Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-3.
Lecture-10.3.
Downsampling by a general factor M-a Z-domain analysis.

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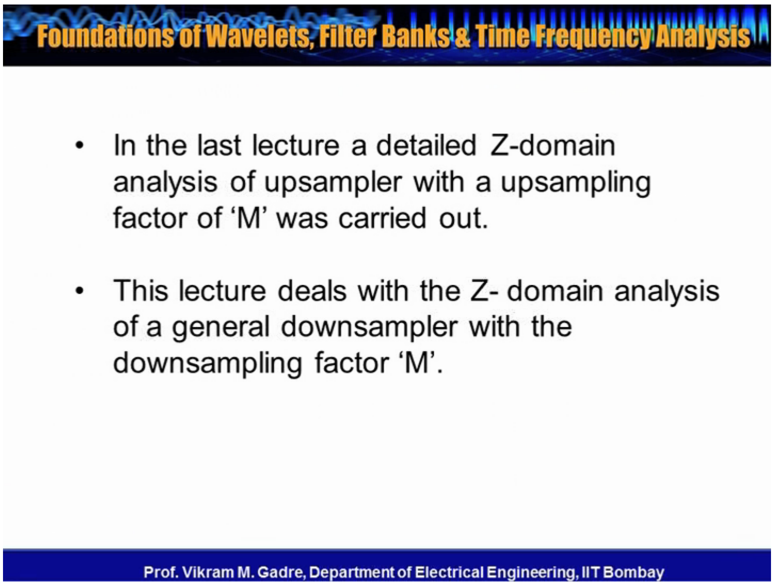

