Foundations of Wavelets, Filter Banks and Time Frequency Analysis.

Professor Vikram M. Gadre.
Department Of Electrical Engineering.
Indian Institute of Technology Bombay.
Week-1.
Lecture -1.2.

Origin of Wavelets.

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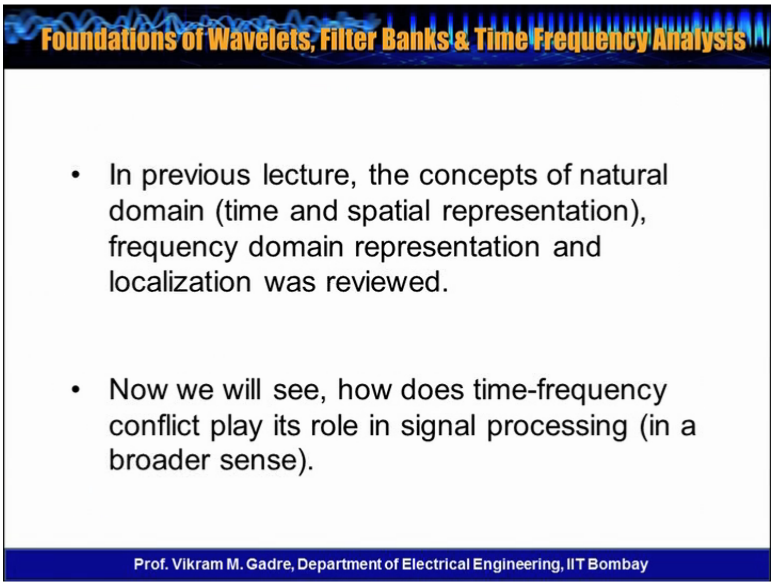

Foundations of Wavelets, Filter Banks & Time Frequency Analysis

- In previous lecture, the concepts of natural domain (time and spatial representation), frequency domain representation and localization was reviewed.
- Now we will see, how does time-frequency conflict play its role in signal processing (in a broader sense).

Prof. Vikram M. Gadre, Department of Electrical Engineering, IIT Bombay

Now this is where the whole story starts and perhaps is the most fundamental inspiration for this course on wavelets and multirate digital signal processing. I will 1st talk about where wavelets come from. Well, Fourier transforms deal with waves, sine waves to be more precise, we recognise the merits of sine waves, sine waves have many nice properties, for one, they occur naturally in many different circumstances. For example, an electrical engineer recognises the sine wave as naturally emerging from an electricity generation system where there is electromagnetic induction.

If there is a perfectly circular rotating device in a magnetic field and if all is perfect in the generating system, we would be generating a perfect sine wave from the brushes. So sine wave is a good idealisation to work with for electrical engineer, but that is not the only point. Sine waves are in some sense the most analytic, the smoothest possible periodic functions. They also have the power of being able to express many other waveforms, that means they form a good basis from which many other waveforms can be generated.

There are many other nice mathematical properties. If I take 2 sine waves with the same frequency, possibly different amplitude and phases, I would get back the sine wave of the same frequency, of course with a 3rd amplitude and phase. If I differentiate a sine wave, I get back sine wave of the same frequency and naturally if I make a combination of these operations, namely differentiation or even integration for that matter and linear combinations and if I restrict myself to sine wave of a particular frequency, I remain within the domain of sine wave of that particular frequency, this is something beautiful about sine waves.

It is not easy to find waveforms which obey this. And as I said before, sine waves form a good basis, so they form good building blocks for being able to express a wide variety of signals. For all these reasons, the sine wave has been very popular in the 1st course on signals, systems, discrete time signal processing and what have you, communication. But as I said right in the beginning of this lecture, one of the reasons why we are not so happy with sine waves is that they need to last forever, begin from - infinity and go all the way to + infinity.

Otherwise, if you truncate a sine wave, if it is one-sided for example, and if you look at what happens to an electrical system when you apply a one-sided sine wave, by a one-sided sine wave I mean, sine wave which starts from some point, 0 up to then it starts from some point and continues afterwards the sine wave, the response is very different from what would be the response for a sine wave that starts at - infinity. In general, there would be transients which are not periodic and then all these beautiful properties of sine waves and their responses go away.

So if I really wish to be able to apply the basic principles of signals, systems and discrete time processing that I have learned in a basic course, I need something unrealistic, I need a sine wave that lasts forever. How can I be more realistic in my demands? By accepting that I cannot deal with waves but it is more appropriate to deal with wavelets, so that is where the word wavelet comes from, small waves, waves that do not last forever, functions that are not predominant forever, they are significant in a certain range of time, perhaps only exist in a certain range of time and insignificant outside.

