Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
Professor Vikram M. Gadre.
Department Of Electrical Engineering.
Indian Institute of Technology Bombay.
Week-3.
Lecture- 9.1.
Relating Fourier Transform of Scaling Function to Filter Bank.

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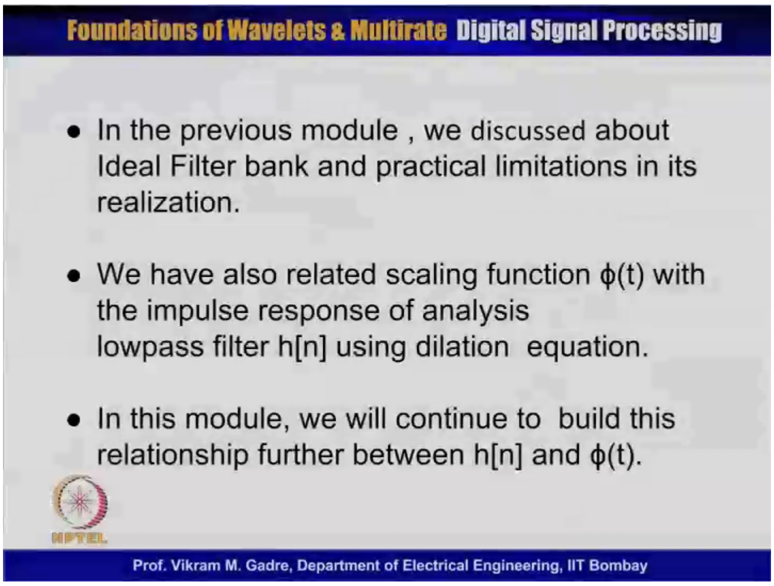

A very warm welcome to the lecture on the subject of wavelets and multi-rate digital signal processing. We continue in this lecture to build further on the relationship between the filter bank and the scaling function and wavelet functions. Let me put before you some of the important conclusions that we have drawn towards the end of the previous lecture. We had said that there is a generic dilation equation that relates the filter bank to the scaling function and the filter bank to the wavelet.

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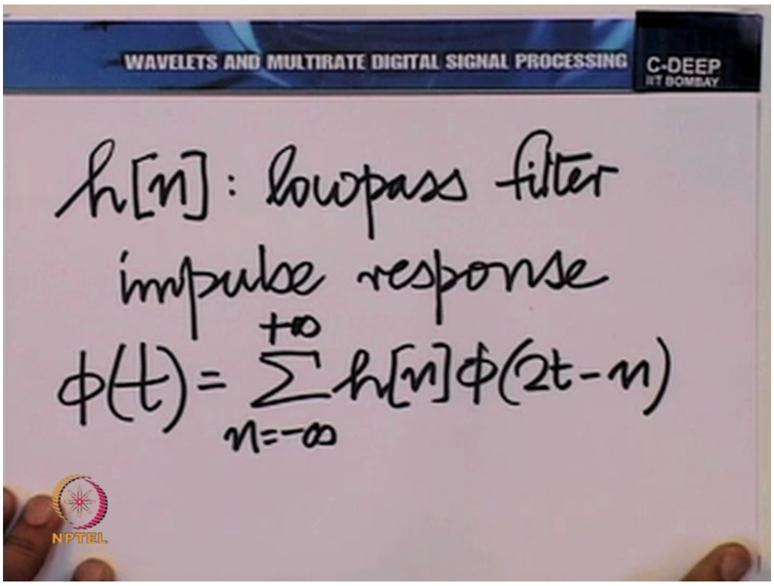

In fact, if n is the low pass filter impulse response we had said that phi t obeys a dilation equation like this and as far as the wavelet is concerned, we had said that if we take the high pass filter in the filter bank…

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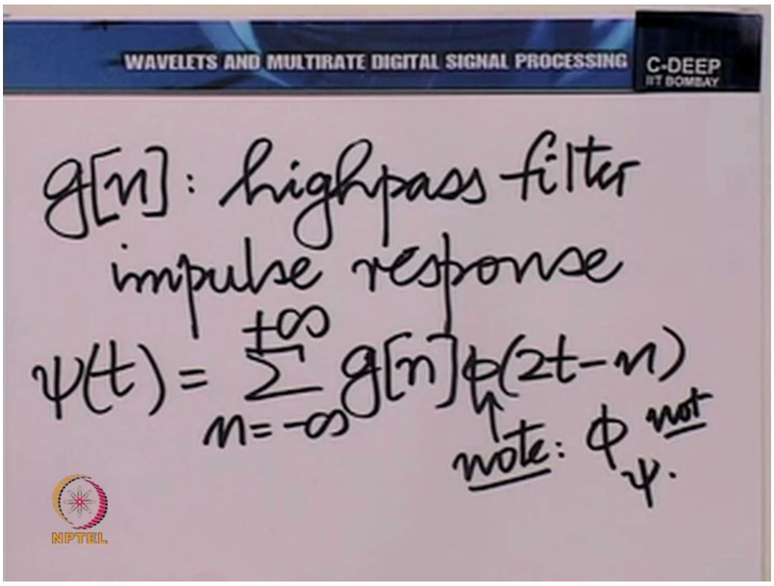