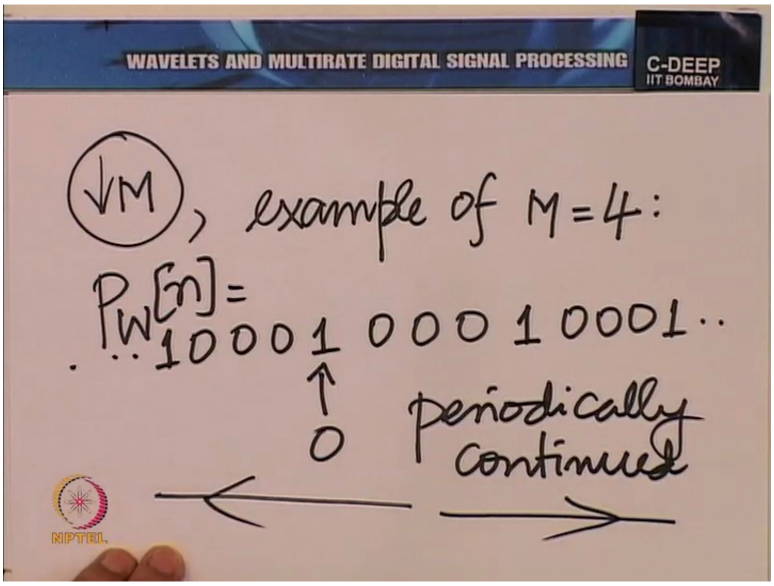

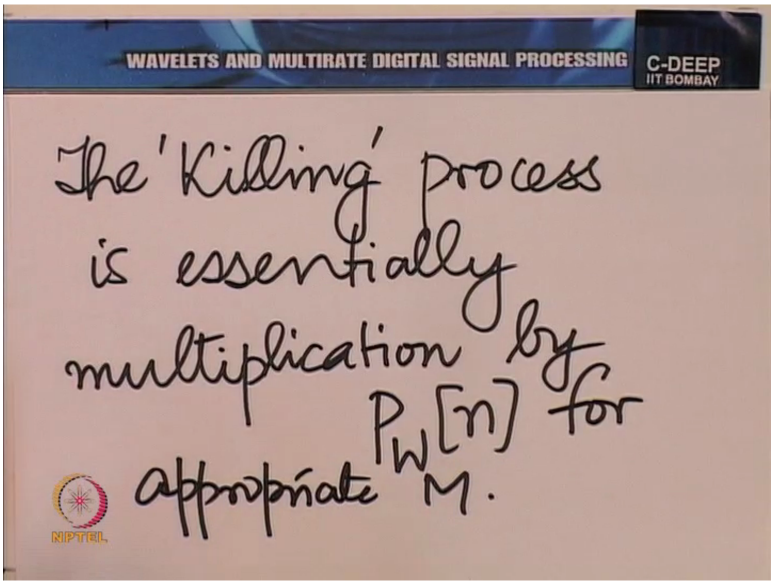

But you can have a similar window periodic sequence for downsampling by any other factor. So suppose you were to downsample by M in general, again let us take the example. Now for variety of M equal to 4 this time, then the corresponding PWn would be 1 located at 0, then 3 zeros, 1 located at 4, then 3 zeros again, 1 located at 8 and so on, on this side and of course 3 zeros and then 1 on this side is periodically continued, continued this way, continued this way.

So what you are saying in effect is the process of killing, the killing process is essentially multiplication by PWn for the appropriate M. So if you want to downsample by 2, you have an appropriate PW n, periodic with a period of 2 with 1 located at all multiples of 2 and 0s elsewhere. In general when you want to downsample by M, you have the periodic sequence PWn with a period of capital M, 1s located at all multiples of capital M and zeros everywhere else, simple enough.

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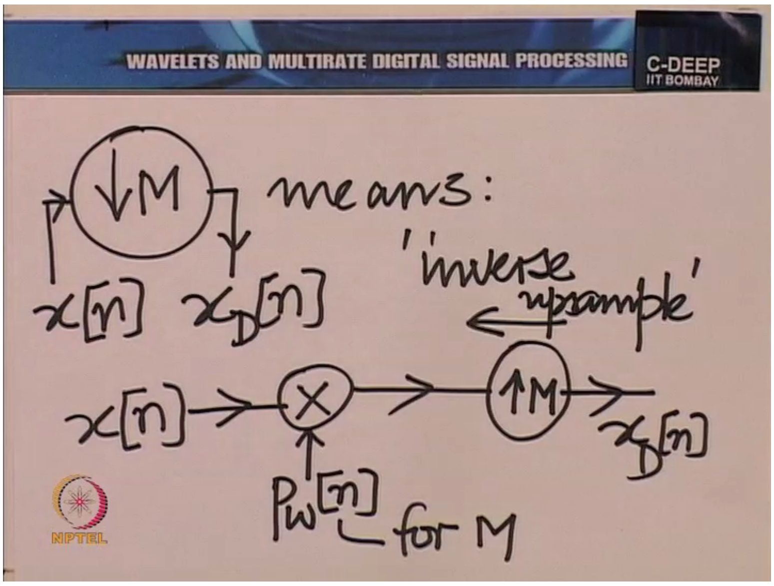

What about the compression step? The compression step can be viewed as an inverse expander, an inverse upsampling step. So maybe we should say downsampling by M means or in other words, let us actually put down sequences here, let us say X n has been downsampled to produce XDn and that means Xn has 1st been multiplied by PWn for downsampling by M, for M in brief. And then now look here, you know what we are saying is XDn is here and I am saying upsample by M but this way, that means it is a reverse or it is an inverse upsampling operation.

Now if we view the downsampling operation this way, it is again very clear why it is not invertible. This part is invertible but this part is not because you cannot divide by PWn, PWn has zeros in it, you cannot divide by that, that is another way of viewing it. And of course this will also get manifested in the Z domain but subtly. Now you know to manifest this in the Z domain, this operation of multiplication by a sequence is difficult to capture in the Z domain.

So what we need to do is to translate this multiplication by PWn into multiplication by other sequences which are easy to interpret in the Z domain. Now the only convenient multiplication by sequence operation which can be handled in the Z domain is multiplication by an exponential. So what we need to do is to replace this multiplication by PWn by a linear combination of multiplication by exponentials. How do we do that?

