Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-2.
Lecture-5.2.
Angle Between Functions and Their Decomposition.

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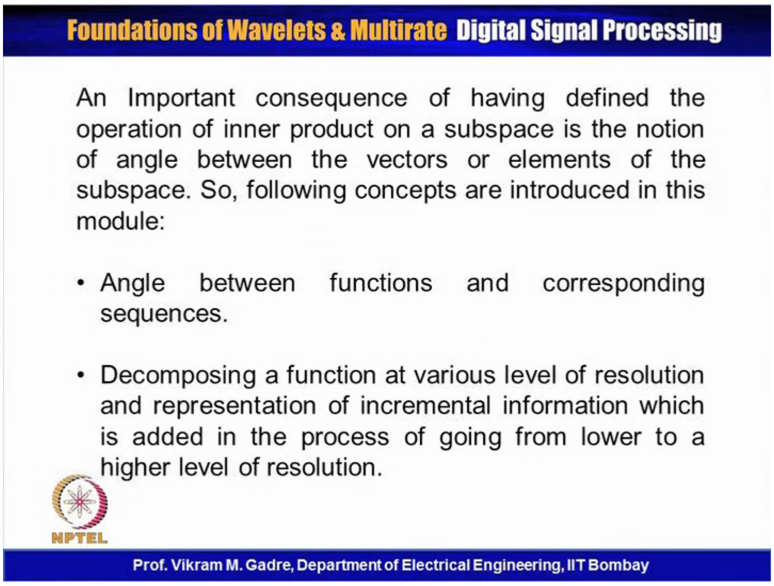

So, we can talk about the angle.

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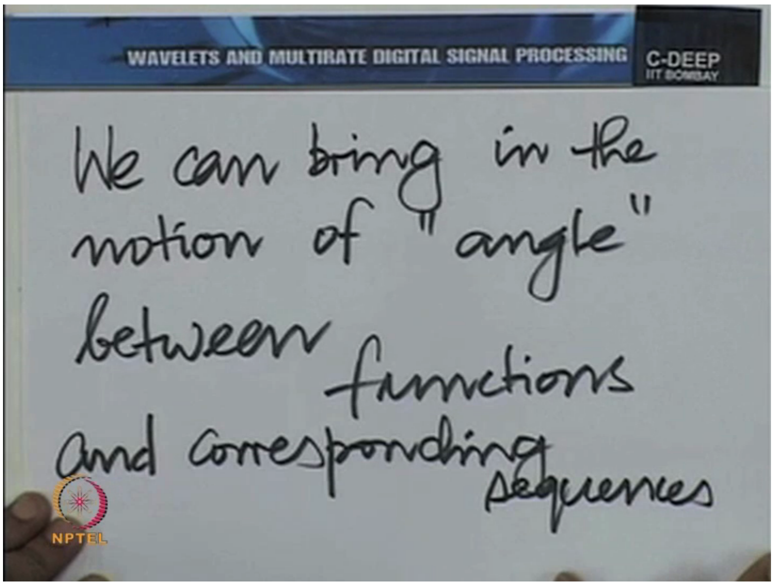

In fact once we bring in the notion of angle between functions, we can also bring in the notion of angle between the corresponding sequences.

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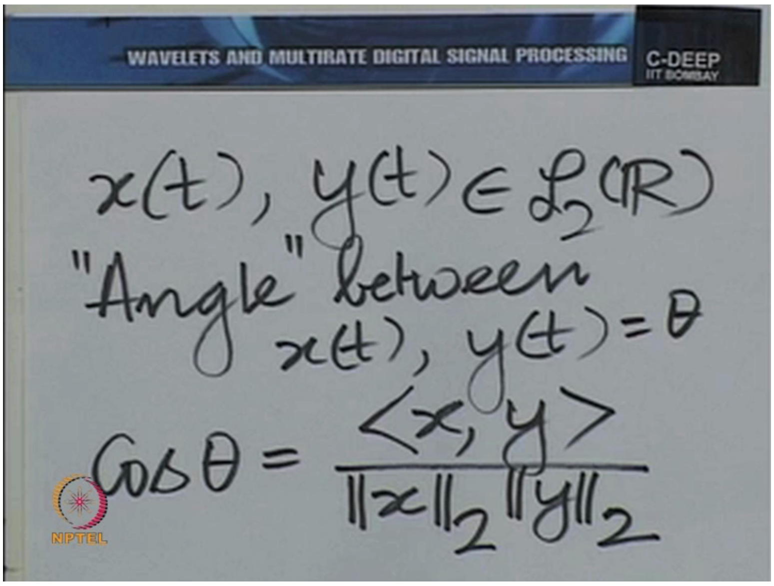