So if G of n is the high pass filter impulse response then psi of t is summation n going from - to + infinity. G of n phi 2t – n. Again phi, not psi. So this is not surprising. What we said was that after all phi t belongs to V1. Psi t also belongs to V1. So therefore, both phi t and psi t should be expressible in the basis of V1. And that is what we have essentially written down.

What is noteworthy is that the coefficients of the impulse response of the low pass filter and the high pass filter act as the coefficients in the expansion in terms of the basis.

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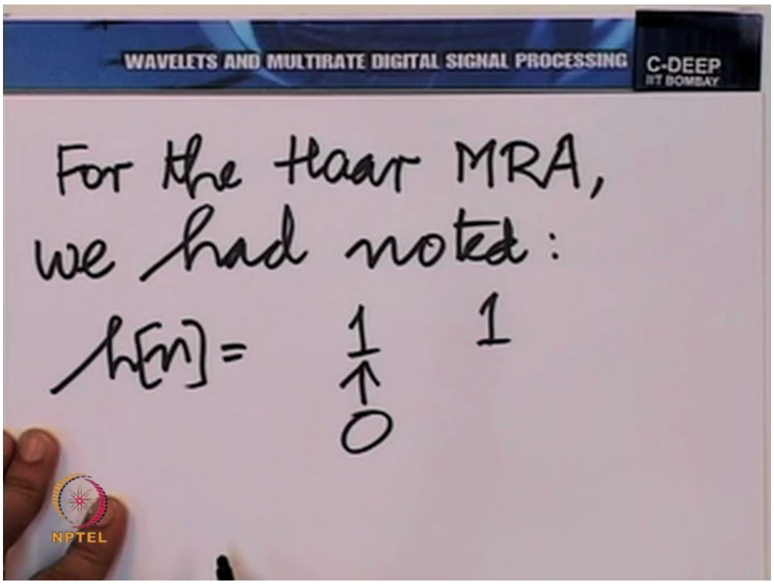

Now in particular for the Haar MRA, we had noted… Hn is this sequence. Recall that this is a way of denoting finite linked sequences and this means at n equal to 0 the value of the sequence is 1 and then points after and before, take values as shown. So for example here, if this is n equal to 0, this is going to be n equal to 1 and of course the other points which are not shown are automatically 0.

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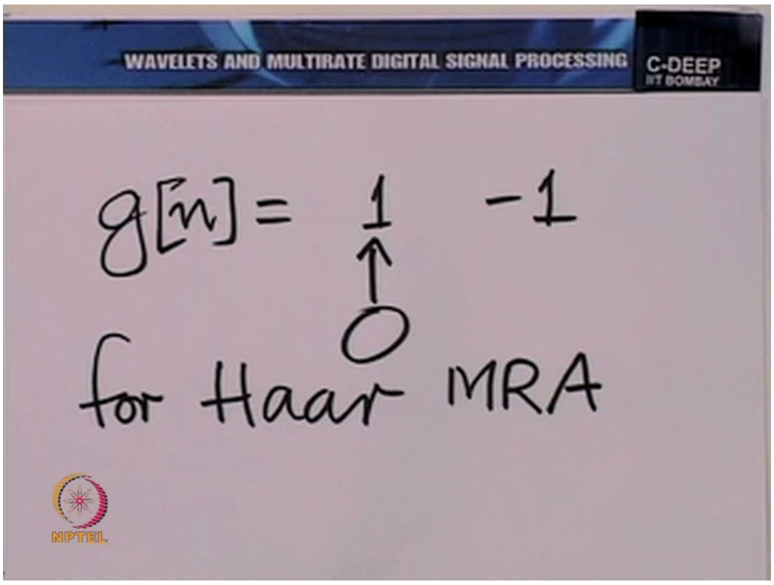

Gn is this for the Haar system and in fact we said that what these equations told us was something much deeper than the containment of phi t and psi t in V1. In a sense, these equations tell us how to go from the filter bank to the wavelet and from the filter bank to the scaling function. We have just hinted at this in the previous lecture but now, we make that idea very very concrete. So let us begin by looking at the Fourier domain. As I said last time, we need to take the Fourier transform because that is where we shall see something very interesting. So let us take the 1st of the 2 dilation equations.