So we have a certain support over which one might want to use them, one might want to consider them to exist and so on. A much more realistic assumption and that really is what we call a wavelet, not a wave but wavelet. For example, you could if you wish, think of truncating a sine wave to a rectangular region, that means suppose we take a sine wave to last from 0 to 1 millisecond as an example. It could be an example of a wavelet, later we will see that this is not a very good example, but yes, in principle, a wavelet, a wave that does not last forever, a simplistic explanation of what wavelets means.

But that is not the whole story, our whole objective was to talk about the other domain too. So going back to the example of audio signals, if I thought of the audio signal as comprising of many sine waves to come together to form an audio piece, then I wish to be able to do something simultaneously in 2 domains and that is the key idea here. So for example, to put in plain language, I should be able to say when there was this 2 second audio clip, out of which there were 5 notes being played, each note was played for different intervals of time, namely the 1st note was played for 0.4 seconds, the 2nd note was played for 0.7 seconds, the 3rd note only for 0.2 seconds and so on.

So I need to be able to segment in time but one I am talking about being able to identify notes, I am also talking about being able to segment in frequency and lo and behold, that is where the conflict arises. A very basic principle in nature sense, if I wish to be able to segment in time and frequency simultaneously, I am going to run into trouble, nature does not allow it beyond a point. And that is something very fundamental. It pops up in many different manifestations in different subjects.

In modern physics, they call it uncertainty, the uncertainty of position and momentum, in signal processing we call it uncertainty, the uncertainty of the time and the frequency domain. So to put it simply, though not very accurately, the shorter you play a note, the more difficult it is to identify it, not very far from intuition. If you play a note for a long time and listen to it for a long time, you are likely to be able to identify it better, commonsense tell us that. But what commonsense does not tell us is that you can never quite go down to identifying one particular frequency precisely.

So if I wish to be able to come down to a point on the time axis, then I need to spread all over the frequency axis and if I wish to be able to come down to a point on the frequency axis, I need to spread all over the time axis. That is of course the strong version of this restriction, but there is a weaker and a little more subtle version and that is as follows. Even if I am not quite interested in coming down to a point on the time axis, I am content with being in a certain region, so I say, as I said in the 1st 0.4 seconds out of the 2 seconds clip, I was playing note number 1, that is some frequency number-one, I would be able to say this at least to a certain degree of accuracy, that is what I am trying to point out here.

What the principle of uncertainty tells us is that this can be done to a certain degree of accuracy, you can identify that note to a certain degree of accuracy. Well the uncertainty also tells us in a more subtle form is that if I even choose to relax to a certain region of time, so I say, well, in this region of time, tell me the region of frequency which were predominant, even then there is a restriction on the simultaneous length or measure of the time and frequency regions.

And of course they have a tussle with one another, the smaller I make that time region, the larger than frequency region becomes, that means the more I want to focus in time, the less I am able to do so in frequency. This is indeed something that arouses a lot of thought. It may seem something far from our interest at 1st glance but when we look at it carefully, we realise it is something very fundamental to what we often desire and that is what I am now going to explain to you with a couple of more examples.

We live in an age where we use mobile telephones, in fact more fundamentally digital communication. What are we asking for in digital communication when we look at it from a signal or system perspective transform domain perspective or time and frequency perspective? Going right down to brass tacks, what we are asking for in digital communication is I should be able to transmit a sequence of bits, binary values, 0 or 1. And how do I transmit that sequence of binary values? I choose maybe one or 2 possible waveforms in the simplest scheme.

So corresponding to 0, I have one waveform, corresponding to the bit 1, I have a different waveform. To make life simple, the 2 waveforms have the same time interval. So for example we talk about, I mean all of us hear about computer networks and they talk about the speed of the network. They say, well, this network can operate at a speed of 1 megabit per second, what does it mean? It means that in 1 second I can transmit 10 raised to the power of 6 bits, so you have 1 microsecond to allow each bit to be transmitted.

Give it a thought, here we are talking about time, what are we saying about frequency? Now let us go to the mobile communication context. I have so many different mobile operators, obviously each operator will want its own privacy, so what is being communicated on the network of operator 1 should not interfere with what is being communicated on the network of operator 2. Now where is this separation going to occur, not in time, after all there are many different people simultaneously using mobile bought from both of the operators, so the separation cannot be in time.

