Foundations of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-1.
Lecture -2.1.
Dyadic Wavelet.

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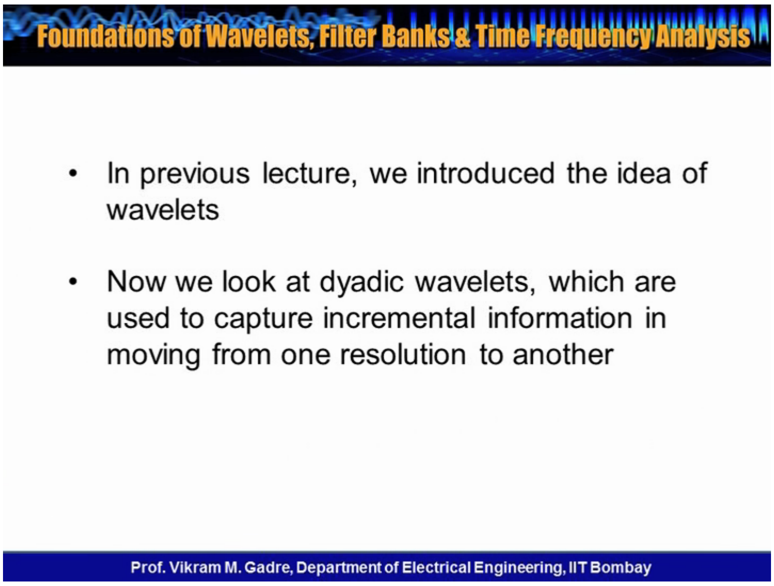

Today we shall begin with the lecture on the subject of wavelets and multirate digital signal processing in which our objective would be to introduce the Haar multiresolution analysis about which we had very briefly talked in the previous lecture. Before I go on to the analytical and mathematical details of the Haar multiresolution analysis or MRA, as it is called for short, let me once again review the idea behind the Haar form of analysis of functions.

Recall that Haar was a mathematician or mathematician-scientist if you would like to call him that. And the very radical idea that he gave was that one could think of continuous functions in terms of discontinuous ones and do so to the limit of reaching any degree of continuity that you desire. What I mean is, start from a very discontinuous function and then make it smoother and smoother all the while adding discontinuous functions until you go arbitrarily close to the continuous function that you are trying to approximate.

This is the central idea in the Haar way of representing functions. We also briefly discussed by this was something important, it seemed like something silly to do at 1st glance but actually is very important. And the reason why it is important, as we mentioned, was if you think about digitally communicating, say for example an audio piece, you are doing exactly that. The beautiful smooth audio pattern is being converted into a highly discontinuous stream of bits.

What I mean by discontinuous is when you transmit the stream of bits on a communication channel, you are in fact introducing discontinuities every time a bit changes. So after every bit interval, there is a change of waveform and therefore this continuity at some level, even if not in the function, in its derivative or in its 2nd derivative, whatever be. Whatever it is, the idea of representing continuous functions in terms of discontinuous ones has its place in practical communication and therefore what Haar did was something very useful to us today.

What we are going to do today is to build up the idea of wavelets, in fact more specifically what are called dyadic wavelets starting from the Haar wavelet. And to do that, let us 1st consider how we represent a picture on a screen and I am going to show that schematically in the drawing here.

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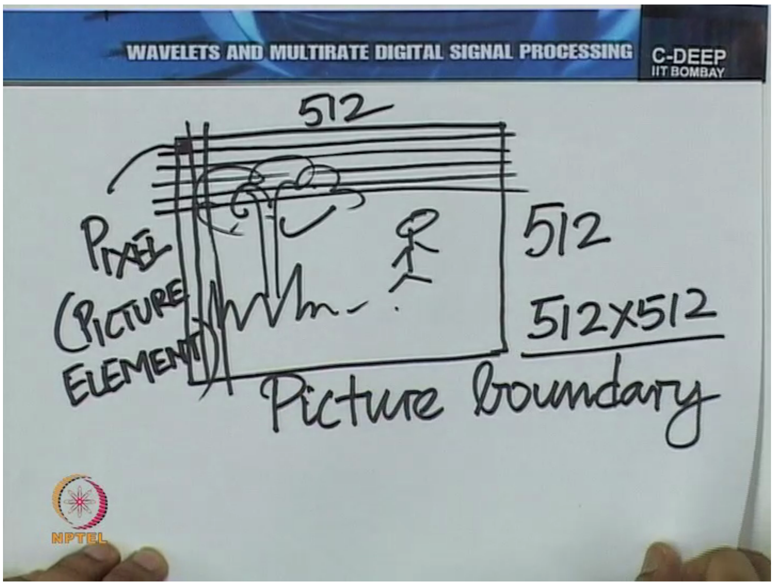

So you see let us assume that this is the picture boundary and I am trying to represent this picture on the screen, whatever that picture might be. So just for the sake of drawing, let me draw some kind of a pattern there, let us say you have a tree and some person standing there and forgive my drawing, but whatever it is, maybe some grass may be here. Now this is inherently a continuous picture. How do I represent it in the computer? I divide this entire area into very small subareas, so I visualize this being divided into tiny, what are called picture elements or pixels.