So, if you have 2 functions, xt and yt, of course belonging to L2R, we are going to confine ourselves that space. Then the angle between xt and yt is essentially defined by the following. you see, let it be theta, we say cos theta is essentially the inner product of xt with yt divided by the norm in x2 of in L2 of x. And the norm of y in L2 multiplied together.

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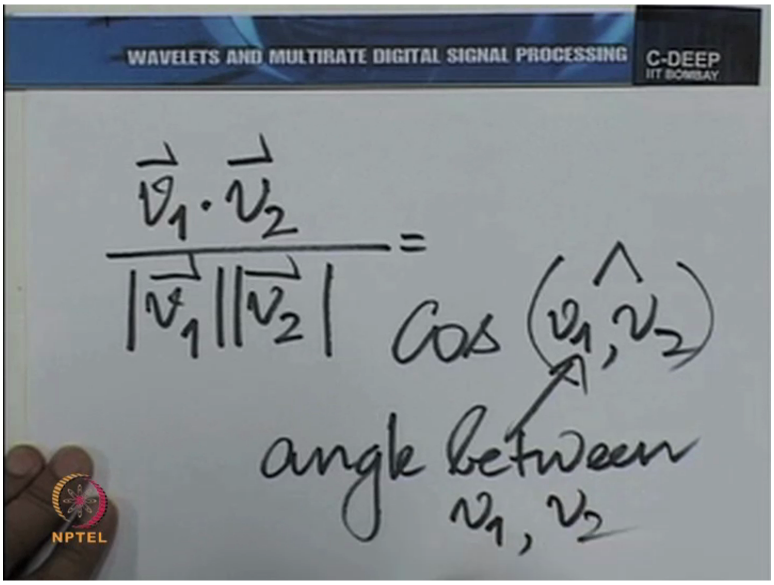

Now this is very similar to the idea of a dot product between 2 vectors. So, if you recall, if you have 2 vectors, let us say V1 and V2, then V1 dot V2, divided by the magnitude of V1 and the magnitude of V 2 gives us the cosine of the angle between V1 and V2. So, in a restricted sense, you do have the notion of angle between functions and whatever you did to construct the angle for the functions can also be done for the corresponding sequences associated with the functions and therefore you have the notion of angle even between the corresponding sequences. And I ask questions and I leave it to you to ponder over the answers, are those 2 angles the same, do they actually match? I think they should, should not they?

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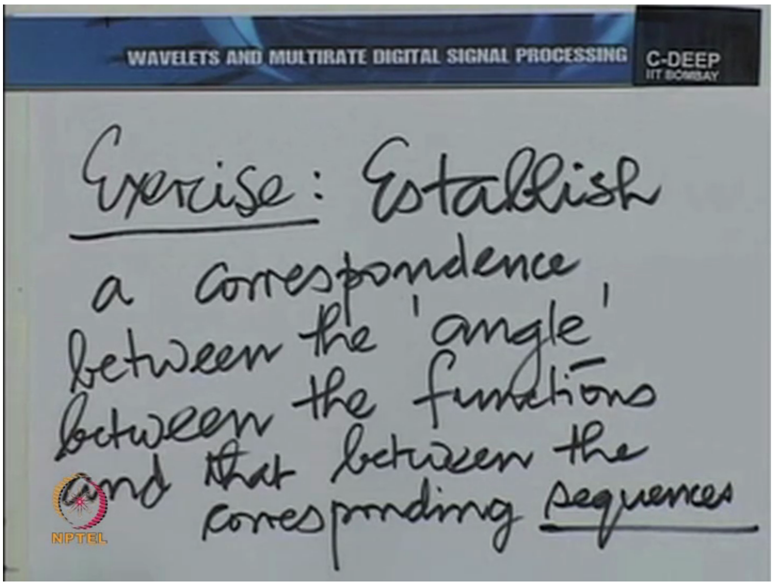