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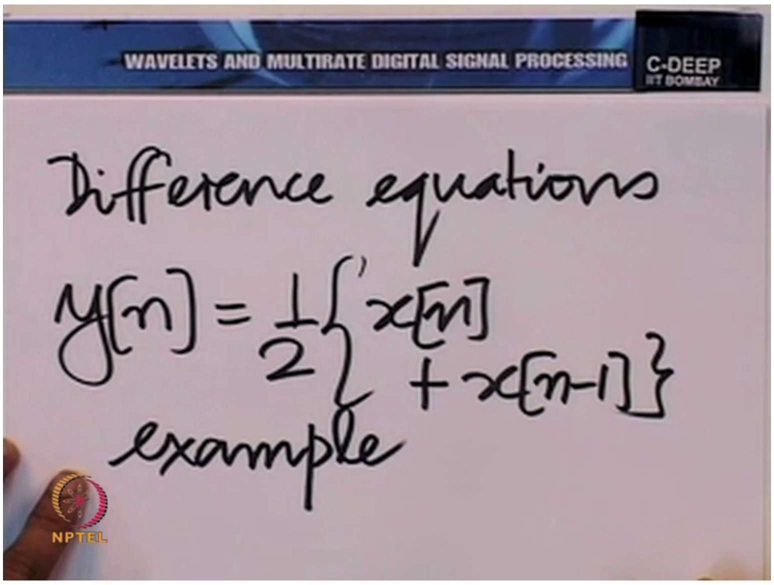

You know, incidentally, just as you have differential equations you have difference equations, you have dilation any questions here. You know we often encounter differential equations. Example could be Yt is A1 dx t dt let us say + A2 xt, it is a differential equation. We have difference equations. For example, Y of n is half x of n + x of n - 1 is an example of a difference equation which describes a discrete system. And now we have a dilation equation. This is a new class of equations.

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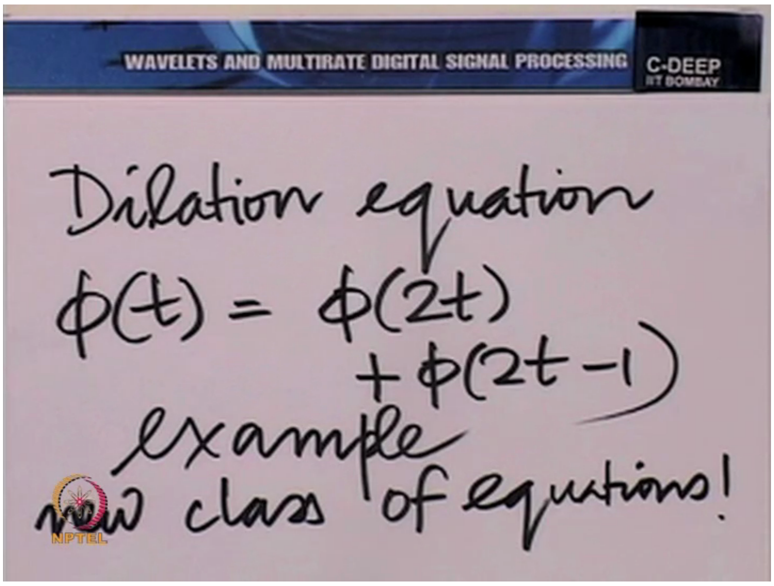

It is a new class of equations. And this new class has a reason from our discussion of wavelets, in fact from the relation between wavelets and multirate filter banks. Anyway, put this a little aside let us come back to the issue of relating that filter completely in generative terms to the scaling function and the wavelet. So let us take this very general dilation equation.

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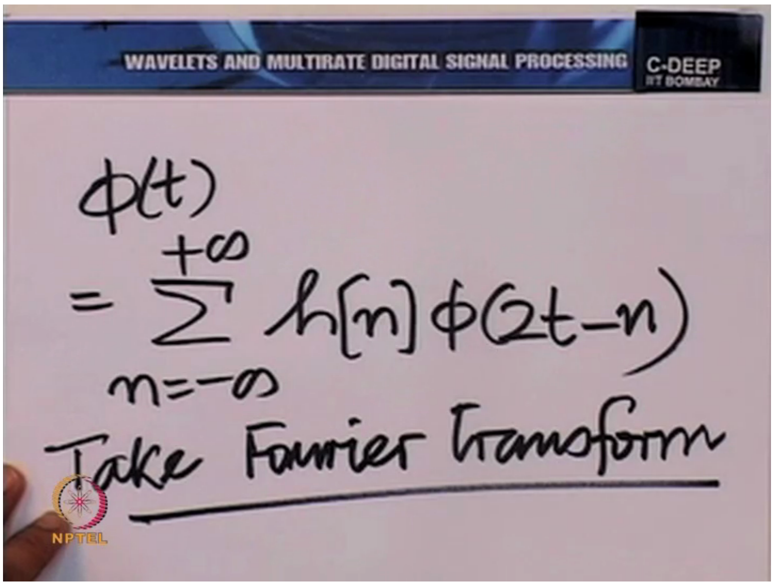

Phi of t is summation n going from – to + infinity H of n phi of 2t - n. And we take its Fourier transform on both sides.

