Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-3.
Lecture-11.1.
Z domain analysis of 2 channel filter bank.

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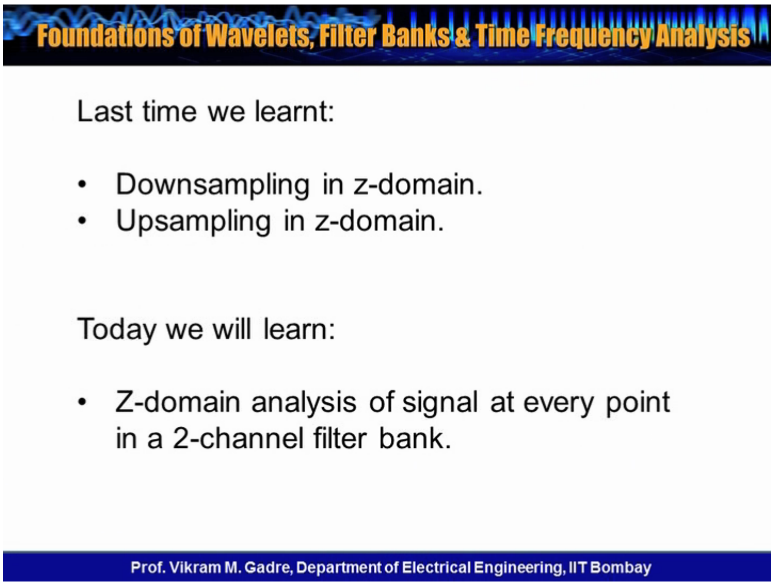

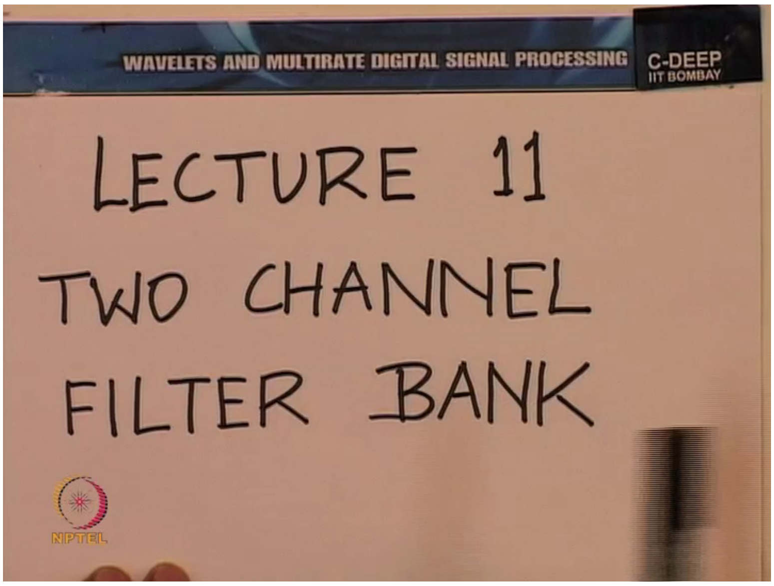

A warm welcome to the lecture on the subject of wavelets and multirate digital signal processing, in which we proceed further on the theme of the 2 channel filter bank. You will recall that in the previous lecture, the 10^{th} lecture, we had looked that what happens when a sequence process a downsampler and an upsampler. I would like to put before you the effect in the Z domain once again when we cross a downsampler and when we cross an upsampler. So today's lecture as I said is entitled the 2 channel filter banks.

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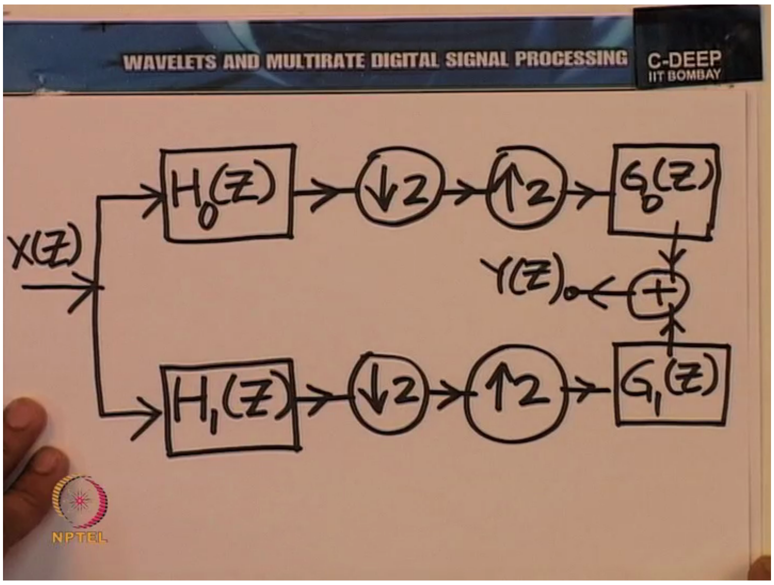