And I leave it to you as an Exercise to actually show that they do. So, Exercise, establish a correspondence between the angles, between the functions and the corresponding sequences. Anyway, our correspondence has become deeper and deeper and whatever we have been doing with the functions, we discovered can now be done with the sequences.

Now the next step is to ask can we also think of decomposition in terms of decomposition of the sequences. So, for Example, let us go back to that function in V1 that we had a few minutes ago.

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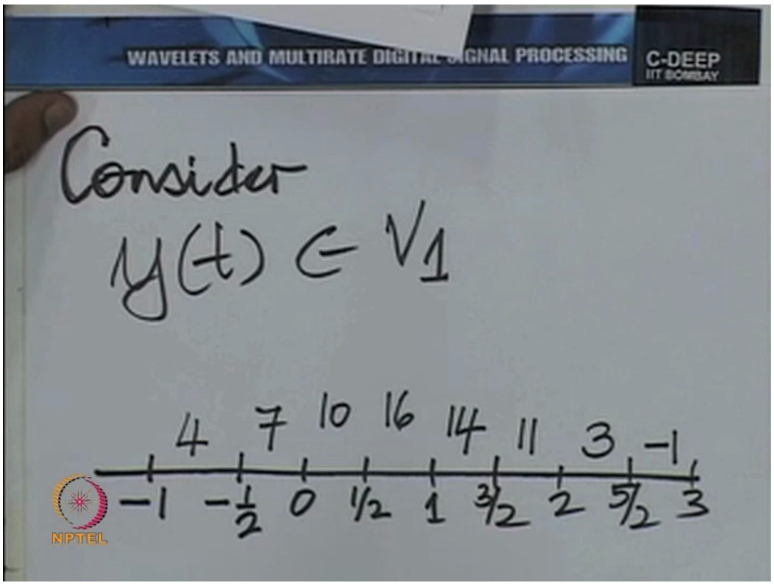

We had this function in V1 and we could then decompose V1 into V0, the orthogonal sum of V0 and W0. Now, just for a minute, let us keep aside the discussion of this particular function and let us look at a typical function in V1, a typical function in V0 and a typical function in W0. So, let us look at typical functions in V1, V0 and W0.

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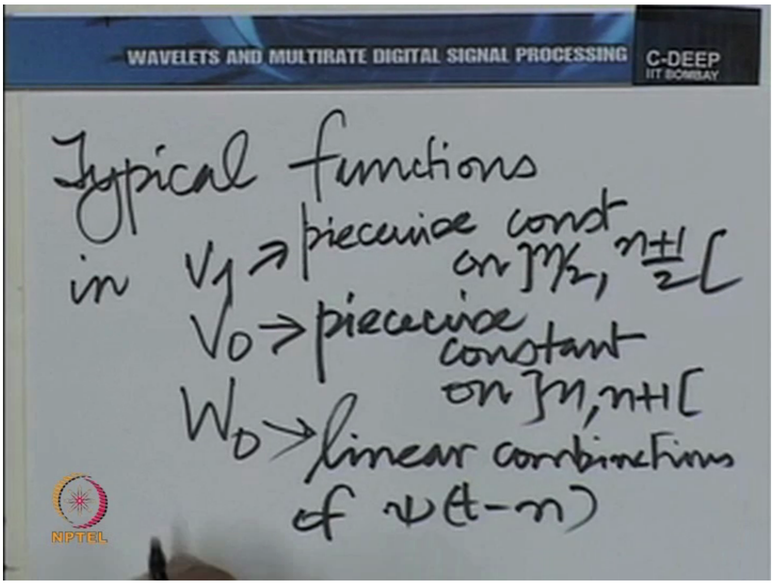

