Foundations of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-4
Lecture -12.2.
Applying Perfect Reconstruction and Alias Cancellation on Haar MRA.

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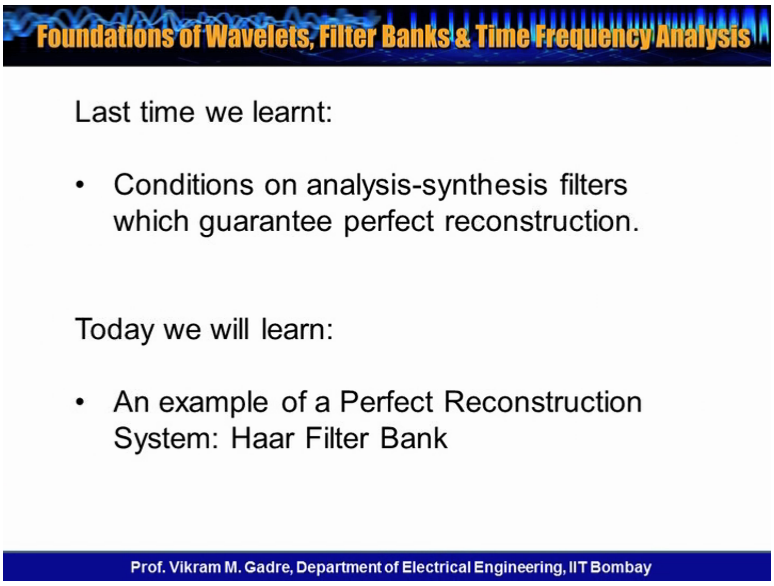
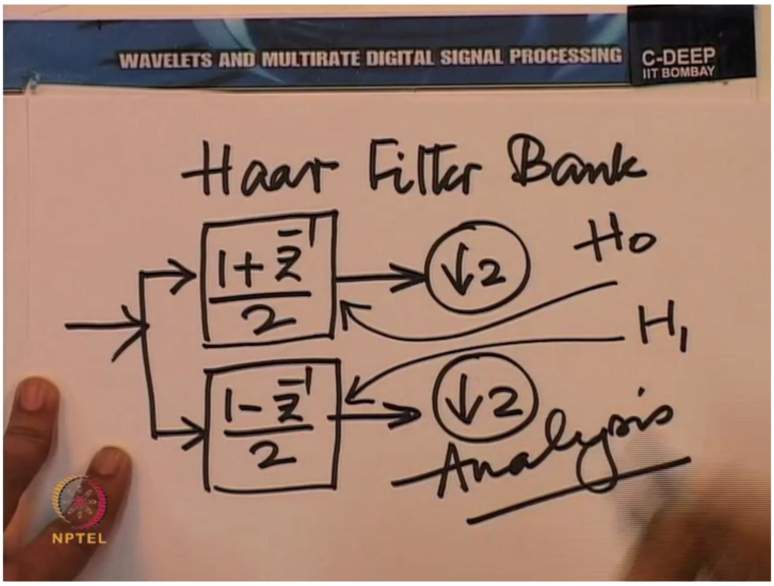
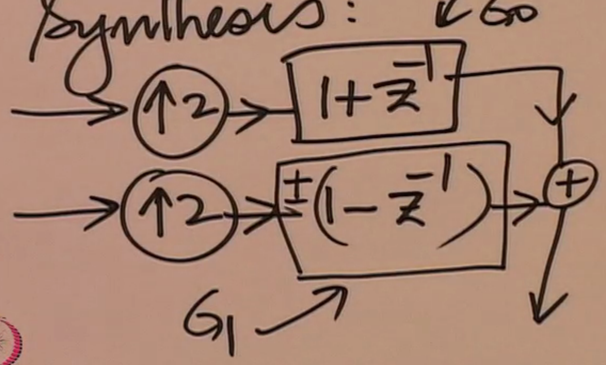




Synthesis:



WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING C-DEEP IIT BOMBAY

$$\tau_1(z) = \frac{1}{2}\left\{ G_0(z)H_0(-z) + G_1(z)H_1(-z) \right\}$$

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$$\text{RHS} = \frac{1}{2}\left\{ (1+z^{-1})\left(\frac{1-z^{-1}}{2}\right) \pm (1-z^{-1})\left(\frac{1+z^{-1}}{2}\right) \right\}$$

We want $= 0$

So in the Haar case we had the following filters, analysis, so this is H0 and this is H1. Synthesis, you know, remember on the synthesis side, at that time we had said we will allow for a + - ambiguity here, let us keep that ambiguity and you will see why that ambiguity is needed, this is G0 and this is G1. now let us write down Tao 1Z here. now 1Z by definition is of course G0Z H0 - Z + G1 Z H1 - Z and with our definitions of G0, H0, G1 and H1 we have the right-hand side becoming simply half G0 is 1 + Z inverse, H0 - Z is 1 - Z inverse by2.

Now here we have a + - ambiguity. G1 Z is of course 1 - Z inverse and H1 - Z is 1 + Z inverse by2. You know when you look at this carefully, you will notice why we want this ambiguity there. We want this to become 0 and therefore it is obvious that the - sign should be chosen, the + sign will not give us 0.