And we intend in this lecture to talk about this structure which we have been repeating oft and again. But today we intend to analyse it threadbare so to speak. Now remember that H0 was a lowpass filter, H1 a high pass filter, you know it is helpful to get this structure firmly embedded in our consciousness. It is a structure often used in the implementation of a discrete wavelet transform, for example of the Haar multiresolution analysis. So it does no harm to repeat the structure more than once, we have been looking at the structure almost in every one of the last 2 to 3 lectures, but no harm done.

Anyway, here is the structure once again and what we intend to do finally today the structure is to relate the Z transforms at every point in this structure. Let us start doing that and to do that let us 1st recall what happens when we go past a downsampler and an upsampler. So as I said, let us look at the upsample 1st, recall from the previous lecture.

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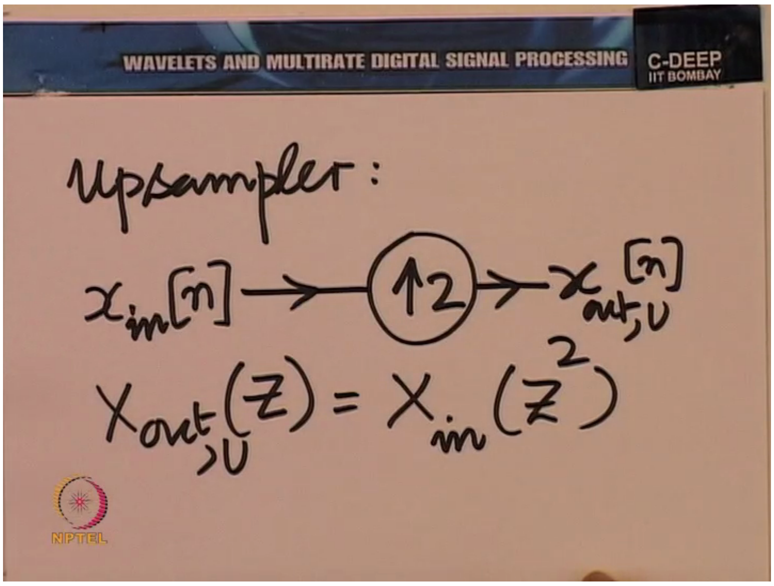

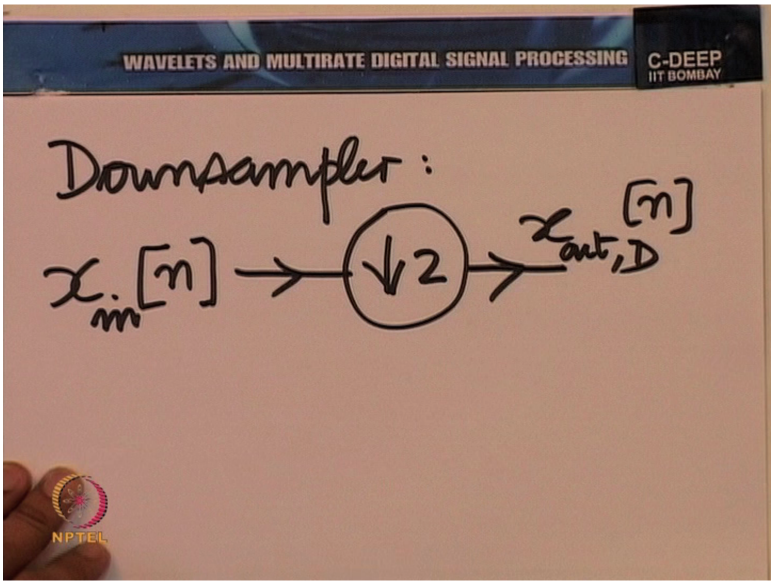

Noted from the previous lecture that if you have X in n going to an upsampler by factor of 2 to produce Xout n here, then Xout Z in this case, maybe to be specific we could say Xout U to denote upsampling here, so Xout U Z is X in Z squared in this case. Let us similarly see what happens when we go past a downsampler. So for a downsampler by a factor of 2, remember we had analysed in detail the case of a downsampling by factor of 2, we had indicated how to generalise, but it was a little more complicated.