Now, these are piecewise constants on the standard unit intervals. These are linear combinations of Psi t - n for integer n. And these are piecewise constants on n by 2, n +1 by 2 for integer n. Suppose we take a given function in V0 and a given function in W0, now what would the dot product be?

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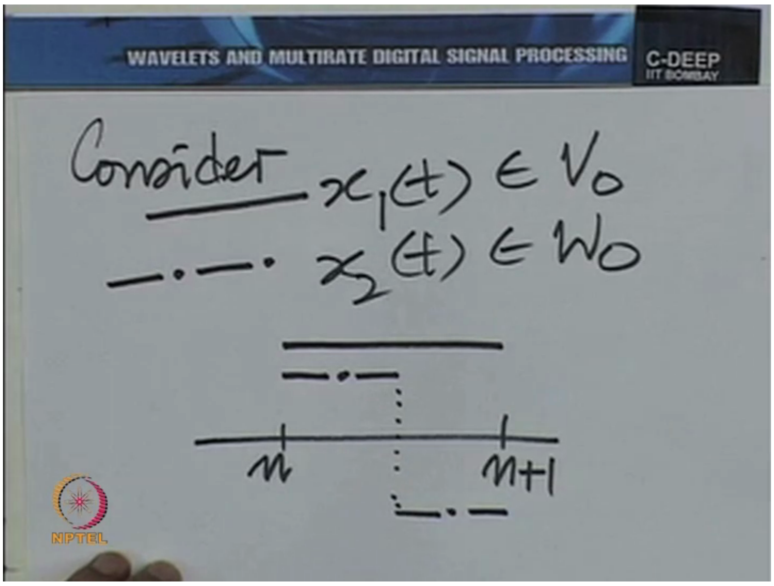

So, what I am saying is, suppose we consider, say x1t belonging to V0 and x2t belonging to W0, let us focus our attention on a particular interval. Let us say n to n +1, let us draw the function x1t by a solid line and this one by a dot dash line.

What would a typical function look like, x1 t function would look like this and x2 function might look something like this. If I multiply these functions together and integrated, you can visualize the integral piece by piece on each of these intervals n to n +1. The integral of any one of these pieces is obviously 0 because the positive and negative areas are equal. And this can be seen to be true of all the intervals and therefore obviously these 2 functions are perpendicular or orthogonal because their dot product is 0.

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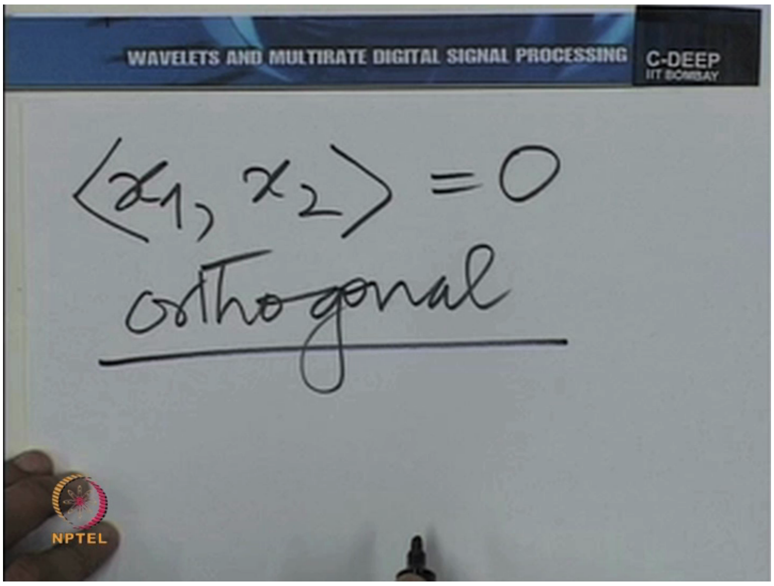

So, the dot product of x1 and x2 is 0, they are perpendicular or orthogonal. you know, we do not use the word perpendicular anymore, when we talk about functions, we should use the word. What is more, take any particular function in V1.