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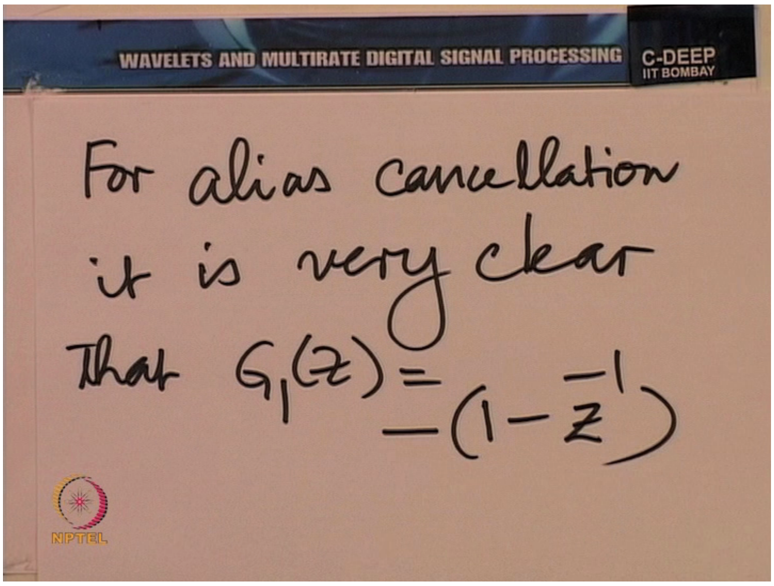

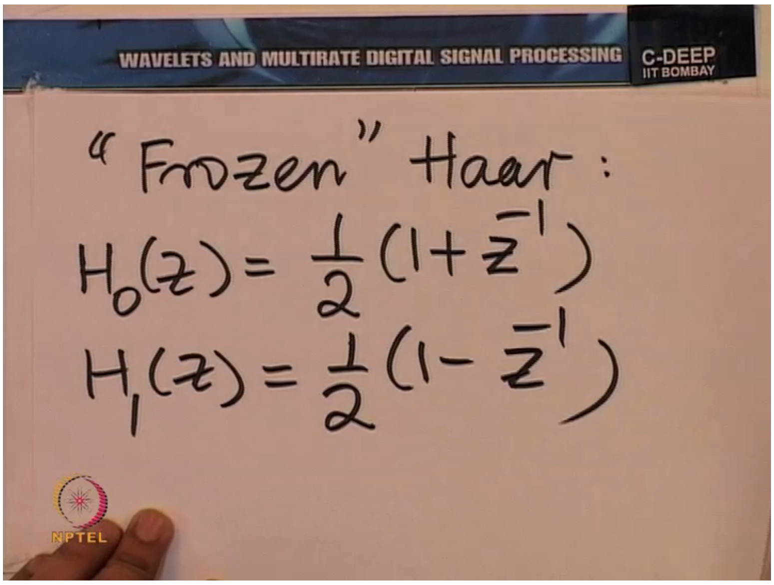

$$G_0(z) = (1 + z^{-1})$$

$$G_1(z) = -(1 - z^{-1})$$

Verify $T_0(z)$:

WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING C-DEEP IIT BOMBAY

$$T_0(z) =$$

$$\frac{1}{2}\left\{ G_0(z) H_0(z) + G_1(z) H_1(z) \right\}$$

$$= \frac{1}{2}\left\{ \frac{(1+z^{-1})^2}{2} - \frac{(1-z^{-1})^2}{2} \right\}$$

WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING C-DEEP IIT BOMBAY

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \left\{ (1+z^{-1})^2 - (1-z^{-1})^2 \right\}$$

$$= \frac{1}{4} \cdot (1 + z^{-1} + 1 - z^{-1}) (2z^{-1}) \} =$$

So for Alias Cancellation, it is very clear that G1 Z must be equal to - 1 - Z inverse and not +. So therefore now let us freeze our G0, G1, H0 and H1 for the Haar case. And now let us verify the perfect reconstruction condition or verify Tao 0Z. Indeed Tao 0Z is obviously G0Z H0Z + G1 Z H1 Z and when we expand this we get 1 + Z inverse the whole square by 2 - 1 - Z inverse the whole square by 2 and this is easy to evaluate, it essentially gives us half into half and we can use the A + B into A - B kind of expression and then we have 1 by 4 A + B is 1+ Z inverse + 1 - Z inverse and A - B is 1 + Z inverse -1 + Z inverse, so 2 Z inverse.

And here of course Z inverse cancels and here we have Z inverse surviving and all in all this is equal to 1. Simple and elegant, in fact it is 1 but with a factor of Z inverse. So C0 is equal to 1 and you have a Z inverse there. The Tao 0Z all in all is Z inverse. What does this mean essentially?