So here we have Xout Dn if you please and in fact we have noted that this operation of downsampling by 2 was equivalent to 2 operations. And let me spell out those operations once again following which I shall write down the Z transform here. So we have agreed that this operation of downsampling by 2 can be split into modulation with half 1 raised to the power n + -1 raised to the power of n 1st, followed by inverse upsampling by 2, inverse, you know. We have noted that upsampling by 2 or upsample by any integer factor for that matter was an invertible operation.

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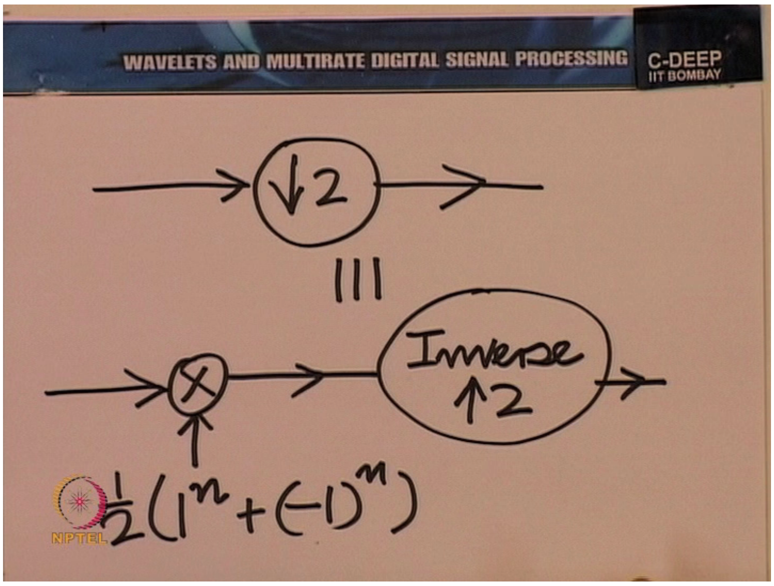

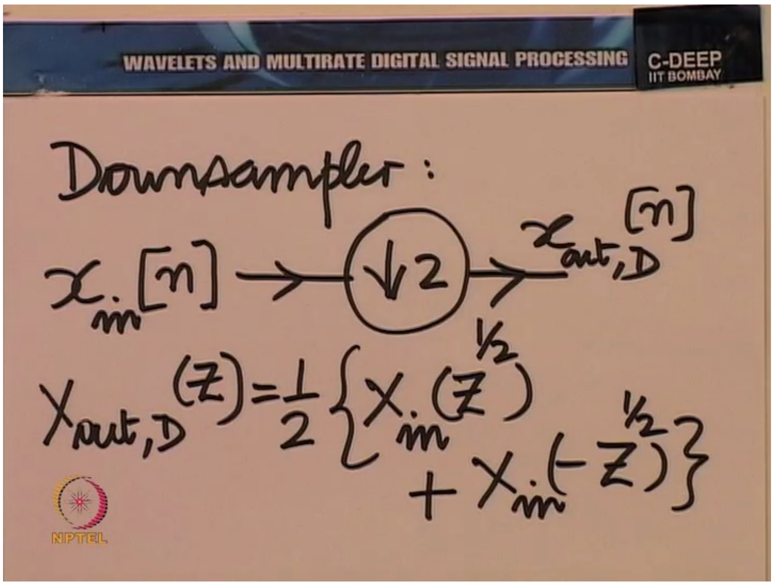

And in fact this is also evident in the Z domain. And therefore we can talk about the inverse here. Now based on this we have write, we have written down the Z transform. So we had said Xout D Z here is half X in Z raised to the half + X in - Z raised to the half. We had derived this towards the end of the previous lecture. And of course if you did not wish to go beyond the inverse upsampler, at this point instead of Z raised to the power half you would simply have Z.

So much so then for the Z domain effect of downsampling and upsampling. Now what do we intend to do next? We intend to use these relationships to build up the relation between the Z transform of the output and the Z transform of the input. So essentially we wish to establish the input output relationship in this 2 band or 2 channel filter bank in the Z domain. Now of course I must make a remark here before we embark upon this activity. We are assuming that the Z transforms exist at all points.

There are rare situations or maybe not so rare where they may not be true. But for the moment we shall ignore those situations, they are not very frequent in practical applications and therefore we can comfortably begin with the assumption that the Z transforms exist at all points. Of course I do not mean the Fourier transforms exist, they may or may not. Anyway coming back then that 2 channel filter bank that we began with earlier on in this lecture, I put it back before you here.