We may argue that separation can be in space, so in one region you may have mobile from one operator and in other region, in space I mean, mobiles from the another operator, so far so good. But that is also not always true, it is very common to see mobile is purchased from different operators operating in the same room, so there is no separation in time, no separation in space. So where is the separation then? The separation has to be in a domain which is not so easy to see but once we have done a course on signals and systems, reasonably easy to understand and that domain is frequency.

So we say, well, operator 1 has the bandwidth allocated to him, operator 2 has the other bandwidth allocated to him. Now, when we say this bandwidth, maybe a certain region of the frequency axis of size let us say 2 megahertz, when we say this region of 2 megahertz is allocated to operate 1 and another region of 2 megahertz is allocated to operator 2, are we not talking about a segmentation in a different domain? In fact there we are talking about simultaneous segmentation.

We have a segmentation in time because you want to transmit different bits in different time segments and you want to have a separation in frequency because what is transmitted by operator 1 should not interfere with what is transmitted by operator 2. So here is a very common, though not so obvious example, of simultaneous desire of a localisation in time and frequency. And then the audio example which is of course little more obvious, little easier to understand, this example is equally common, at least in the scenario today but perhaps not so easy to understand.

But a little reflection makes it very clear to us, there is a desire to localise in 2 domains simultaneously. Well, let us go to a biomedical example. Very often when one analyses an electrocardiographic waveform, what one wishes to identify are features in the ECG signals. Now I do not intend to go into the medical details, but there are different segments in a typical ECG signal, they are often indexed by letters P, Q and so on. Without meaning to focus on specific details of an ECG signal, let us try and understand the connection to time and frequency localisation.

When we talk about an ECG signal, all features are not of the same length in time, some features are kind of shorter, some features are longer. In fact to go away from an ECG signal, biomedical engineers often talk about what are called evoked potentials. So we provide a stimulus to a biomedical system or to a biophysical system and we evoke a response and the waveform corresponding to that response is called evoke potential. It can be studied as an electrical signal.

Now evoke potential again typically has quicker parts in the response and slower parts in the response. Naturally we expect the slower parts of the response would be predominantly located, if you think of the frequency domain, in the lower ranges of frequency. And the quicker parts of the evoke potential waveform would be located in the higher ranges of frequency. Now here is again an example of time-frequency conflict. Suppose I wish to be able to isolate the quicker parts, is it all right to simply isolate the higher frequency content in a certain signal and which comes from an evoke potential and suppress the low-frequency part?

Well, you see if we try and suppress the low-frequency part, then we have already suppress the slower parts of the response and if we try and suppress the higher frequencies, in, in a bid to emphasise the slower parts of the response, we have suppressed the quicker parts of the response. So if we think conventionally in terms of the frequency domain, maybe high pass filtering or lowpass filtering, nothing works for us. If we do high pass filtering, then we have effectively suppressed the slower parts of the response, if you do lowpass filtering, we have suppressed the quicker parts of the response.

So we need a different paradigm or different perspective on filtering, we need to identify in different parts of the time axis, which regions of frequency axis are predominant and therefore in a certain sense identify different parts of the frequency axis to be emphasised in different time ranges. This is another perspective again on the time-frequency conflict and all this is going to lead us in the direction of building up this course on wavelets.

We shall of course understand some of these concepts a little better as we progress in the lectures, but for the time being I have given you these 3 examples with the intent of bringing before you, perhaps not completely, but at least in a way to inspire your imagination, the whole idea of time-frequency conflict. Or more generally the conflict between 2 domains, domains of analysis and presentation of a signal and of course then going further, even of a system.

In a 1st course we understand the domains very well, we understand there is a time domain, we understand there is a frequency domain, we do well because we keep them apart, it makes life easier. But what we are trying to bring out through these 3 examples, the audio example, digital communication example and the biomedical waveform example, whether it is electrocardiographic waveform is or the evoke potential waveforms, what we are trying to bring out is that one normally needs to consider the 2 domains together, time and frequency.

And when we try and do so, there is a certain, very fundamental conflict that we have to deal with. That conflict called uncertainty appears as I said in different manifestations in different subjects and we are going to look at that principle, the uncertainty principle as it applies to signal processing in great depth at a certain stage in this course. But before that, we are going to consider one particular tool for analysing signals, analysing situations with the recognition that we need to be local and not global.