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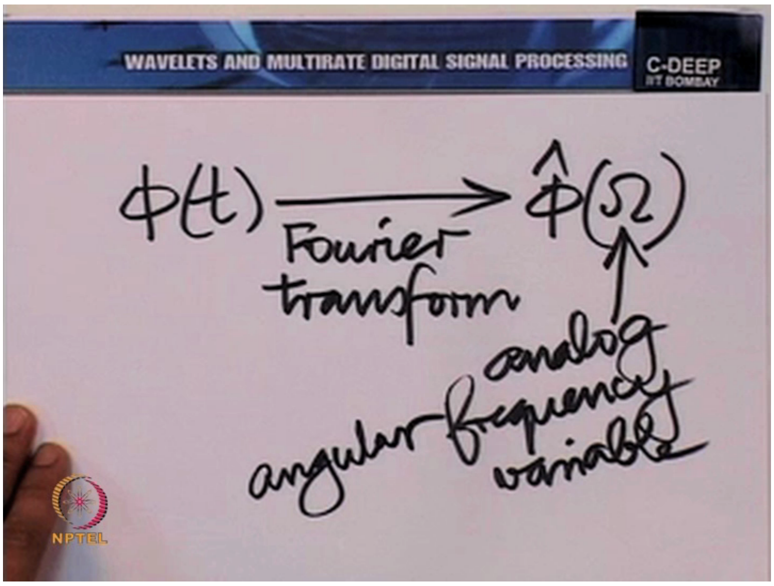

Indeed, let us denote the Fourier transform of phi t as phi cap omega. Now remember, this is the analog frequency variable or the frequency variable corresponding to the continuous time context. So I should say analog angular frequency variable to be very precise. And we know the relation between phi t and phi cap omega.

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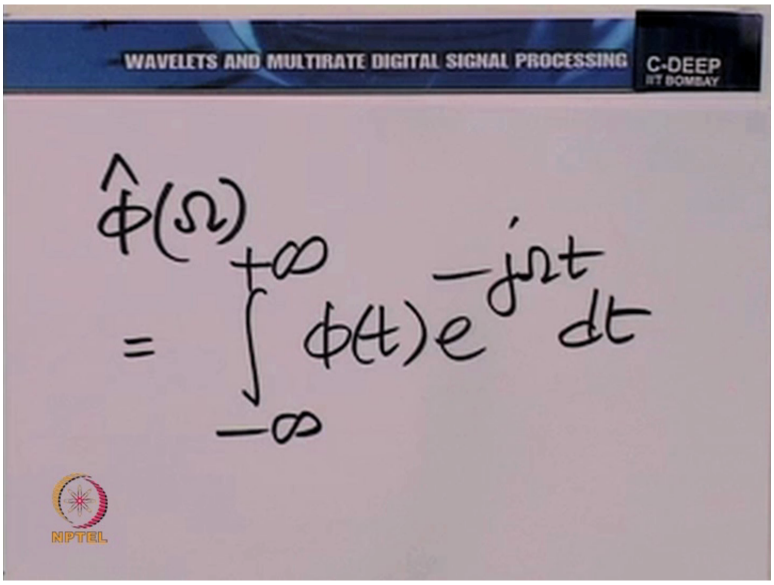