So each small area here is a pixel, a picture element so to speak. And there are for example, suppose I make 512 divisions on the vertical and 512 divisions on the horizontal, I say that I have a 512 cross 512 image, that many pixels and in each pixel region I represent the image by a constant.

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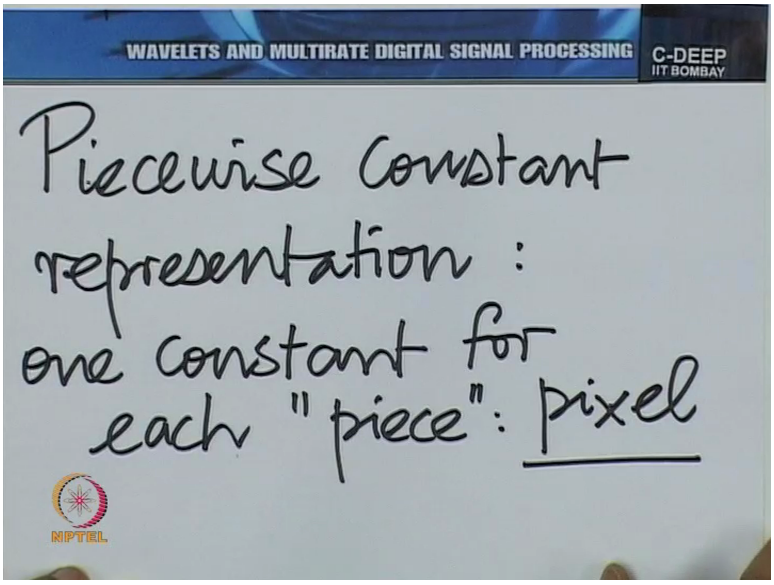

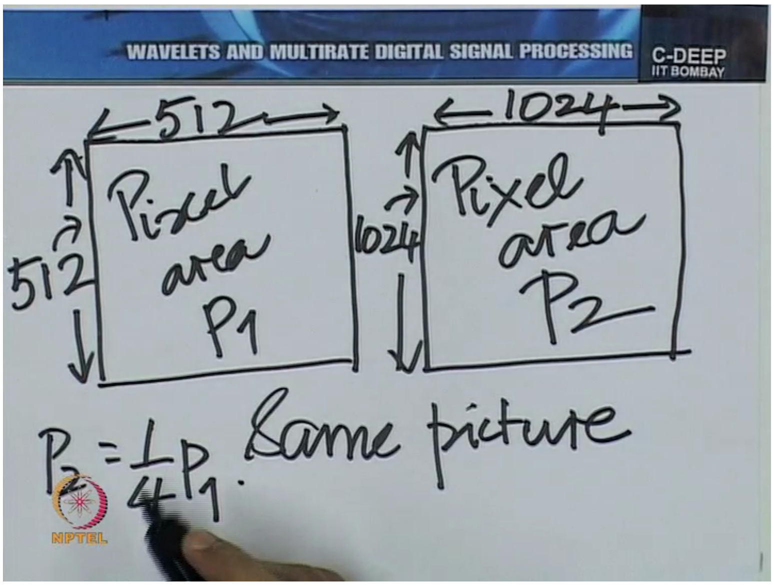

So the 1st thing to understand is there is a piecewise constant representation, let us write that down, there is a piecewise constant representation of the image, one constant for each piece and that piece is the pixel or the picture element. Now suppose I increase the resolution, so I go from a resolution of 512, so I take the same, what I mean is, I take the same picture, same picture. In this case I make a division 512 cross 512, in this case I make a division 1024 cross 1024.

Now obviously the pixel area here, let us say the pixel area here is P2 and the pixel area here is P1, it is very easy to see that P2 is one 4^{th} of P1 and therefore I have reduced the area by factor of 4. Naturally if I use a constant to represent the value or the, you know the intensity of the picture on each pixel here and do the same here, what you see in this picture is going to be closer to the original picture in some sense than what you see here.