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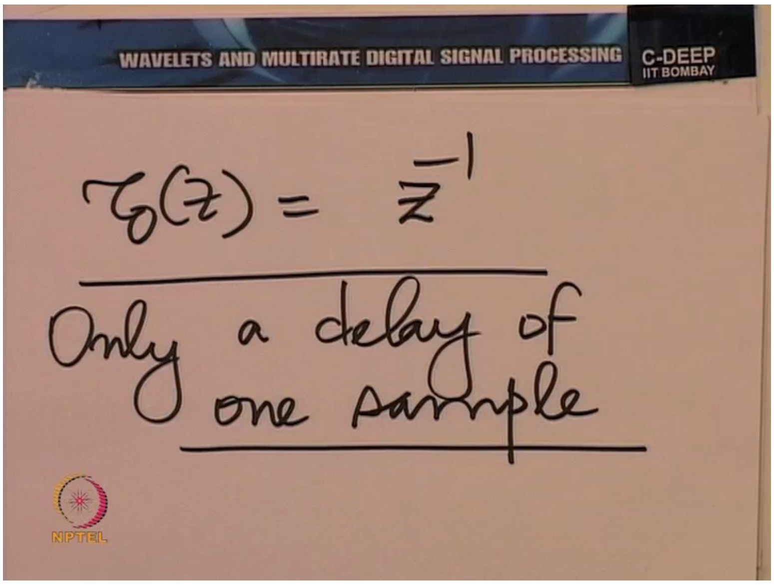

Only a delay of one sample. The constants have already been accommodated, so C0 becomes 1. Now why was this delay required? As I said this delay is required on account of causality. If we did not want this delay to be there, we would need noncausality either on the analysis or on the synthesis side. So for example if I do not want this is Z inverse term, I must multiplied the output by Z, in other words I must shift the output backward by one sample. That means G0 and G1 would now become noncausal filters.

Wherever causality is not an issue, so for example suppose you are dealing with spatial data, then this is not a problem, we can get Tao 0Z exactly equal to1 without the Z inverse term. But where causality is an issue, as it is when you are dealing with time data, then we can do this. Now in fact in this case, let us also dissect the situation and understand a little better. Let us put down the condition that we have written for Alias Cancellation in this specific and simplest case and see if it holds here.

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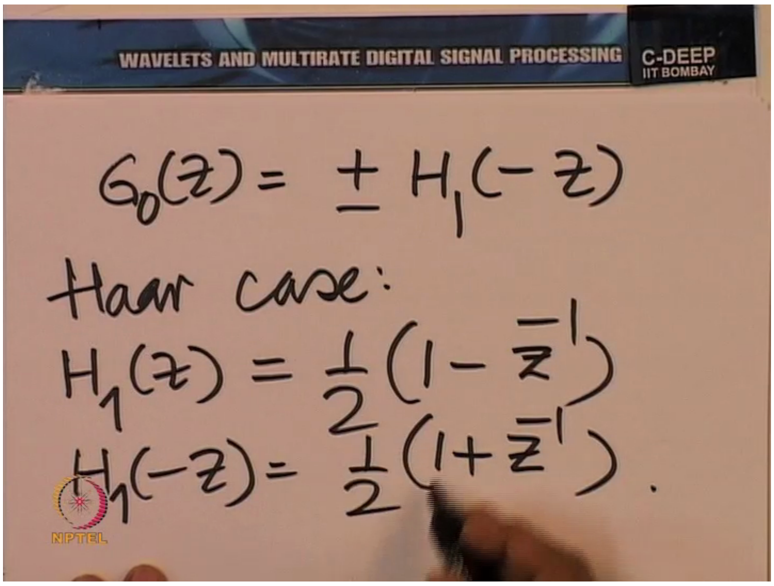