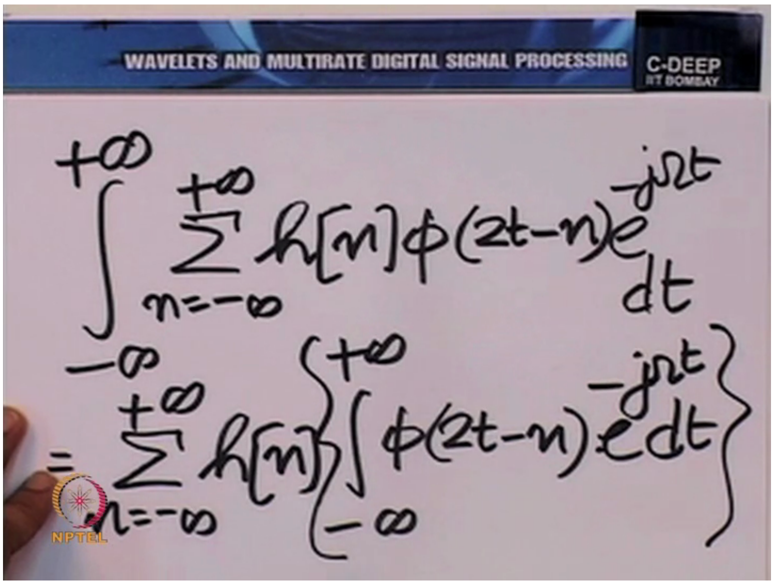

So we have phi cap omega is integral from – to + infinity phi t e raised to power – K omega t dt. And we operate this on both sides. So we write down, integral from – to + infinity summation H n over n times phi 2t - n e raised to power – j omega t dt and integrated all the way from – to + infinity. So we have this integral here.

Now let us you see if this converges which it does because it is the Fourier transform of phi t we could interchange the order of summation integration. So we would have, this is equal to summation n going from – to + infinity, H of n integral from - to + infinity phi 2t - n e raised to power – K omega t dt. So we have isolated the part that operates with dt here. Let us evaluate that part separately.

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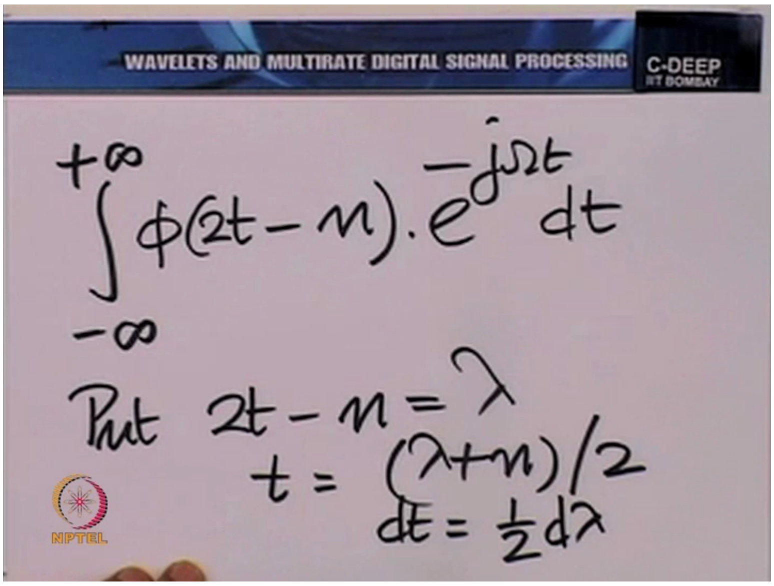

Put 2t - n equal to Lambda whereupon we have t is equal to Lambda + n by 2. And of course one can also write down dt, dt is essentially half d lambda.

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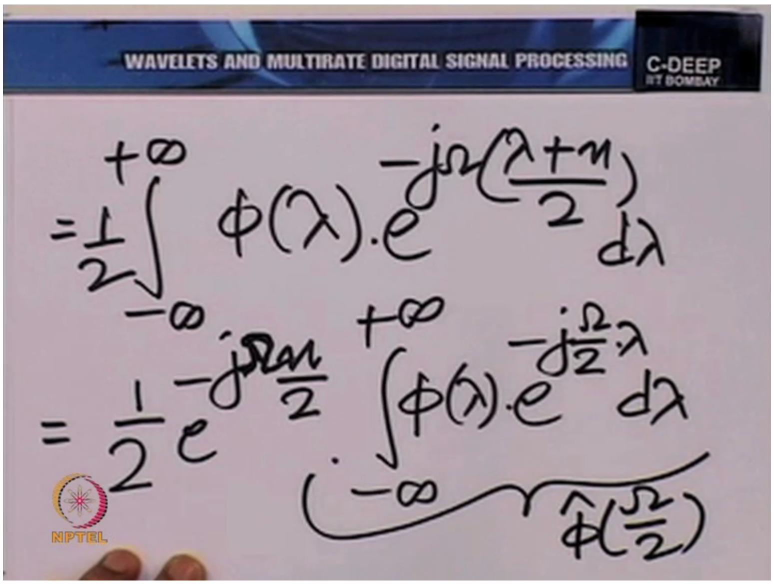

And substituting this, we have the integral becomes integral from – to + infinity phi of Lambda e raised to power –j omega Lambda + n by 2 d lambda and a half outside. And we can do a little more work on this. So we keep the terms dependent on Lambda inside and we have J Lambda or rather J capital omega here, n by 2 emerging outside. This is J capital omega n by 2 and this is – infinity to + infinity phi Lambda e raised to power –j omega by 2 Lambda d lambda and this is familiar. This is essentially phi cap evaluated at omega by 2 as one can see, the Fourier transform evaluated at the point capital omega by 2.