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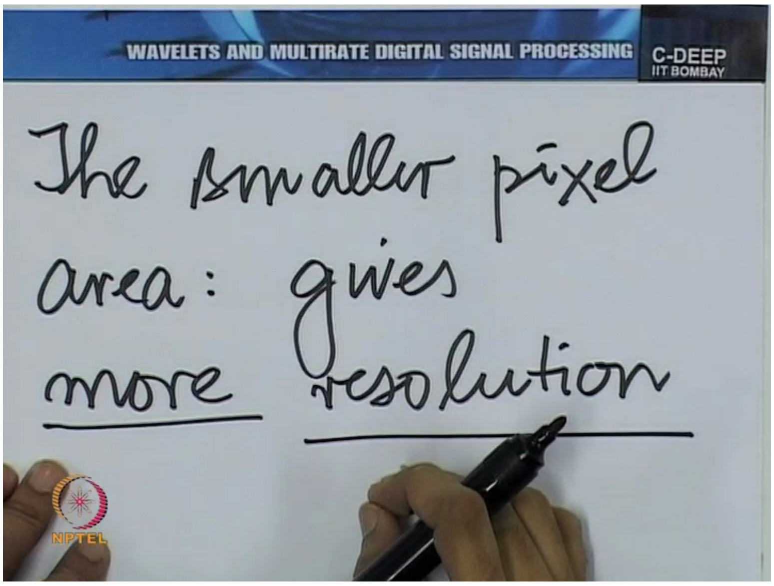

So in other words we can capture this by saying, the smaller the pixel area, the larger the resolution. Now this is the beginning of the Haar multiresolution analysis. The more we reduce the pixel area, the closer we are going to go to the original image. Even though this captures the idea that we are trying to build, it is not quite the idea of the Haar MRA. The Haar MRA does something deeper and that is what I am now going to explain mathematically in some depth. Now here I gave the example of a two-dimensional situation which apparently is more difficult than one-dimensional but it is easier for us to understand physical.

We can more easily relate to the idea of a piecewise constant representation in the context of images or pictures but the same thing could be true of audio for example. So you could visualize a situation though seemingly more unnatural where you record an audio piece by dividing the time over which the audio is recorded into small segments. Now let me show that pictorially, it would be easier to understand.

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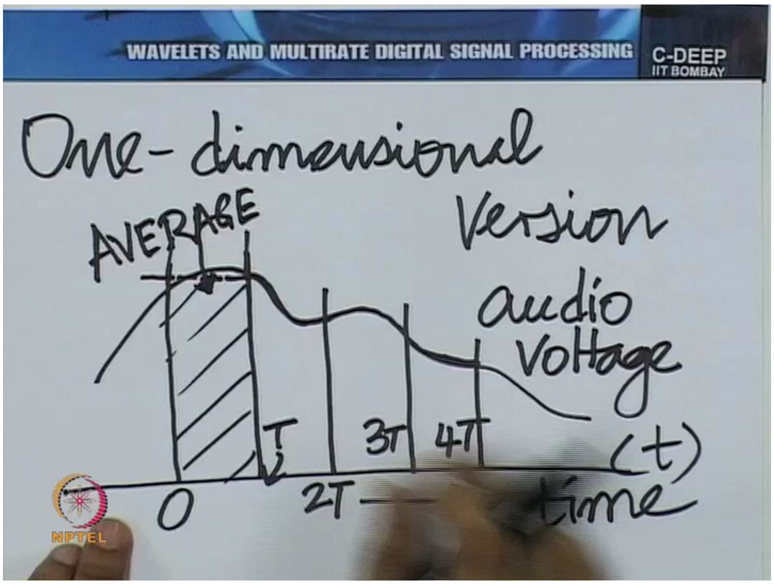

So suppose for example you had this waveform here, the one-dimensional version. So suppose I have, this is the time axis and I have this waveform here, assuming that this is the audio waveform, audio voltage recording, let us without any loss of generality assume that this is the 0 point in time, so let Time be represented by t and let this be the 0 point in time. Now let me assume that I divide this time axis into smaller intervals of size T here, this point is T, this point is 2T and so on.

I make a piecewise constant approximation, that means I represent the audio voltage in each of these regions of size T by one number. Now what is the most obvious number or what are the set of most obvious numbers that one can use to represent this waveform in each of these time intervals? For example, in this time interval, or for that matter in any of the time intervals, it makes sense to take the area under the curve and divide by the time interval to get the average of the waveform in that time interval and use that as a number to represent the function.