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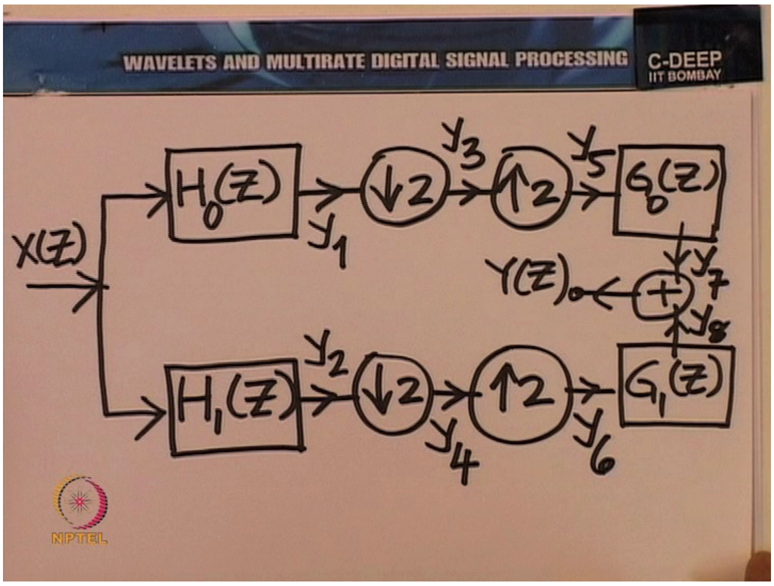

I intend now to relate YZ and XZ in this diagram. And towards that objective, let us number the Z transforms at different points. In fact let us give these sequences names. So let us call this Y1, let us call this Y2 and so on. And the very output we shall just call Y without any subscript. So accordingly this becomes Y3, this Y4, this is Y5, Y6, Y7, Y8 and then of course Y9 is simply Y. So with this then let me write down the expressions for each of these. So let us begin with Y1.

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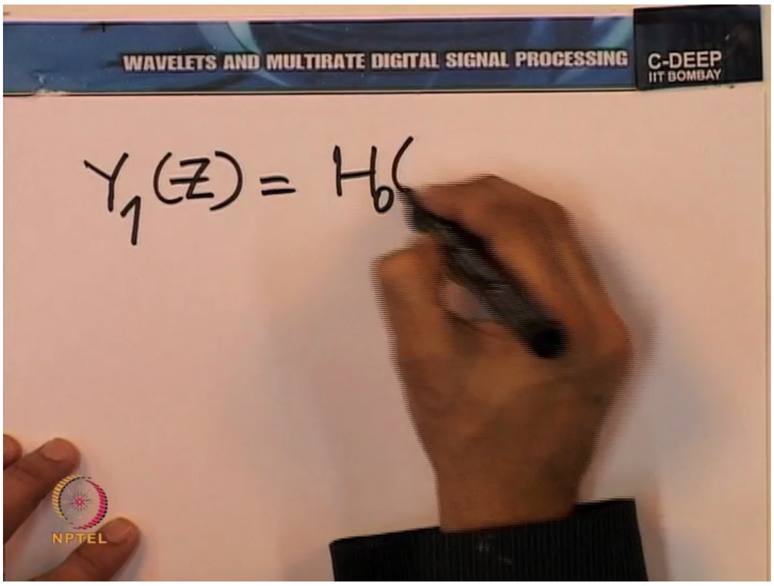

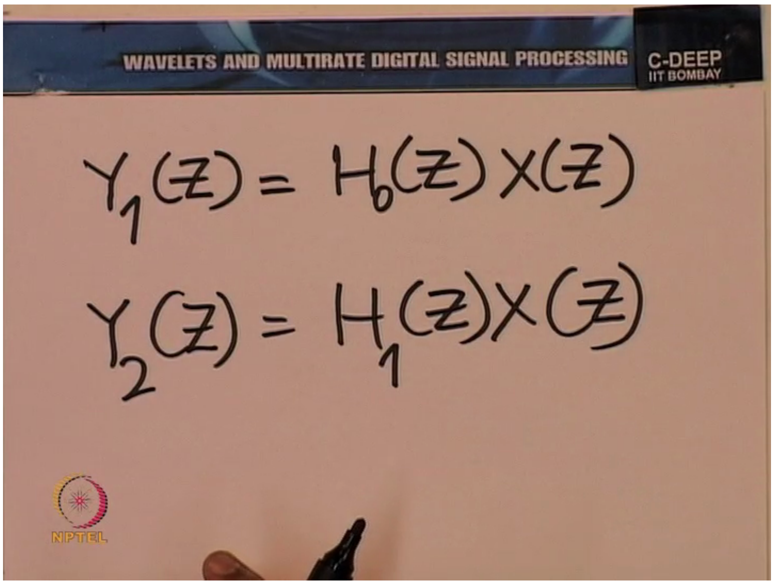

Y1 is easy and so is Y2, in fact if we look at the Z domain relationships Y1 Z is simply XZ times H0Z. And similarly Y2 Z is simply XZ times H1Z. When we filter with a filter of system function H0Z, the effect is to multiply the Z transform of the input by the system function of the filter to produce the Z transform of the output. And therefore Y1 Z and Y2Z are easy. Y1 Z is H0Z XZ, Y2 Z is H1Z XZ.