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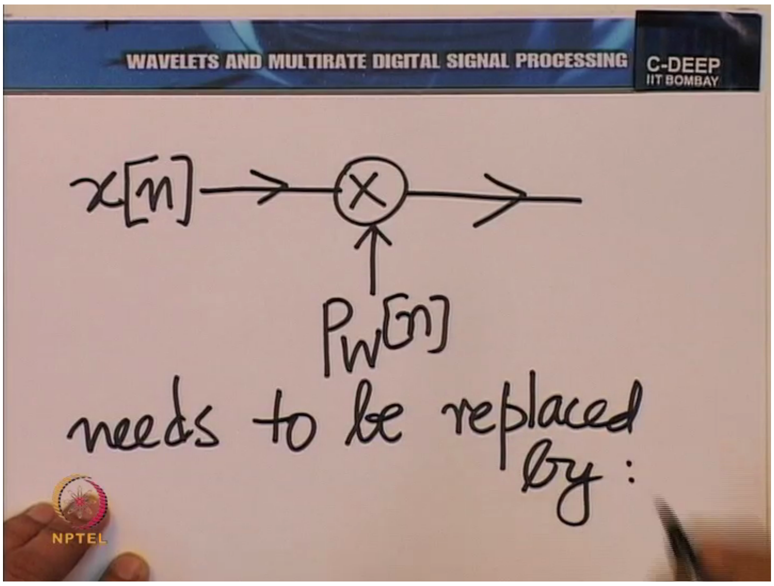

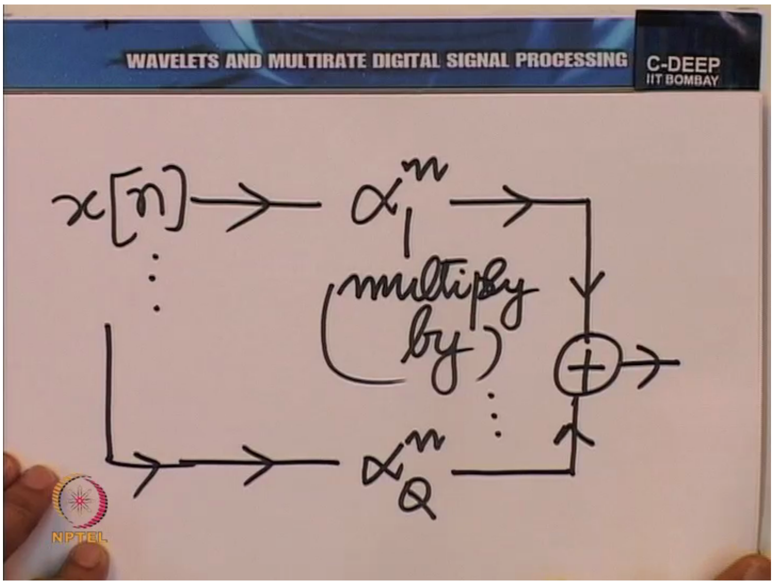

So let me put down our objective little more clearly. This multiplication by PWn needs to be replaced by Xn multiplied by exponential alpha 1 to the power of n, so I think maybe you know I will say multiply by alpha 1 to the power n +, say Alpha Q to the power n, Q such terms, same Xn. So you know multiplication by alpha 1 to the power of n or Alpha to the power of n in general is easy to handle in the Z domain.

So the other way of saying it is we need to express, we need to express PWn as a linear combination, let us say CK times Alpha K to the power n, K going from let us say 1 to Q. Now you know this is not at all difficult to do. In fact, if you think about it, that is exactly what the discrete Fourier transform does in discrete time signal processing. The discrete

Fourier transform is essentially expresses a sequence, in fact a periodic, time-limited sequence as a combination of exponentials.

