine waves are the most continuous in some sense, the smoothest function that you can think of. The derivative of a sine wave is a sine wave, integral of a sign wave is a sine wave, when you add 2 sine waves of the same frequency, they give you back a sine wave of the same frequency.

So sine waves are the smoothest function that you could deal with and even if you had somewhat discontinuous function on the interval from 0 to T and if you use this mechanism of Fourier series decomposition, you would lined up expressing a discontinuous function in terms of extremely smooth analytic functions. What would you be doing in the Haar approach that we discussed a few minutes ago? Exactly the dual.

Even if you had this continuous audio pattern, you would decompose it into highly discontinuous functions which are piecewise constants. Constant on intervals of size T in the resolution T, on intervals of size T by2 at the resolution T by 2 and so on and so forth. now this tells in the Fourier series presentation you have this proponent-opponent kind of argument, that is, for a given class of functions, even if they are discontinuous, even if they have a lot of non-analytic points and so on, for a reasonably wide class of functions.

Remember in the Fourier series, that wide class of functions is captured by what are, by what are called the Dirichlet conditions, now I would not go into those details here. But there are certain kinds of conditions, very very mild conditions which a function needs to obey before it can be decomposed into the Fourier series or in other words before the Fourier series can do this job of representing that discontinuous function in terms of highly continuous and analytic smooth functions.

So similar set of conditions thus exist even for the Haar case, I mean, if one really wishes to be finicky, one does need to restrict oneself to a certain subclass of functions, where again that restriction is not really serious in most physical situations. For the time being in this course, we may even just ignore that restriction.