Now we have before us the relationship that emerges between Y3 and Y1 and similarly between Y4 and Y2, as I said downsampling by 2 is equal and to modulation and then an inverse upsampling by 2. So you know it is easy then to jump from Y1 to Y5 1st and Y2 to Y6 because if you think of this downsampling by 2 operations as a modulation followed by an inverse upsample by 2, the inverse upsample by 2 cancels with this upsample by 2 leaving only the modulation here.

So this is the strategy which we might find useful in some circumstances in analysing multirate systems, particularly when you have a downsampler followed by an upsampler. You know if you have a downsampler followed by an upsampler of the same factors, it is sometimes easier to go past both of them and then jump back behind the upsampler, we will do exactly that here.

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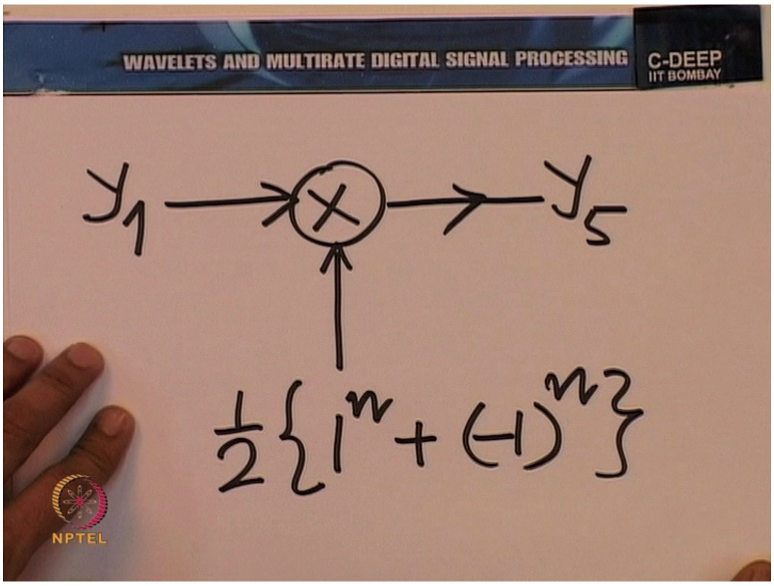