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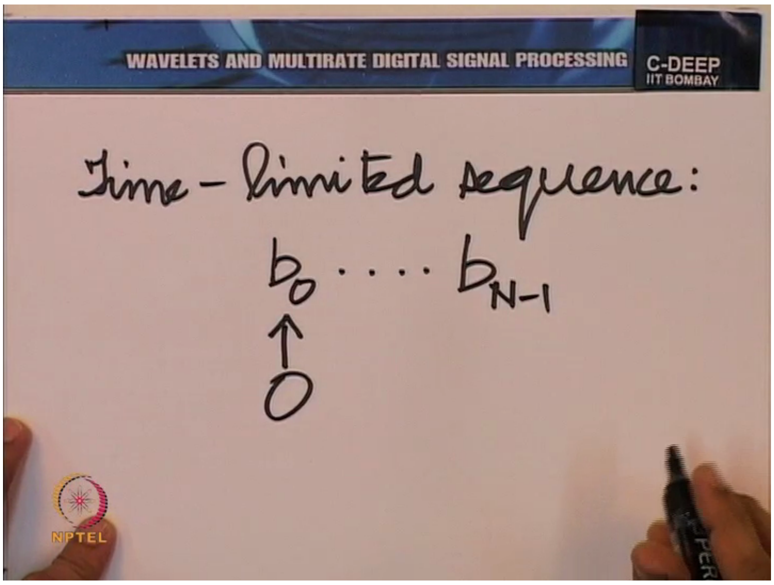

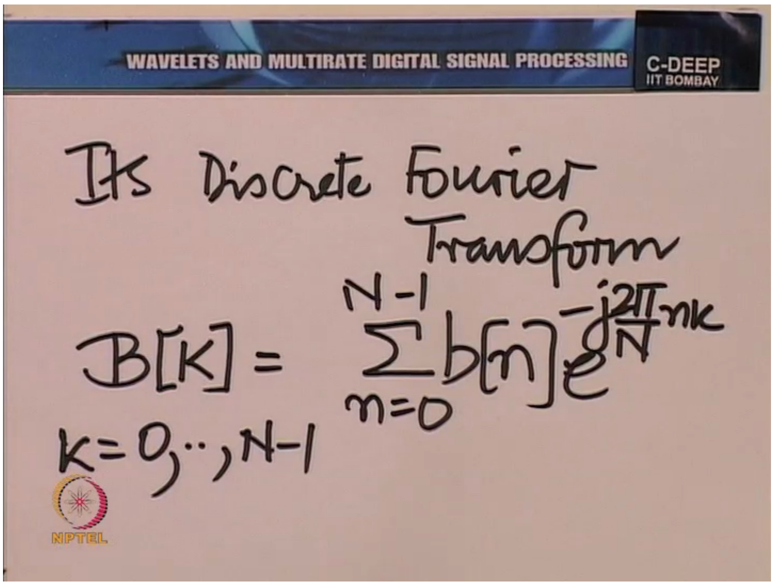

Let me just to refresh our ideas the expression for the inverse discrete Fourier transform before us. So if you recall, you know if you have a limited, a time-limited sequence, let say it has capital M samples, so, you know you have, let us give the sequence a name b0 to bn or bn -1 to be more precise. So if this is the time-limited sequence of capital M samples, its discrete Fourier transform if you recall is essentially capital B of K, of course K is also running from 0 to capital N -1.

With capital B of K defined by summation, n going from 0 to capital N -1, bn e raised to the power - J2 pie by n, capital N times nk. And then we can reconstruct bn from its discrete

Fourier transform, that is easy, bn is 1 by n summation K going from 0 to n -1, capital B of K e raised to the power J2 pie by n Kn now this is true for again n going from 0 to capital N -1.

But after all if you recall what we have here in the discrete Fourier transform expression and in this expression which is essentially an inverse discrete Fourier transform, recall that the discrete Fourier transform is often abbreviated by DFT and the inverse discrete Fourier transform by I DFT. In the idea of D we have exactly the kind of expansion or expression that we desire. We are expressing that limited sequence of capital M samples in terms of its DFT components and exponentials here. So we have exactly what we want.