Here for example, you can visualize that the average will lie somewhere here, I am just showing it in dotted, so average. So intuitively, it makes sense to represent the voltage waveform in each of these intervals of size T by the average of that waveform in that interval, is that right? Let us write that down mathematically.

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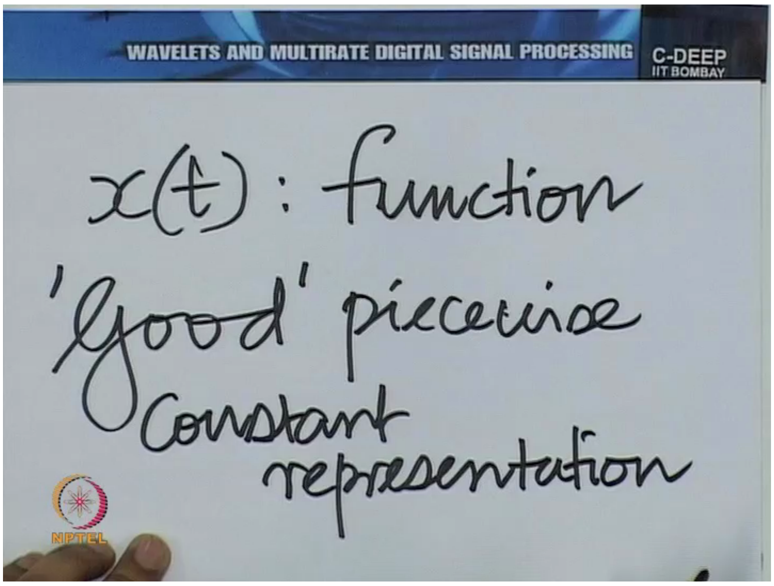

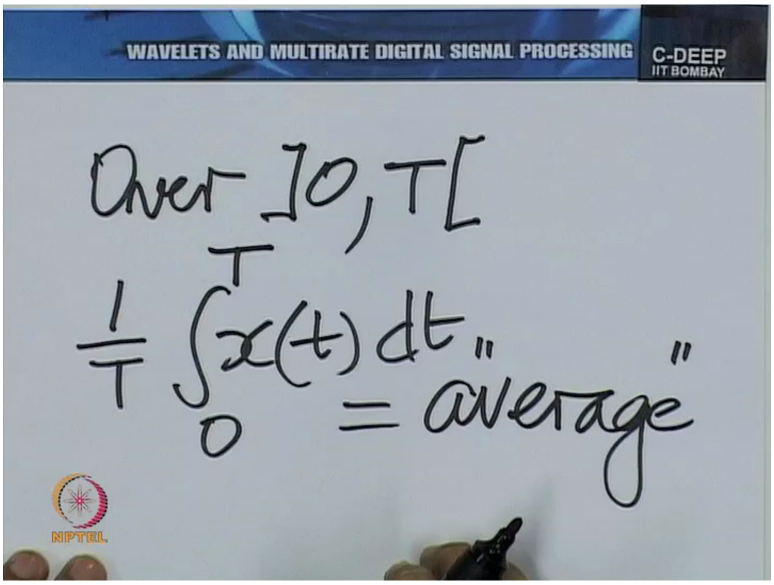

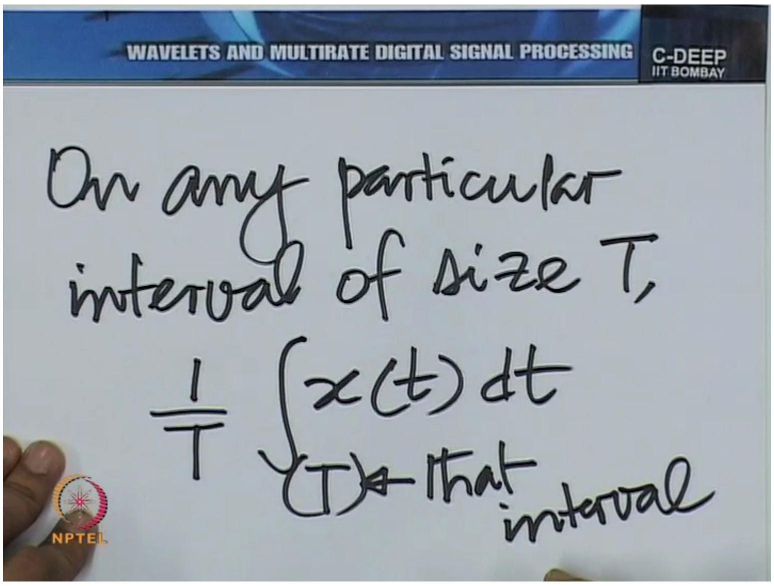

So what we are saying, if you have a function x of time, a good piecewise constant representation is the following. Over the interval of T, over the interval from say 0 to T, now you know strictly it is the open interval between 0 and T, the representation would be integrate x of t dt from 0 to T and divide by T, the average. Now of course on any particular integral of T, the same holds. So we say that on every interval of T, on any particular interval of T, of size T, the average would be obtained by 1 by T integral over that interval of T, when you write it like this, you mean that particular interval of T.