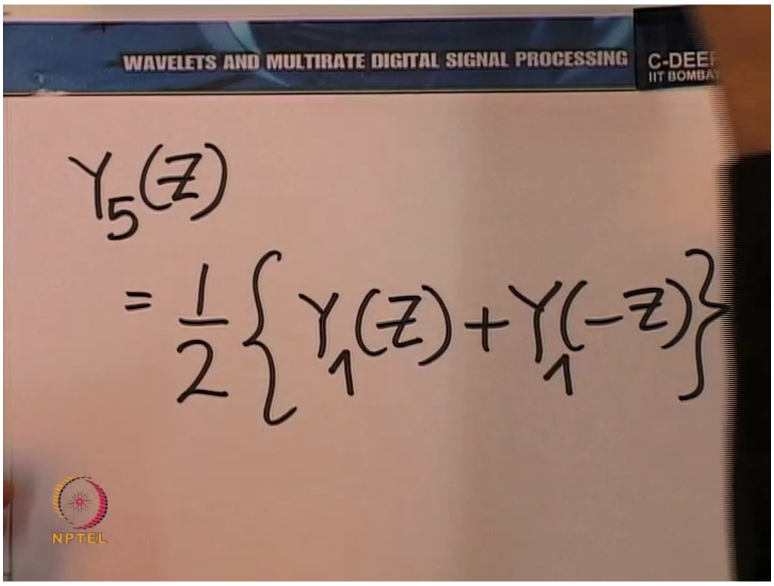

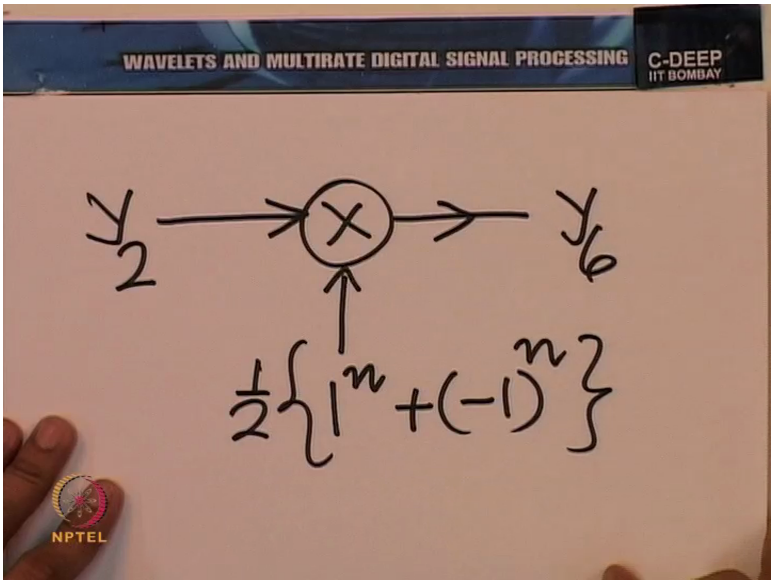

So we will put the relation between Y1 and Y5 and Y1 to Y5 is essentially just a modulation, modulation by 1 raised to the power of n + -1 raised to the power of n multiplied by half. And therefore the Z transforms are easily related. Y5 of Z is half Y1 Z + Y1 - Z. And there is every similar relationship between Y2 and Y6, put this back. Y2 and Y6 are related by the same modulation. So let us write that down to.

So we have Y2 is modulated to obtain Y6 and therefore Y6Z is half Y2Z + Y2 - Z. In fact this jumping across the downsampler and the upsample was useful because it brought us quickly towards the output. Now we are only almost 1 step away from the output. That step is also not very difficult to take. We just have 2 filters followed by a summer which is easy to do in the Z domain. So let us complete that exercise. So now we are here, we know what Y5Z is, we know what Y6Z is.

Y5Z is acted upon by a filter with system function G0 and therefore the Z transform at the output here after G0 is the Z transform of Y5 multiplied by G0. Similarly the Z transform at the output of G1 is Y 6Z multiplied by G1Z. And in fact let me take you one step even further, after all YZ is obtained by adding Y7Z and Y8Z. And therefore we already have YZ before us, let me write down those steps quickly before you now.

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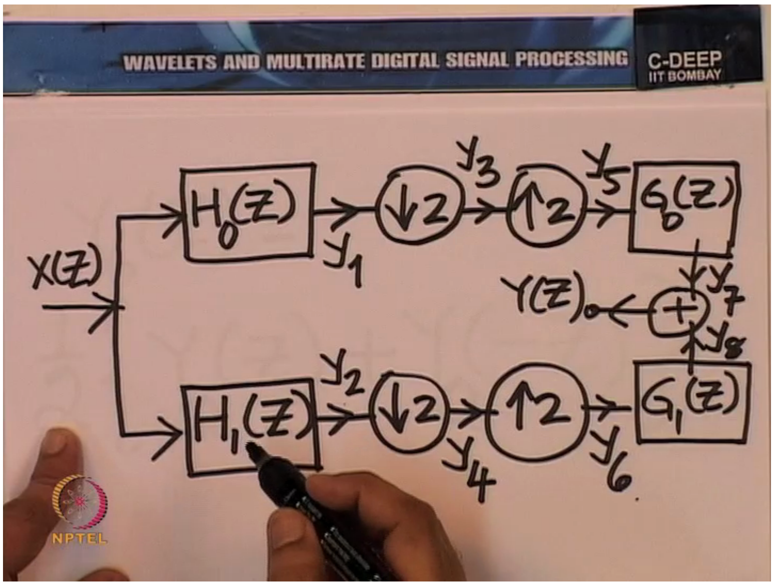