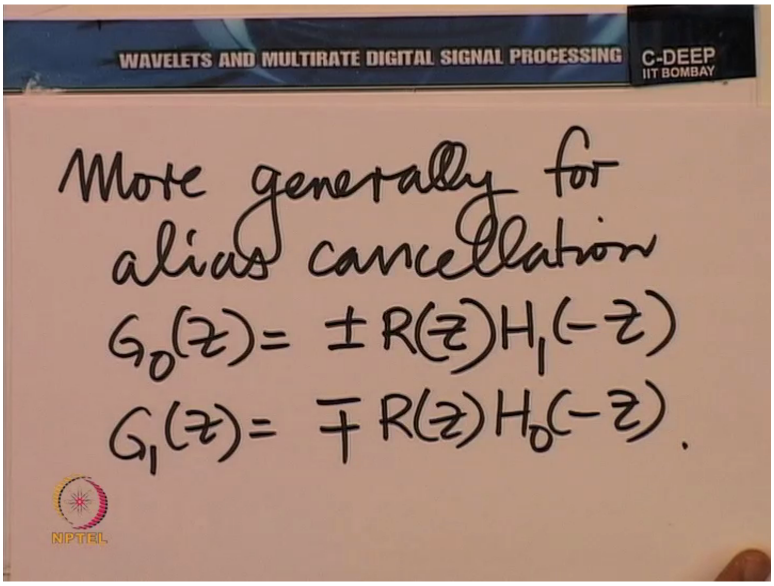

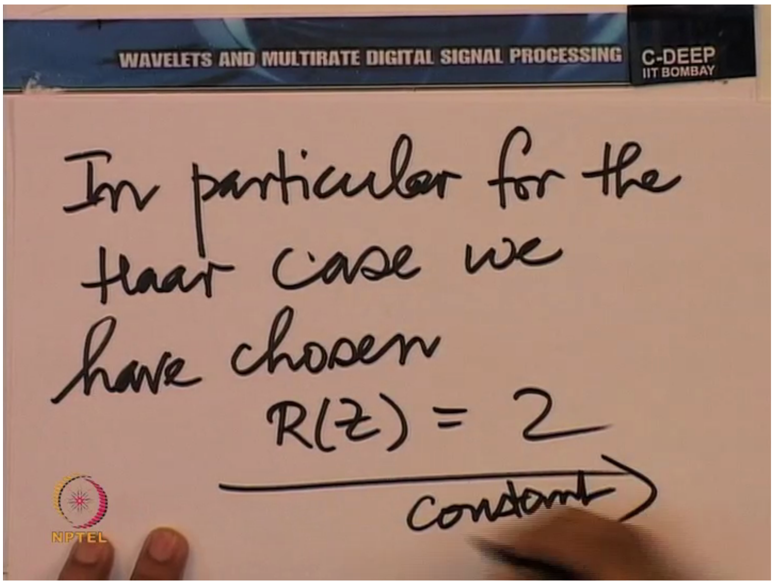

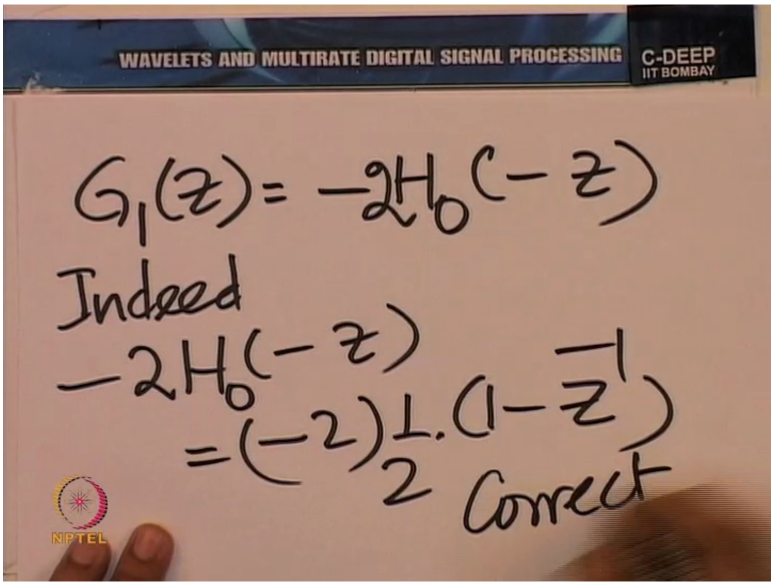

So indeed we had suggested that the simplest possibility of Alias cancellation is when G0Z is either + - H1 of - Z and in the Haar case, we have H1 Z is essentially half 1 + 1 - Z inverse rather and therefore H1 of - Z would be half 1 Z inverse. So you notice that G0H Z is indeed + H1 of - Z but without this factor of half. So you know that factor of half is not an issue at all, you remember that more generally we have written down the following requirements.

We had said that more generally for Alias Cancellation, we need G0Z to be + or - some RZ times H1 - Z and G1 Z to be correspondingly - + RZ times H0 - Z. And in particular you could choose RZ to be a constant. So in particular for the Haar case, we have chosen RZ to be equal to 2, a constant. And in fact I can also check for the 2nd expression. G1 Z in the Haar case should then be - H0 of - Z or rather with a factor of 2, so 2 times. And indeed - 2 times

H0 - Z is - 2 times half into 1 - Z inverse, which is correct. So things have all fallen into place, it is convenient.

I once again point out how beautifully one can understand several concepts at once when one takes the specific example of the Haar. The Haar MRA embeds in it several concepts explained in a simple way. But of course we cannot be content with the Haar and we shall slowly understand why. The 1st step in understanding this is to understand where the Haar is the baby and when we need to grow further. Why is the Haar just the beginning of a family of multiresolution analysis, in what sense is it the simplest case? Towards that objective, let us look at that lowpass filter and that high pass filter from a slightly different perspective.

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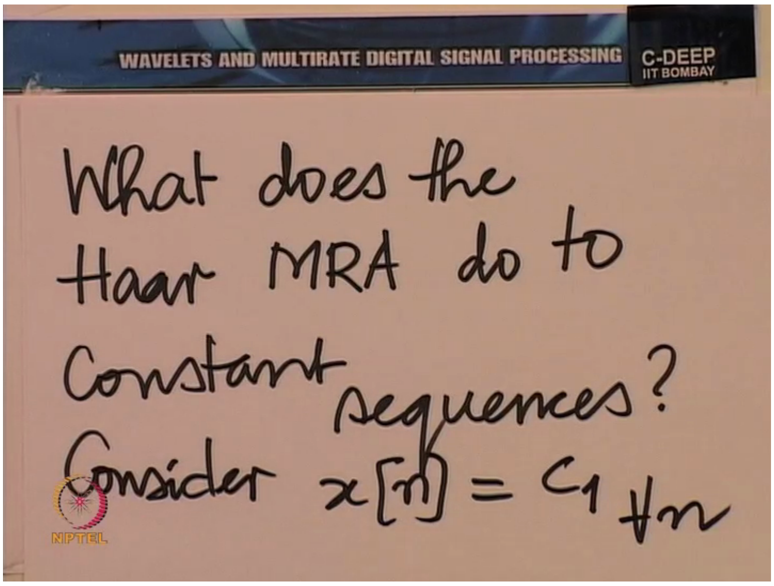