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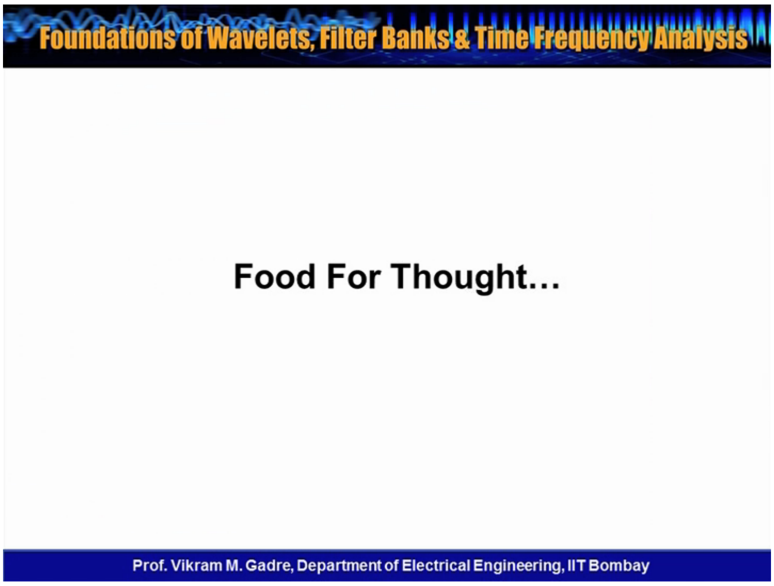

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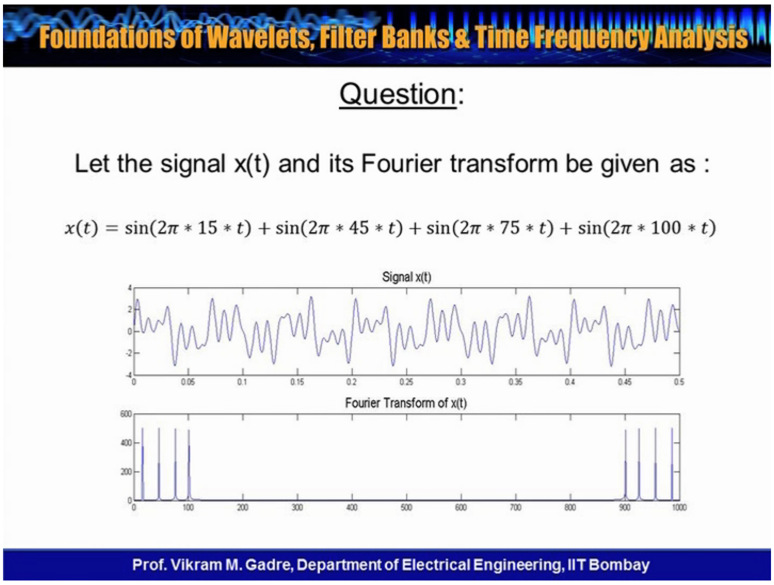
Food For Thought…

Prof. Vikram M. Gadre, Department of Electrical Engineering, IIT Bombay



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Question:

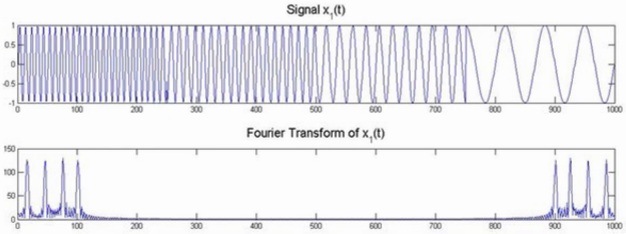
Let the signal x(t) and its Fourier transform be given as :

$$x(t) = \sin(2\pi * 15 * t) + \sin(2\pi * 45 * t) + \sin(2\pi * 75 * t) + \sin(2\pi * 100 * t)$$

Prof. Vikram M. Gadre, Department of Electrical Engineering, IIT Bombay

Let another signal $x_1(t)$ and its Fourier transform be given as :

$$x_1(t) = \begin{cases} \sin(2\pi * 100 * t) for 0 \leq t \leq 250ms \\ \sin(2\pi * 75 * t) for 250 \leq t \leq 500ms \\ \sin(2\pi * 45 * t) for 500 \leq t \leq 750ms \\ \sin(2\pi * 15 * t) for 750 \leq t \leq 1000ms \end{cases}$$



Now compare the Fourier transforms of both the signals x(t) and $x_1(t)$ as shown in the figures. Both of them show four frequency components and at exactly the same frequency locations i.e. at frequencies 15, 45, 75 and 100 Hz.

As can be seen from the plots, the two spectrums are almost identical (after neglecting the ripples and amplitude in signal $x_1(t)$), although the corresponding signals x(t) and $x_1(t)$ in time domains are entirely different.

So the question is, how can two entirely different signals have the same spectrums ??