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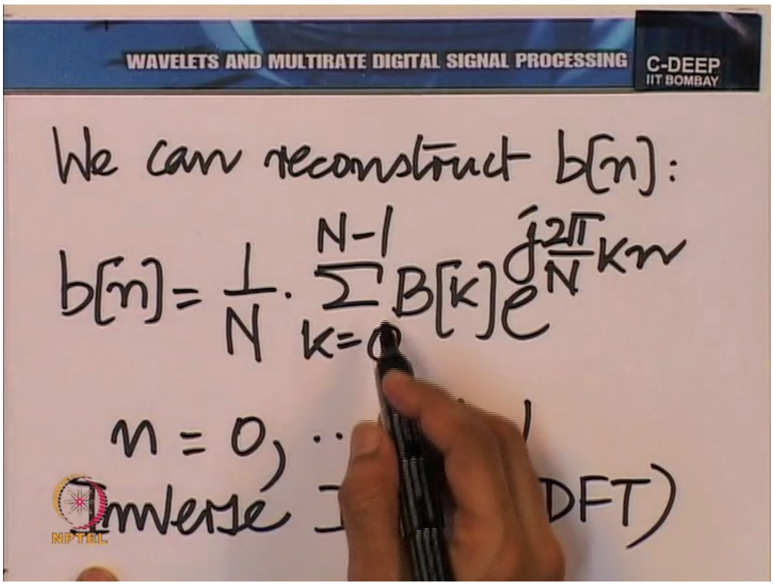

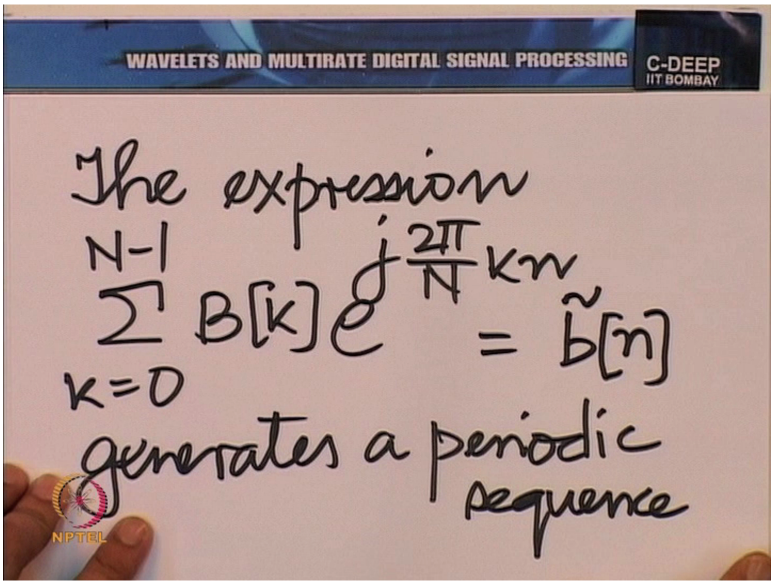

Now we can exploit this, you see what is to be noted here is that although we have talked about a limited sequence, actually you can we radically extend that sequence in capital N.

And whatever expression we have here to reconstruct bn from capital B of K, so bn is reconstructed from capital B of K here, seemingly only at n equal to 0 to n -1 but even if you apply the same expression for all integer n you get a periodically repeated sequence, whatever is in between 0 and -1 is radically repeated in every interval of n successively and before.

And that is easily seen if you substitute n by n + any multiple of capital N here, noting that e raised to the power J 2 pie by n times n is 1, let me do that just for completeness. So what I am saying is if we use this expression, the expression summation K going from 0 to capital N -1, capital B of K e raised to the power J2 pie by n kn generates a periodic sequence. So let, let us call this expression as a function of n b tilde n.

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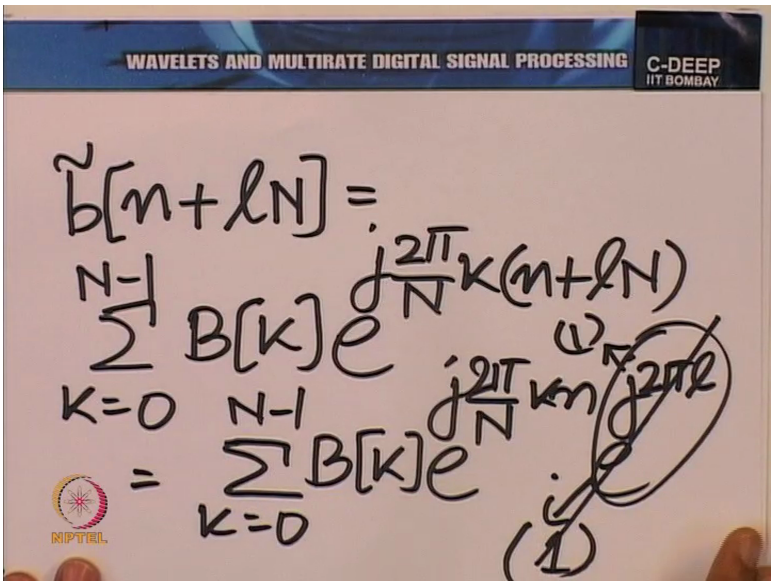