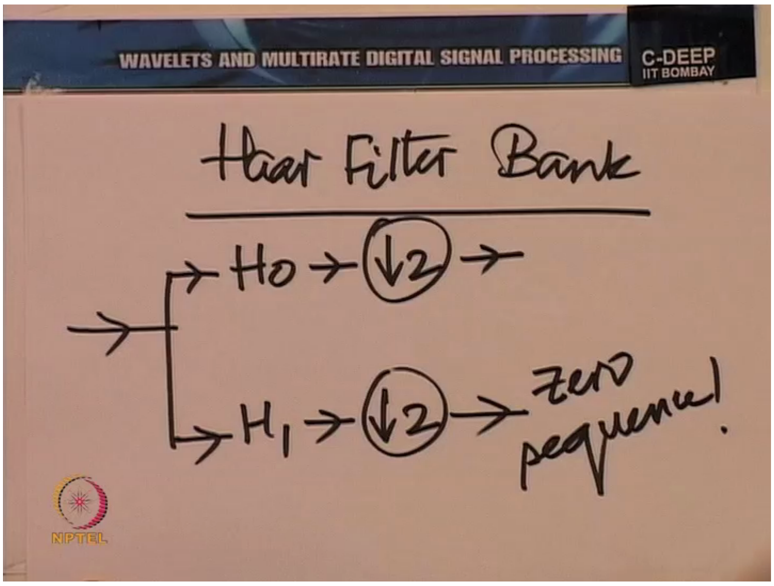

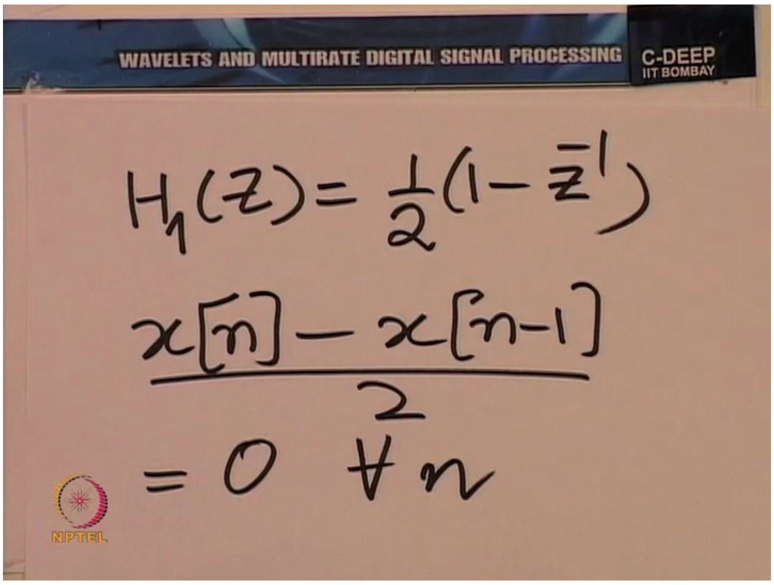

What does it do to a certain class of sequences, let us see that. So let us put the following question. What does the Haar do to constant sequences? In other words, consider XN equal to some constant, say C1 for all n, the extreme case. How would the outputs of the various points in the Haar filter bank look? So it is very easy to see that if you take the Haar MRA, I would keep writing the filters again, I will just show them symbolically. I have H0 here, I have H1 there and if I take just the analysis side, it is very easy to verify that the output here is going to be a 0 sequence.

In fact I will take just a minute and verify it. Essentially H1 Z operates 1 - Z inverse. And this essentially means the operation X n - X n - 1 by 2 which is identically 0 for all n. So this is a very significant observation we are making. We are saying on the Haar filter bank, if there is any constant component in the input, it is destroyed on the high pass branch. This is a slightly different way of looking at the Haar filter bank. In fact now we will go one step further. In the Haar filter bank I had one term of the form 1 - Z inverse.

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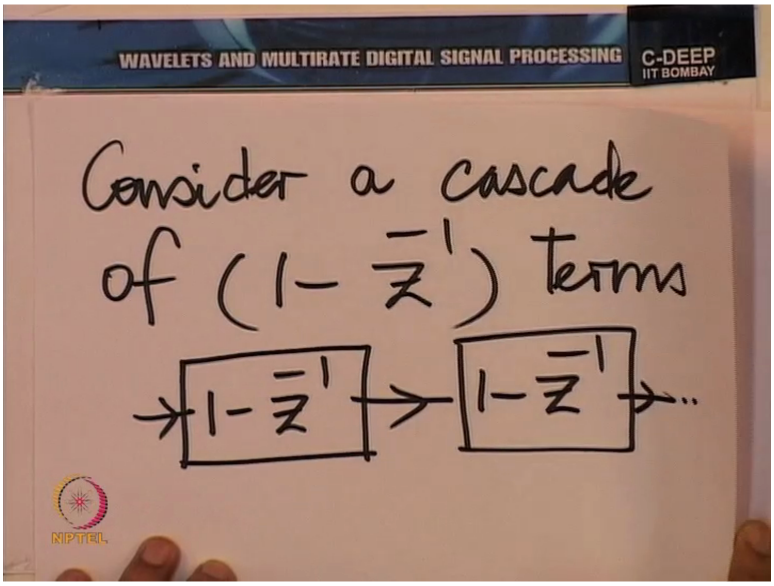