So now we have a very beautiful relationship. You see what we are saying in effect now is that we can express the Fourier transform phi cap omega in terms of itself which is not surprising because you have a recursive dilation equation on phi t. So there is a corresponding dilation equation on the Fourier transform. What is that dilation equation?

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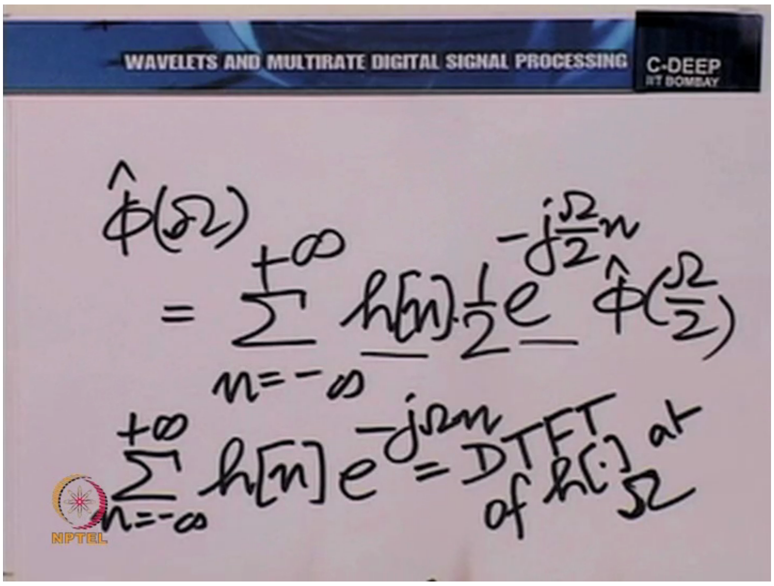

That dilation equation is phi cap omega is summation n going from – to + infinity Hn times half e raised to power –j omega by 2 n times phi cap omega by 2. Now you know this part of the summation, the part of the summation that involves n is familiar to us again.

Indeed we note that the summation n going from – to + infinity H n e raised to power – j omega n would be essentially the DTFT the discrete time Fourier transform of H. Evaluated at capital omega. So all that we have done in this expression is that we have replaced capital omega by capital omega by 2 from this point and we will give it we will again use the notation that we have been using.

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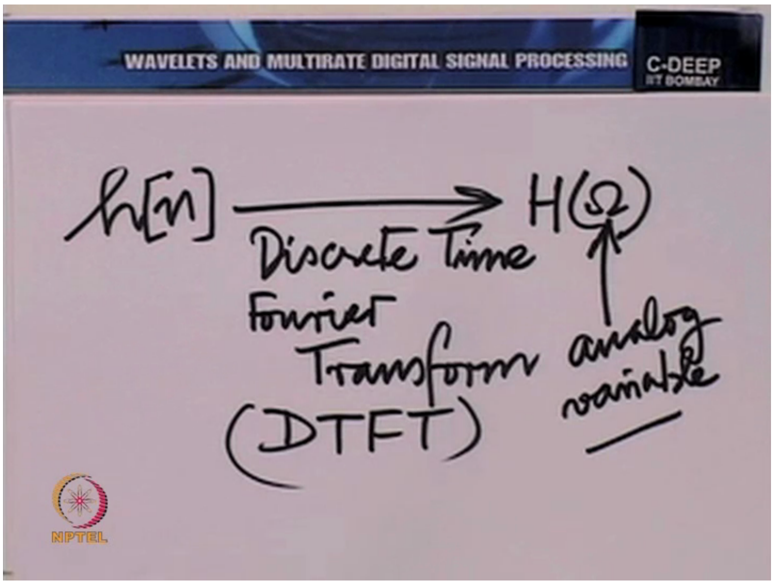

So we are saying, if H of n has the discrete time Fourier transform or DTFT given by capital H of omega. Now note, here I am using the continuous or analog variable. That is because I want to retain my discussion in the analog domain or in the continuous time domain. So I'm substituting small omega by capital omega here for the sake of consistency in notation. And if Hn has the discrete time Fourier transform given by capital H of capital omega then what we have here is the following dilation equation.