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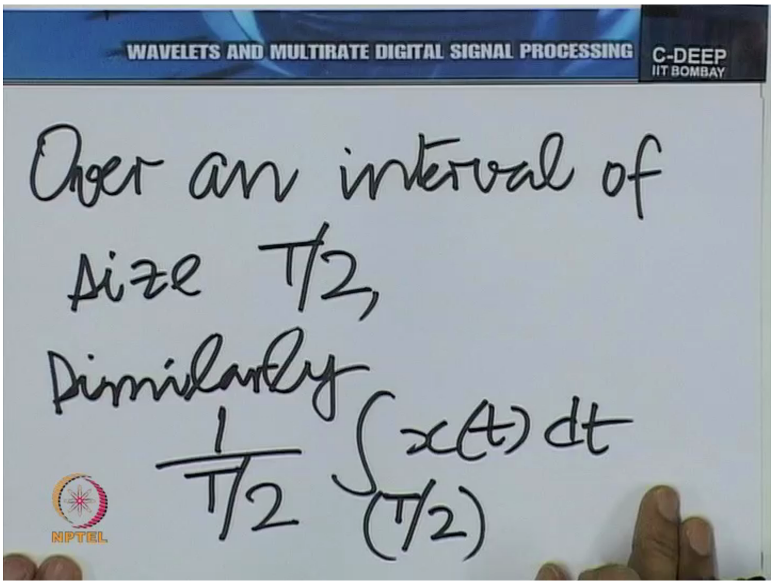

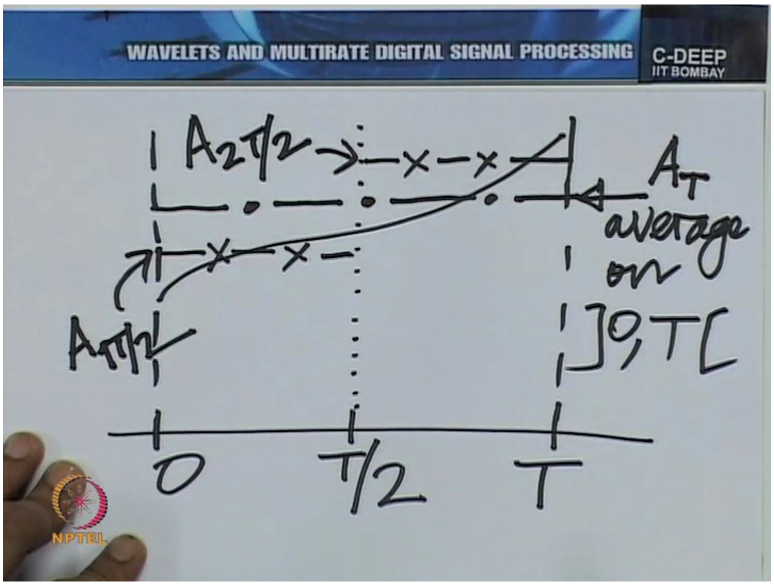

Integral of x t with respect to small t. This is a piecewise constant representation of the function on that interval of size T. Now the same thing could be done for an interval of size T by 2. So over an interval of size T by 2, you would similarly have 1 by T by 2 integral over that interval of length T by 2 x t dt. Now we are going closer to the idea of wavelets. Let us pick a particular interval of size T, in fact again without any loss of generality let us choose the interval from 0 to T and divide it into 2 sub intervals of size T by 2.

So what I mean is take this interval of size T, 0 to T, I am expanding it, so you have this function here over that interval, divide this into 2 sub intervals of size T by 2. 1st take the piecewise constant approximation on the entire interval of T and I show that by a dot and dash line. You can visualize the average will be somewhere here. So this is the average on the entire interval 0 to T. Now I take the sub intervals of size T by 2, so I have this sub interval of size T by 2, I use a dash and cross to write down the average there, so I have dash and cross, dash and cross here.

You can visualize that in this sub interval, the average will be somewhere here. And similarly in this sub interval, you could write down the average something like this. Now let us keep this a main, let us call this average A on T, let us call this average A1 on T by 2 and let us call this average A2 on the interval of size T by 2 and let us write down the expressions for each of these averages. What are the expressions?