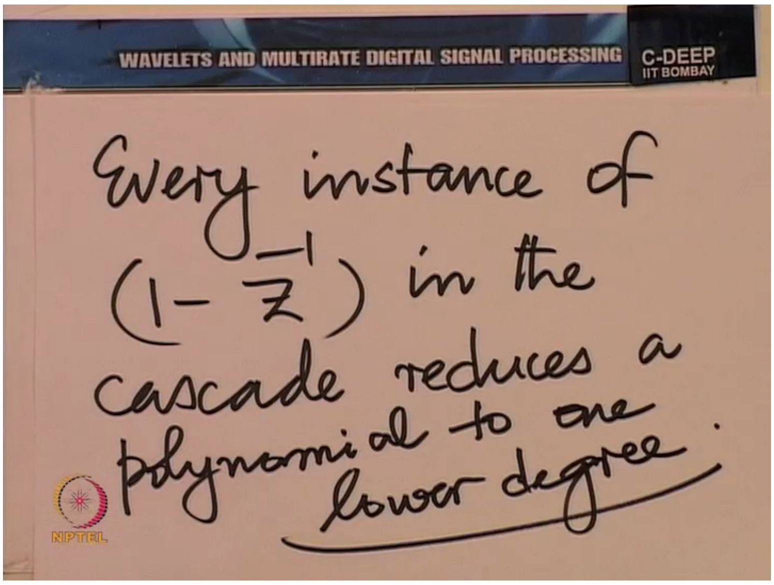

Suppose I had 2 such terms, what would happen? So let us put that down. So we will consider a cascade of 1 - Z inverse terms. So you know you have a system like this, 1 - Z inverse fed into 1 - Z inverse and so on. We shall now prove a very simple and a very elegant result. We shall show that every cascaded, every instance of 1 - Z inverse in the cascade reduces a polynomial sequence to 1 degree lower. So you know I am looking at the situation from a slightly different perspective.

Now I am not talking about frequencies or sinusoids anymore. I am saying suppose you think of an input sequence is having polynomial components. Now where on Earth do you encounter a polynomial kind of expansion? Well, we know about the Taylor series, after all the Taylor series is essentially a polynomial expansion of an input. And when we make a polynomial expansion of the input and we subject a few terms in this polynomial expansion to the action of 1 - Z inverse, we have an interpretation that we are talking about here. So you know, visualize a region in which you are talking about the sequence being, so you know, let the sequence for example come from analytic continuous function and let then that function be expanded in a Taylor series around a certain point which means you have polynomial terms.

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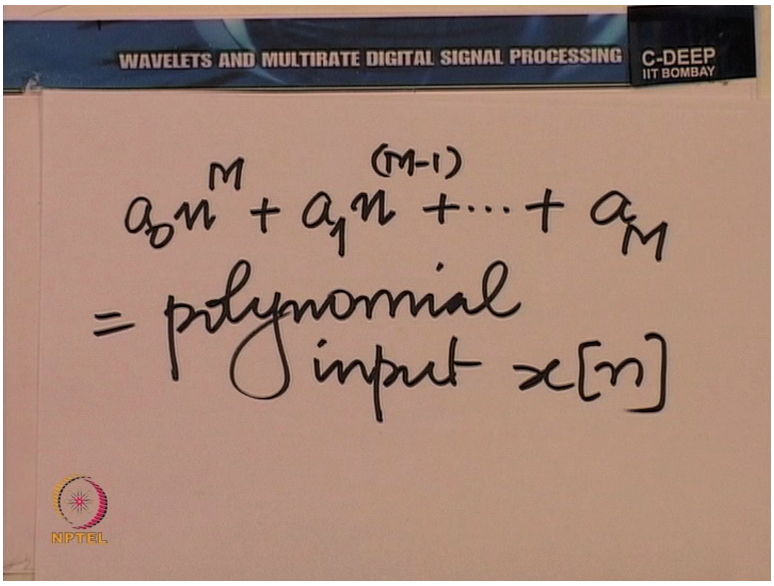