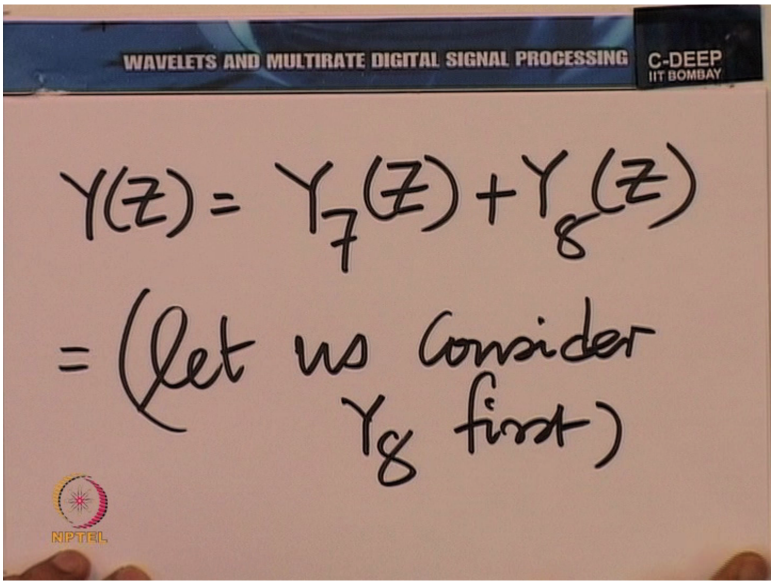

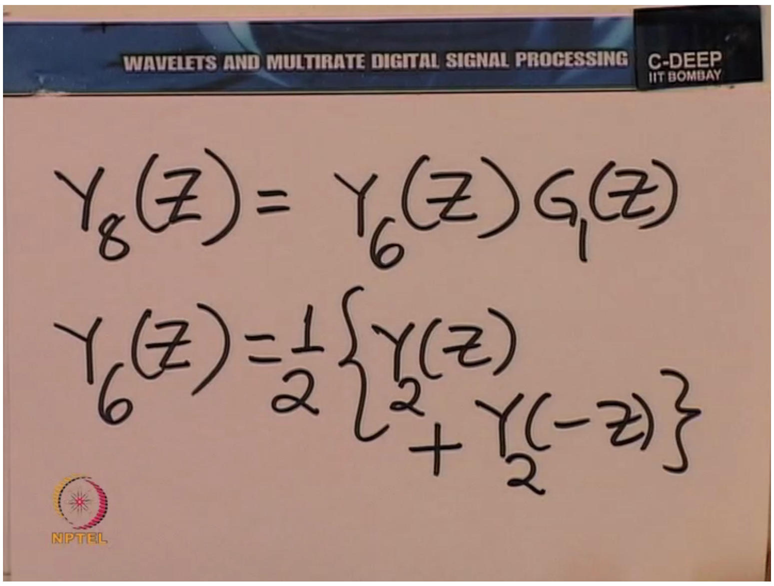

So we have Y7Z is Y5Z times G0Z. And Y8Z is Y6Z times G1 Z, whereupon YZ which is Y7Z + Y8Z becomes, you know let me put before you the expression for Y5 and Y6 once again. So let us take Y6 as an example. Now you are going to involve Y2Z and Y2 - Z. Y2 - Z if you recall is XZ times H1Z as an example. Now when you take Y2 of - Z you would get X of - Z multiplied by H1 - Z. So in fact let me put that down, what I am trying to do is to identify a form of the expression.

So you know this Y2 of - Z has 2 terms embedded in it. So let us focus, let us say for example on this term here, Y8 here. The Y8Z as I said is going to be Y6 times G1Z, Y6Z is going to be related with Y2Z and then Y2Z is further going to be XZ times H1 Z. Whereupon Y2 - Z is going to be X of - Z times H1 - Z. What I am trying to bring out from this argument or these equations is that each of these terms is going to have a contribution from X of Z and a contribution from X of - Z.

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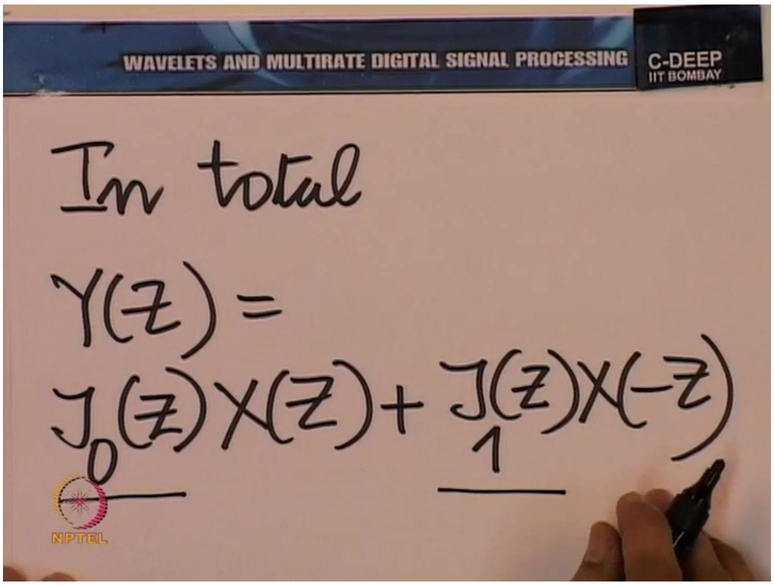

Now later on we shall interpret these terms, X of Z and X of - Z. But for the time being all that we need to do is to identify that there is an X of Z set of terms and X of - Z set of terms, which we can together write down as follows. So we will say YZ in total is of the following form, it is going to have some system function multiplying X of Z and some other system function multiplying X of - Z. And using all the equations written so far, we can write down these 2 system functions, call them Tao 0Z and Tao 1Z. Indeed Tao 0Z has the following form.