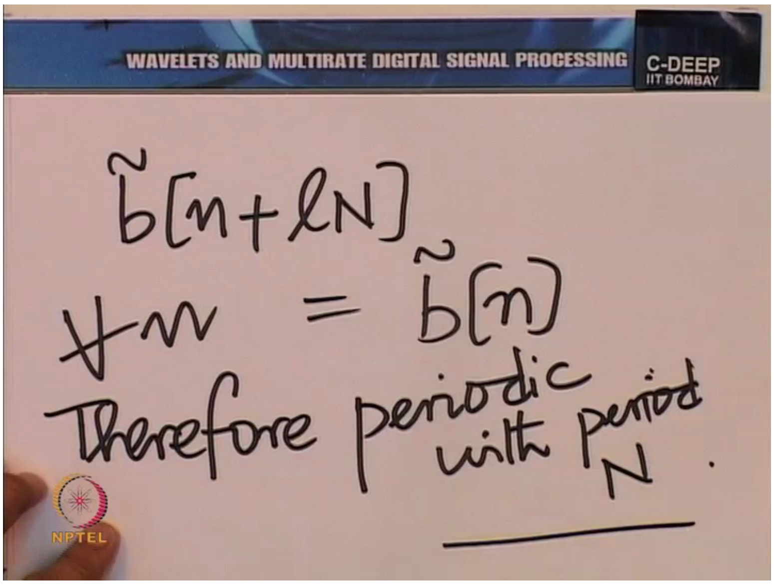

It is very clear that b tilde n + L times capital N is essentially summation K going from 0 to n -1 b K e raised to the power J, 2 pie by nK n + L times capital N. And this can be simplified, if K going from 0 to capital N -1, bK e raised to the power J2 pie by capital N times Kn e raised to the power J2 Pi L and this is essentially 1, so this is essentially 1. So you may remove this term and leave this and that is same as b tilde S.

And therefore what we are saying is in effect b tilde n + L times capital N is b tilde n for all n and therefore periodic with period capital N. So here we have a mechanism for generating a periodic sequence by using the combination of exponentials, let us exploit that. So let us now take the case of capital N equal to 2, there it is easy.

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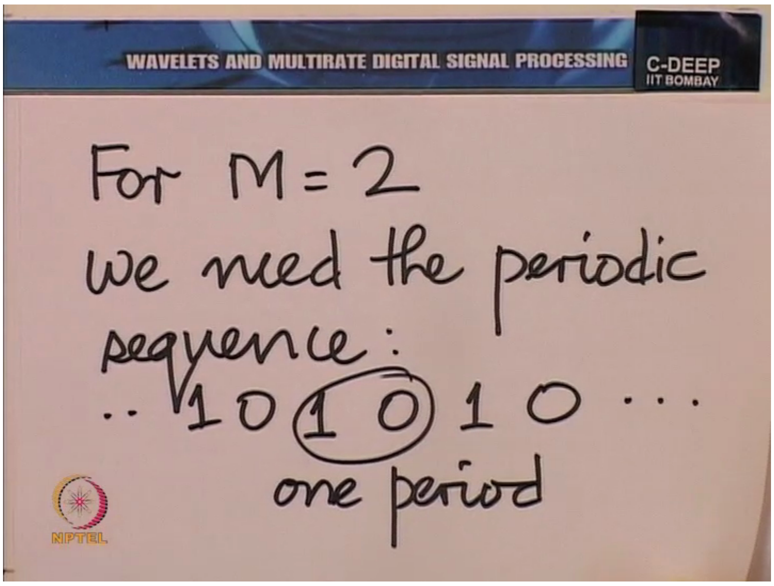