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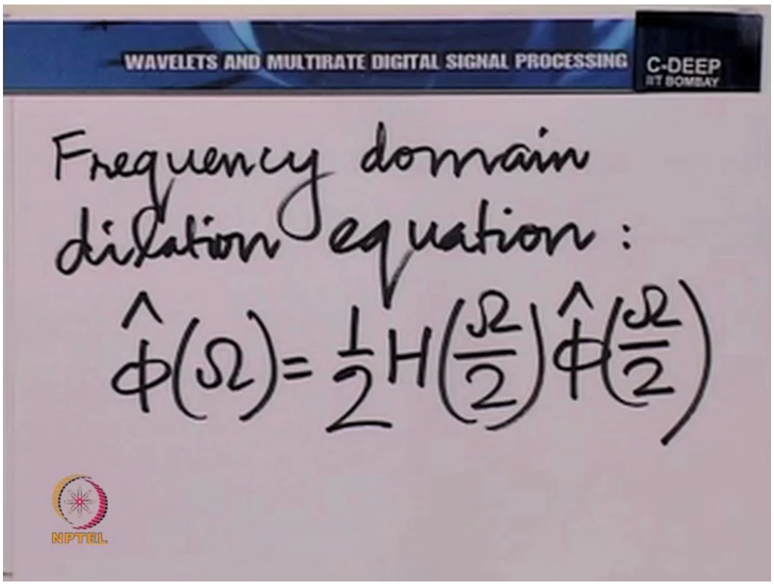

The frequency domain dilation equation this phi cap capital omega is half capital H evaluated at omega by 2 phi cap times evaluated at capital omega by 2. You see the beauty is that a dilation equation which involved summation over many terms has now become a dilation equation involving a simple product. How do we interpret this? The Fourier transform of phi t is the same Fourier transform evaluated at omega by 2.

So, evaluate at omega is equal to evaluate at omega by 2 times DTFT. Now the beauty is what we have done here to go from phi cap omega to phi cap omega by 2 can be done to go one step lower. So the same equations can be rewritten at capital omega replaced by capital omega by 2.

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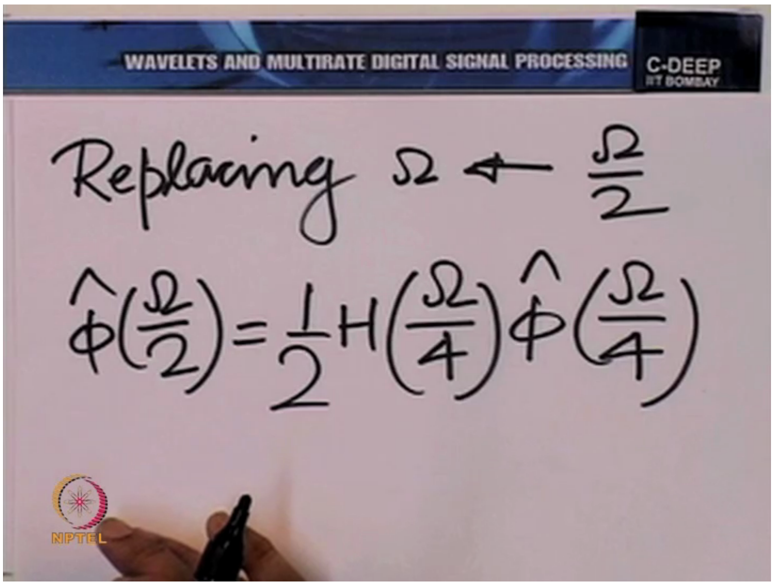

And doing that, we would have phi cap evaluated at phi cap omega by 2 is half evaluated omega by 4 times phi cap evaluated at omega by 4. So now we have a recursive process. Everytime you have phi cap omega by 2, you replace it in terms of a product of phi cap omega by 4 and then a DTFT. So ultimately we have something like this.

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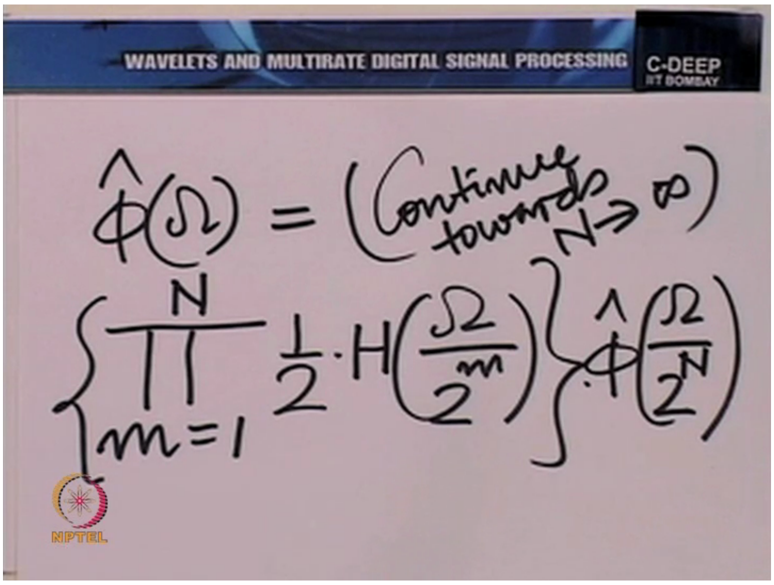