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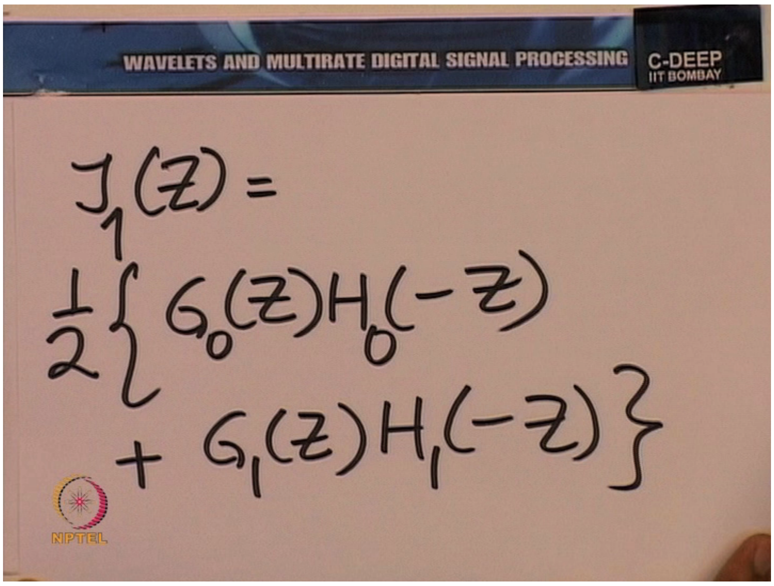

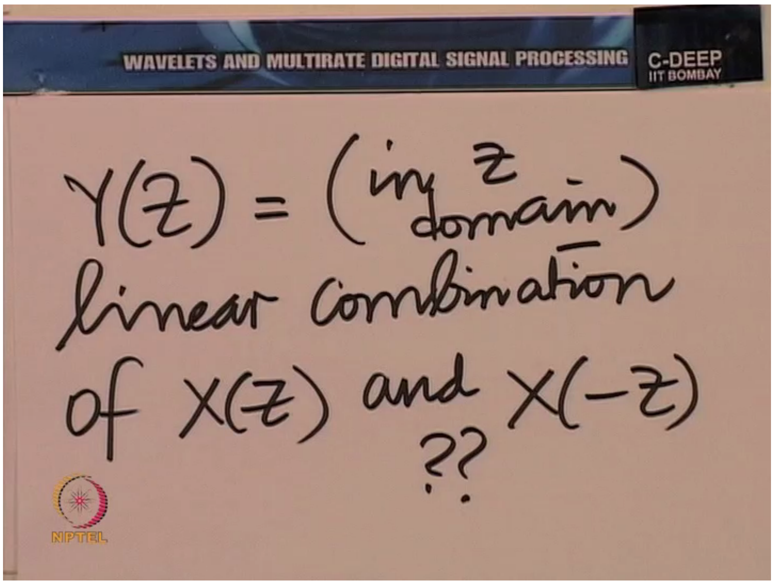

It just it just requires a little bit of algebra to get here, so I would not actually do all the steps, I am sure all of you can do them. And now we write Tao 1Z as well. So Tao 1 Z would be G0 Z H0 - Z G1 Z H1 - Z. Now you know it should be noted that Tao Y 1Z which is what multiplies X of - Z, you involve H0 of - Z and H1 of - Z. So you never involve G0 - Z or G1 - Z. The Z replacement by - Z is only for the analysis side. Anyway, with this little observation we now write down YZ once again for clarity.

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So YZ is a linear combination, I mean in the Z domain you know of XZ and X of - Z. What does this mean? What you mean linearly combining vector XZ and X of - Z? You see if the X of - Z terms were not to be there, then we have a simple expression there. YZ is some function in Z multiplied by XZ, that is the good old linear shift invariant system for you with the system function and in this case if Tao 1Z were to be absent, the system function would be simply Tao 0Z.

Tao 1Z is the troublemaker here and we need to understand 1st what this X of - Z is, what does it do spectrally, what does it do in the time domain? So let us understand that now.