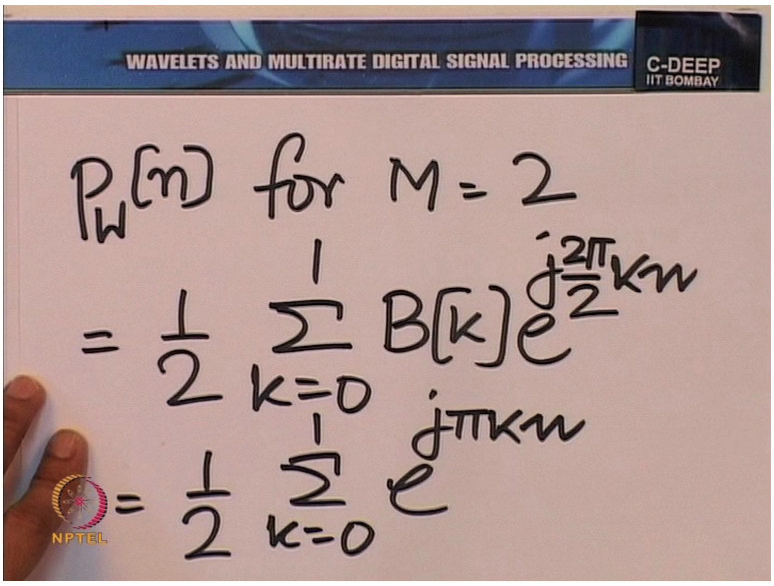

We need the periodic sequence well, 1, 0, 1, 0 and so on, so on and so forth, and we exploit just one period here. And we construct the discrete Fourier transform of this, that is easy. It is 1 times e raised to the power - J2 pie by 2 times, you know of course the discrete Fourier transform is a function of the frequency index K, so this is 0 times K + 0. So in fact b of K is 1, for K equal to 0 and 1. And how do we reconstruct this sequence, the periodic sequence PWn then?

For M equal to 2 is reconstructed using the inverse discrete Fourier transform, so it is summation K going from 0 to 1 b K e raised to the power J2 pie by 2 times Kn and that is easy to do, that is half summation K going from 0 to 1, 1 times e raised to the power J2 pie by 2 is J pie Kn.

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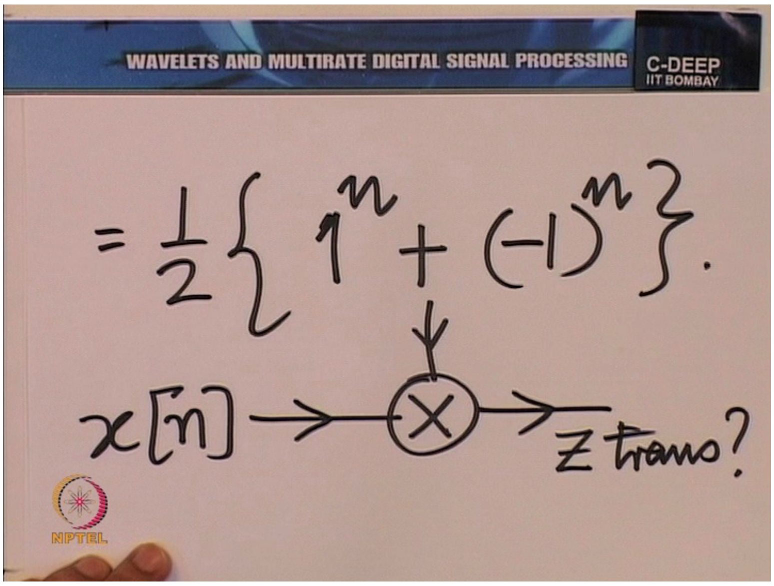