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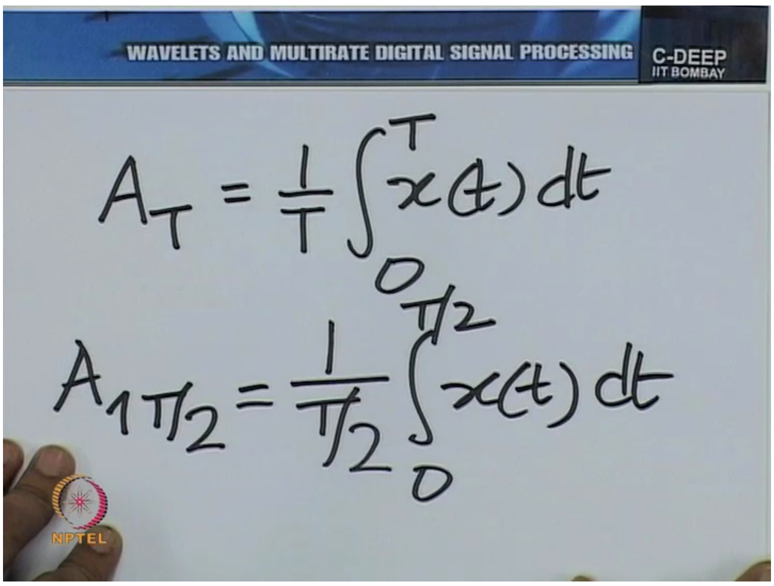

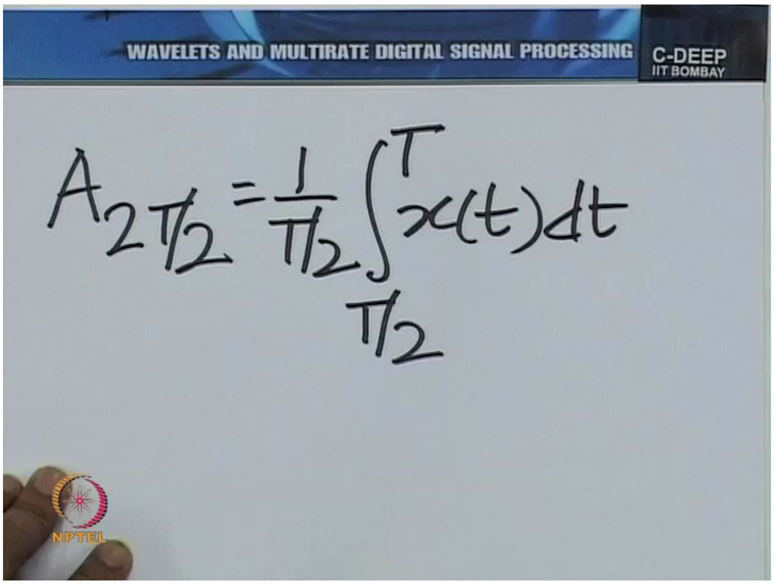

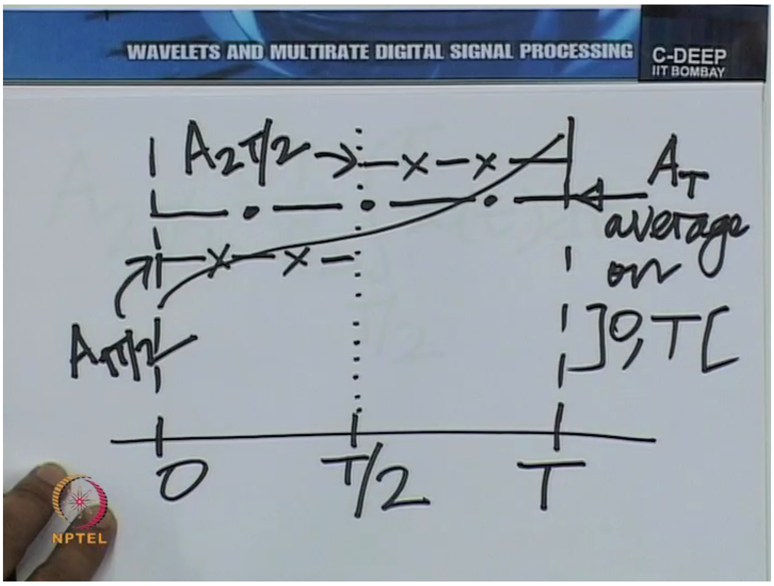

AT is obviously 1 by T integral x of t dt from 0 to T. A1 T by 2 is 1 by T by 2 integral from 0 to T by 2 x t dt. And similarly A2 T by 2 is 1 by T by 2 integral from T by 2 to T, x t dt. For convenience, let me flash all the 3 expressions before you once again. AT is the average over the entire interval of T, A1 T by 2, the average over the 1st interval of T Y2 with this expression and A2 T by 2, the average from T by 2 to T, the 2nd sub interval of size T by 2 with this expression.