We have phi cap omega is like a product. It is a product n going from 1 to n. Capital N if you like. Half H omega by 2 raised to power of n. This product and then multiplied by phi cap omega by 2 raised to power capital N. So we have a product of these discrete, so-called discrete time Fourier transforms here. The only catch is now we need to use the analog frequency variable because we are dealing with analog frequencies here and here. Now we can take the limit or continue towards n going towards infinity. Now what is going to happen when you make capital n go towards positive infinity here? Any finite capital omega is going to be taken closer and closer and closer to 0.

Again if you wish to be very finicky, you should use the opponent-proponent model where you say no matter how small I ask this argument to be, I can make it small enough and so on. But I think we understand well enough that you can make capital N as large as you desire and you get a larger and larger number of terms in this product and you can take this argument to as small a value as you desire whrerever capital omega is finite.

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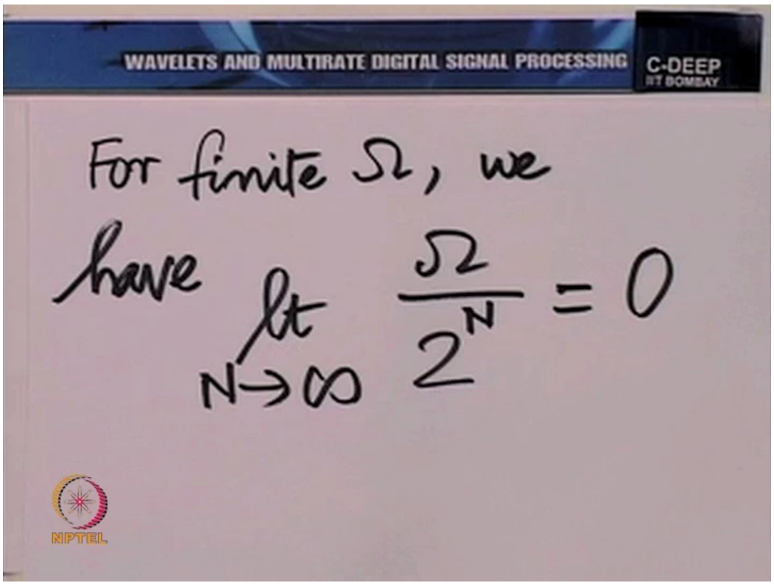

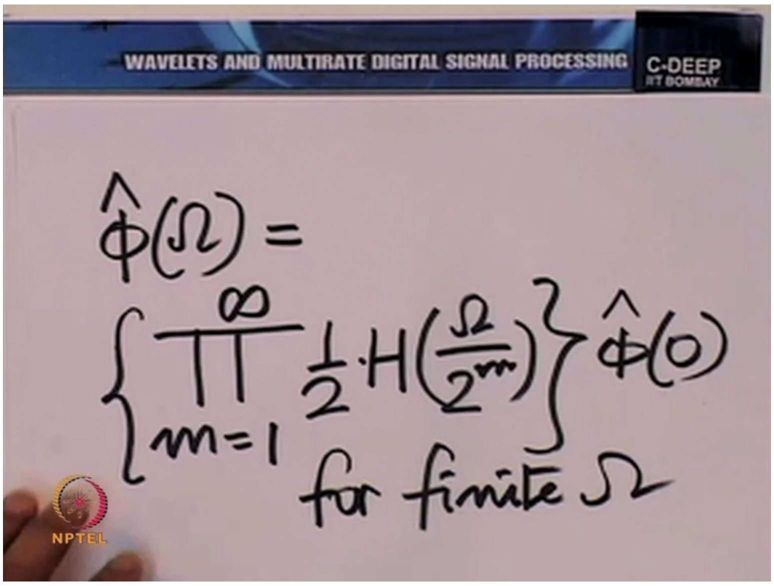

So for finite capital omega, we have the limit as capital n tends to positive infinity of capital omega divided by 2 raised to power of n equal to 0. So therefore at least on the finite frequency axis, the left-hand side is equal to the right-hand side well and the right-hand side has essentially the Fourier transform of the left-hand side at the point 0. So what do we have here?

Let me write that down mathematically. Phi cap omega therefore is essentially a product, m going from one to infinity, positive infinity. Remember, the half occurs with each of these terms in the product. Now we have to be careful and say for finite capital omega but that is not a very serious problem.