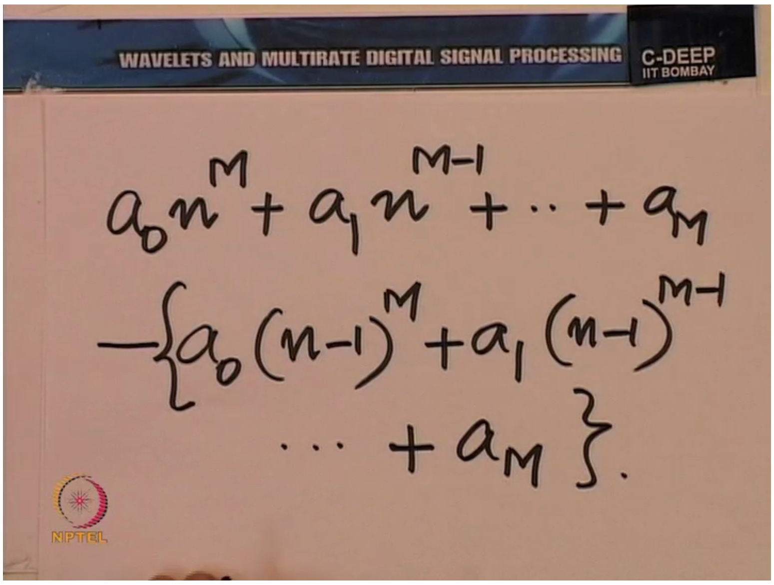

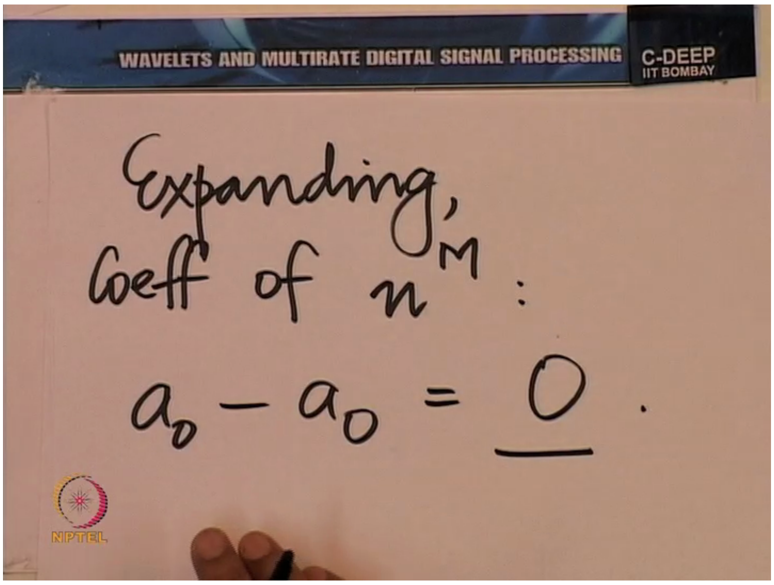

Now let those polynomial terms be subjected to the action of this cascade of 1 - Z inverse, that is the situation in which we should visualize ourselves. It is a different way of expanding an input. Anyway putting that context in perspective, coming back to the polynomial. So we will show that if I feed any polynomial of the form say A0 n to the power of capital M + A1 to the power n -1 and so on up to AM which is the polynomial input sequence.

Everytime we subject this polynomial to 1, 1 - Z inverse, what is going to happen? So let us subject it to 1. The 1st time we subjected, we are doing this. Now what is happening in this process? It is very clear that when we expand this, the coefficient of n to the power of M is easy to evaluate, it is essentially A0, I mean you know you have the term A0 n raised to the power of M coming from here and the term A0 n -1 to the power of M which contributes the coefficient of n raised to the power of M here and that coefficient is again A0, so A 0 - A0 which is 0.

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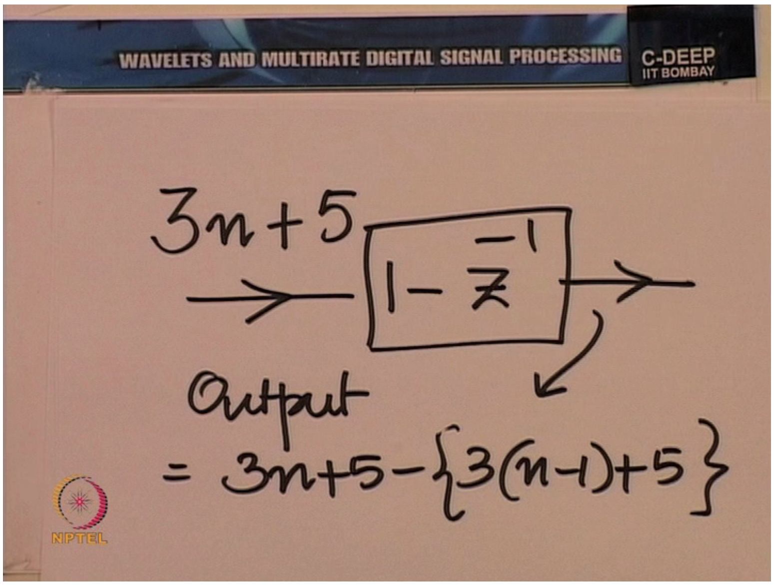

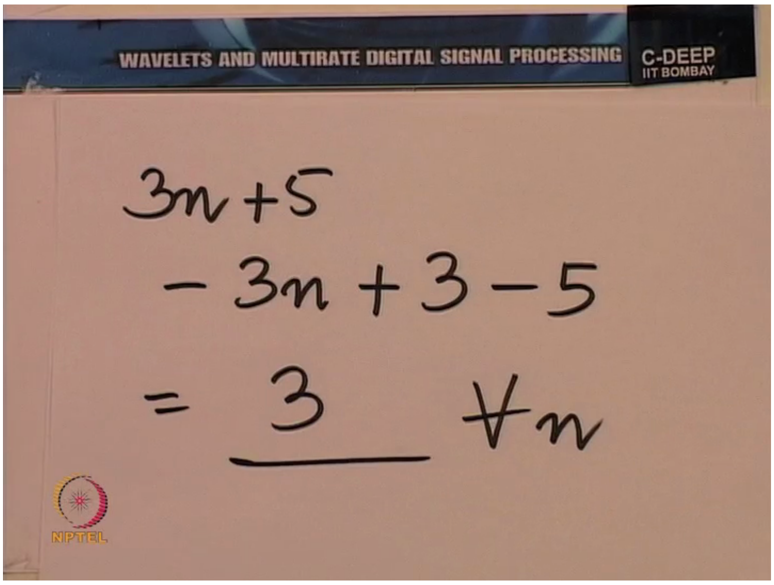

So that is interesting, each time we subject this polynomial sequence to the action of 1 - Z inverse, we are reducing the degree of the polynomial by 1. In fact let me illustrate this by taking a sequence which is polynomial of degree 1 and let us subject it to the action of this filter. So you have essentially something like, let us take a concrete example, so let us take 3 times n + 5 and let us subject it to the action of 1 - Z inverse to fix our ideas. So what you are going to get here is 3n + 5 - 3n - 1 + 5 and that is easy to evaluate, it is essentially 3n + 5 - 3n + 3 - 5 and that is just 3 for all n.