And just to get our ideas straight, here again is the picture. Now the key idea in the Haar multiresolution analysis is to try and relate these 3 terms. So to relate AT, A1 T by 2, A2 T by 2 and it is in that relationship that the Haar wavelet is hidden. So what is the relationship? Now the relationship is very simple, I mean all that we need to do is to notice that we have divided the integral from 0 to T into 2 integrals over 0 to T by 2 and T by 2 to T and then remember there is a slight difference in the constant associated.

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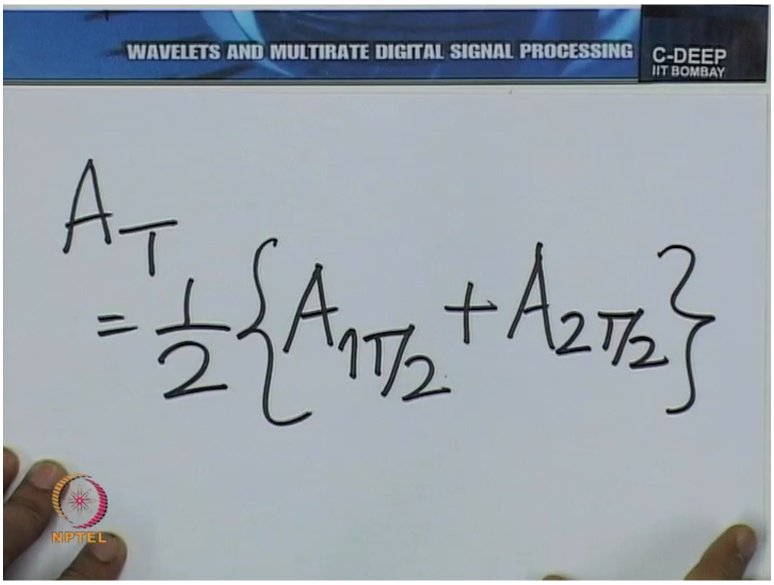

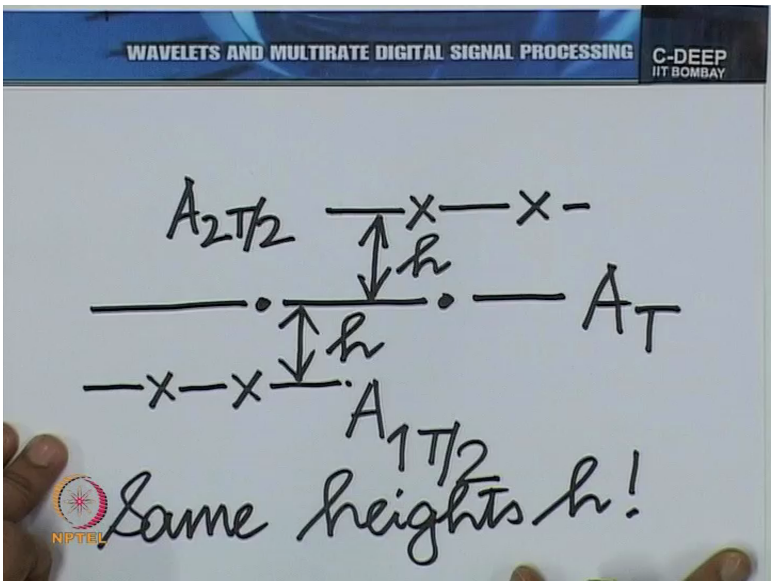

So we have a constant of 1 by T in AT and a concept of 1 by T by 2 in A1 T by 2 and in A2 T by 2 whereupon we have this very simple relationship between the 3 quantities. AT is half, I leave it to you to verify, it is half of A1 T by 2 + A2 T by 2. And how do we interpret this, let me try and, you know kind of focus just on this relationship, in other words, let us just focus on these 3 constants and make a drawing there. So what we are saying is something like this, I have this AT there, I have this A1 T by 2 here and I have this A2 T by 2 there.