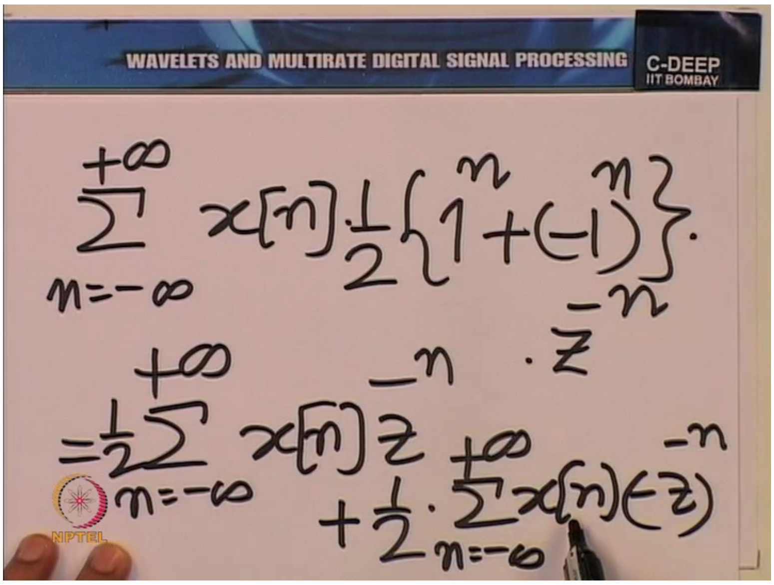

So in fact let us expand it explicitly, that is half with K equal to 0 it is 1 raised to the power of n + -1 raised to the power of n and now we are well set, in fact we are almost there. Indeed what we are saying is if you have X n and if you we are multiplying this by this sequence and you wish to obtain the Z transform here, all that you need to do is to say express the following expression n going from - to + infinity, Xn into half 1 raised to the power of n + - -1 raised to the power of n times X power of - n in the Z domain.

Which is easily seen to be summation n going from - to + infinity, of course half can be brought out everywhere, Xn Z raised to the power of n + half summation over all n from - to + infinity Xn - Z raised to the power - n. So we have done our job, indeed these are very familiar expressions, this is just the Z transform of X and this is the Z transform of X with Z replaced by - Z.

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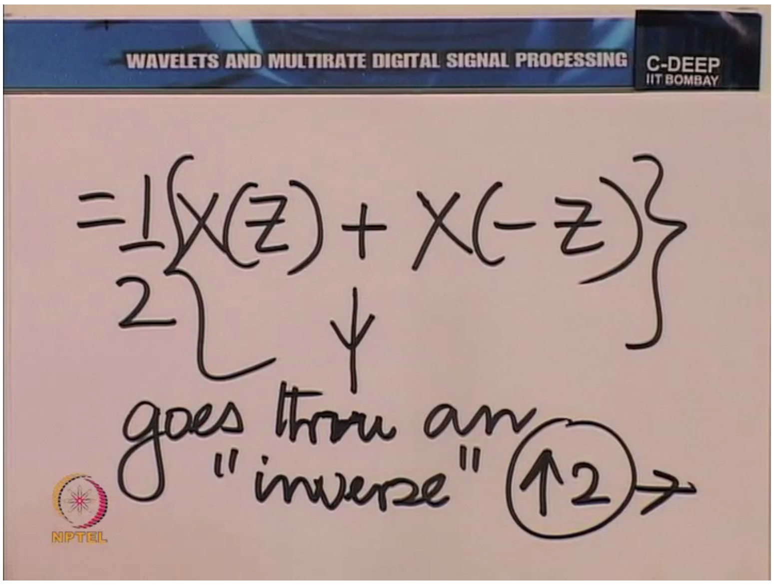

So we have, this is essentially equal to capital X evaluated at Z, + X evaluated at - Z multiplied by half. So we have done our job, now all that we need to do is to note that this goes through an inverse upsample, upsampler by 2 so to speak. And therefore if we put these together, we have the final result Xn with Z transform X of Z, when it goes through a downsampling by 2 becomes in the Z domain XDn with the corresponding Z transform given by half X Z raised to the half + X - Z raised to the half.

We have done what we wanted to in this lecture, we have established the Z transforms of the basic multirate operations in terms of the original Z transforms of the sequences. We shall build on this further in the next lecture to carry out the complete analysis of the 2 channel filter banks. With that then we conclude this lecture and look forward to the complete analysis. Thank you.