So you brought the degree of the polynomial down by one. You had a degree 1 polynomial, now you have a degree 0 polynomial. That is exactly what happens for any degree polynomial. So what we just showed a minute ago, and I will put back that discussion is that the coefficient of the highest power n to the power M vanishes and therefore when this sequence goes into Z raised to the power – 1 - 1 or 1 - Z inverse so to speak, you only have n raised to the power M -1 and lower degree terms left.

So now we have just proved a simple lemma. Each instance of 1 - Z inverse brings a polynomial degree down by one. So in other words, in a certain sense the more 1 - Z inverse terms you have and by the way we will soon see these terms are going to be on the high pass branch, not on the lowpass branch. You know, 1 - Z inverse when we substitute Z equal to E raised to the power J Omega vanishes at Omega equal to 0. Let us verify that.

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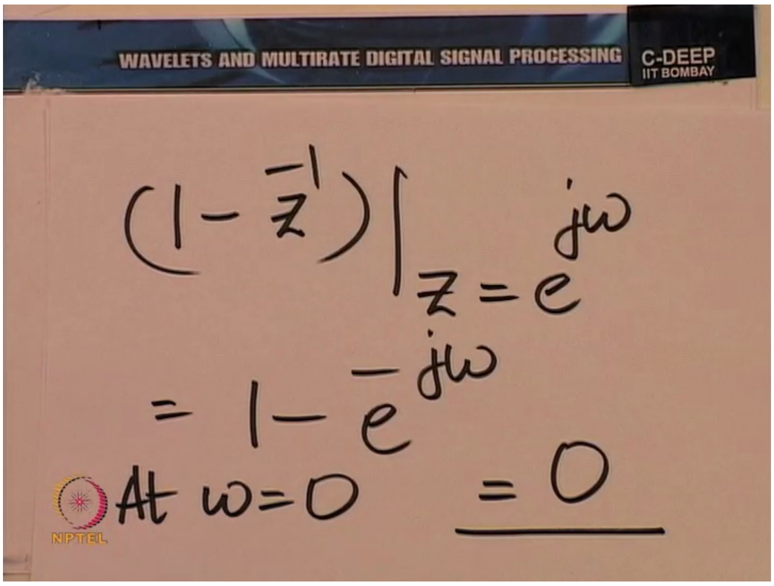

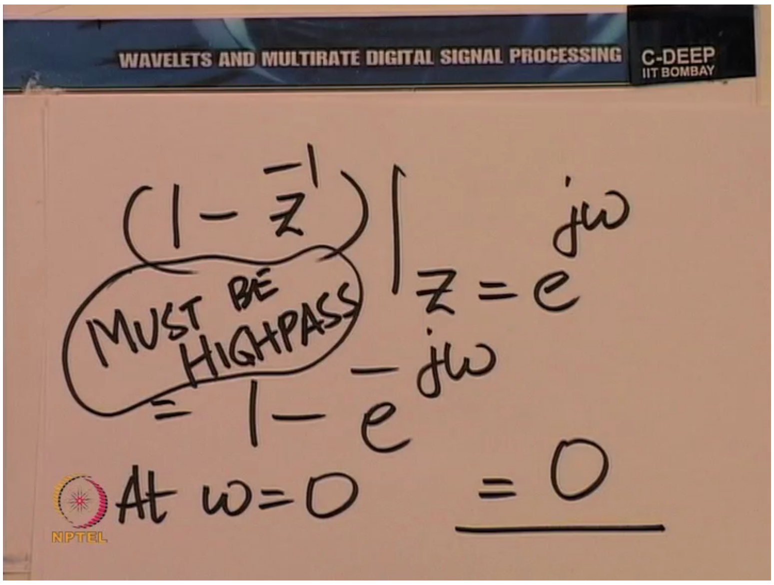

So when we take 1 - Z inverse and substitute Z equal to E raised to the power J omega, 1 - E raised to the power J omega and when we put omega equal to 0, we get this is equal to 0. So in other words this is 0 DC so to speak. 0 at 0 frequency, 0 response at 0 frequency, this cannot possibly be lowpass in its behaviour, so it must be high pass. In other words, if you do want terms of this kind, 1 - Z inverse, they must only be present in the high pass filter, they cannot be terms present in the lowpass filter, otherwise you know you would have a 0 response is 0 frequency, ridiculous for a lowpass filter.

In fact what we are now going to build up is a whole family of multiresolution analysis in which you have more and more 1 - Z inverse terms in the high pass branch. And that is in fact very well-known as what is called the Daubeshies family in multiresolution analysis.