And we are saying this plus this by 2 gives you this, in other words this is as much higher above AT as this is lower, what we are saying is these 2 heights are the same, that is what this relationship implies. Now another way of saying it is, if I were to make a piecewise constant approximation on intervals of size T, how would they look? So let me just sketch them.

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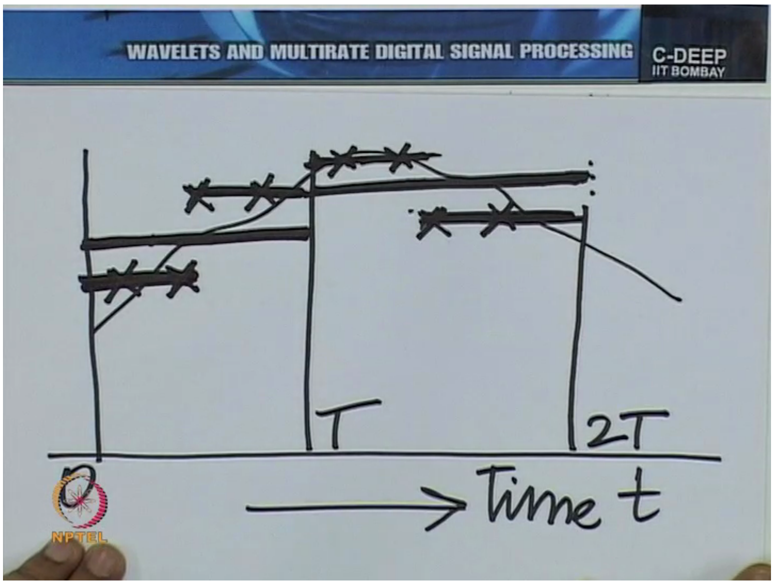

So I take this function once again here, I have this function here, I have divide it into sizes of, intervals of size T, let me show just 2 intervals for the moment. So this is how the function would look when you would make a piecewise constant approximation on intervals of size T and when you do it on intervals of size T by 2, it would look like this, something like this. Now, this is a function, so let me highlight it, now let me darken it. This is in its own right a function, piecewise constant function, the one which I have darkened here.

And this is in its own right, the darkened part is in, is in its own right an approximation to the original function here. Similarly let me now darken this and put some other mark on it, let us keep the crosses, so I will darken this and I will put crosses on it, this is also another function. So dark and cross function is another function, that is in its own right an approximation too. So let us give them names.

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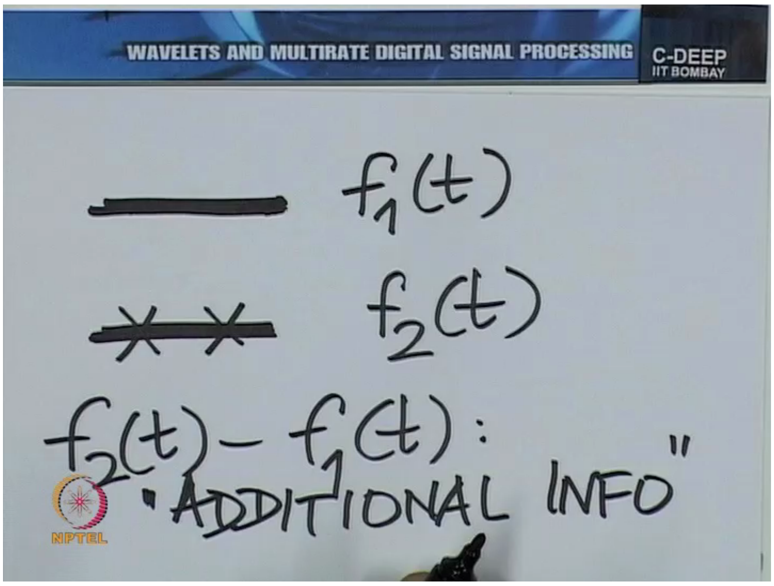

Let us call this function, just the dark one as f1 t and let us call this function, the one which we have shown with the dark and cross as f2 t, f2 t - f1 t is like additional information, what we are saying is, instead of a piecewise constant approximation on interval of size T, when we try and make a piecewise constant approximation on intervals of size T by 2, we are bringing in something more. Go back to the original case of the picture, we have inherently underlying a continuous two-dimensional picture, a continuous two-dimensional scene.

When we make an approximation with a 512 cross 512 resolution, then we have actually brought in one level of detail, when we go to 1024 cross 1024 representation, the level of detail is 4 times more. What is the additional detail that we have got in going from 512 cross 512 to 1024 cross 1024? In effect when we take this difference f2 t - f1 t, we are answering that question.