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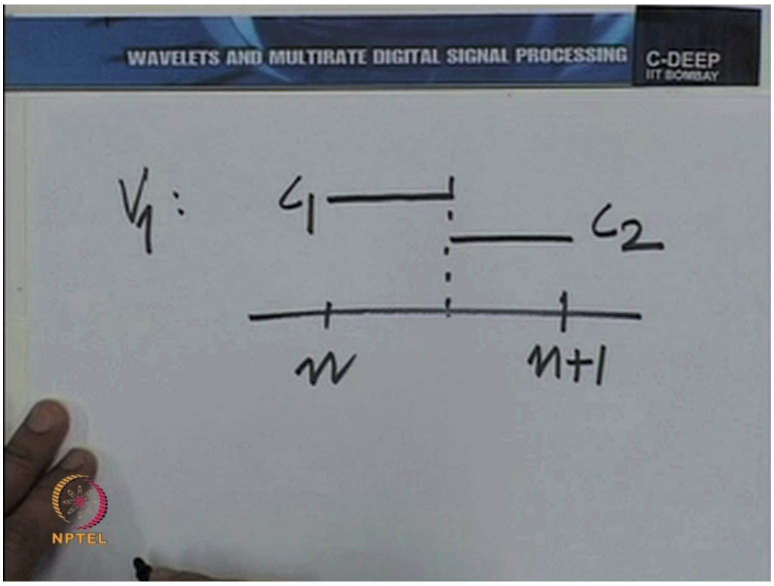

So, let us take the function, let us say, again focus on any one particular interval of function V1, let us take the interval n to n +1 and let the function have the value, let us say C1 for the 1st half interval and C2 for the 2nd half. It is very easy to see that this can be treated as a function belonging to V0 + a function belonging to V1 where the corresponding function belonging to V0 looks like this.

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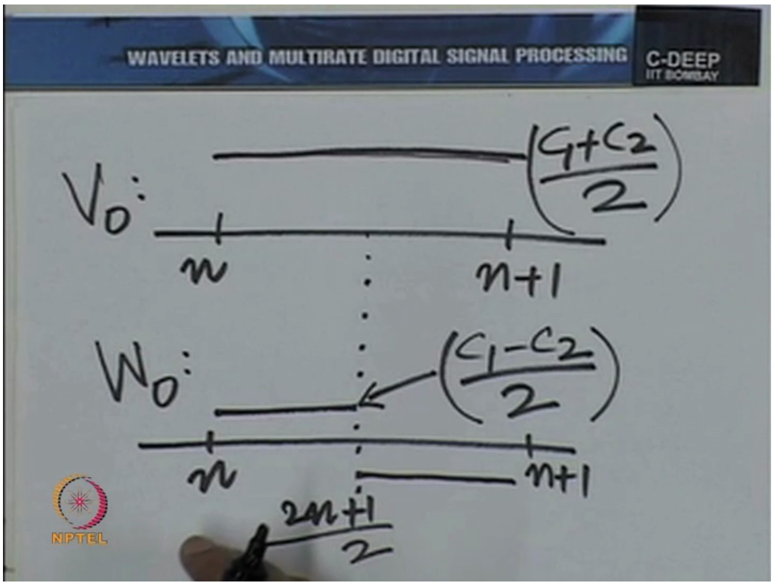

So, this function is equal to this function C1 + C2 on the interval C1 + C2 by 2 over the interval n to n +1 + the function C1 - C2 over the 1st half interval and the negative of the same thing over the 2nd half interval.

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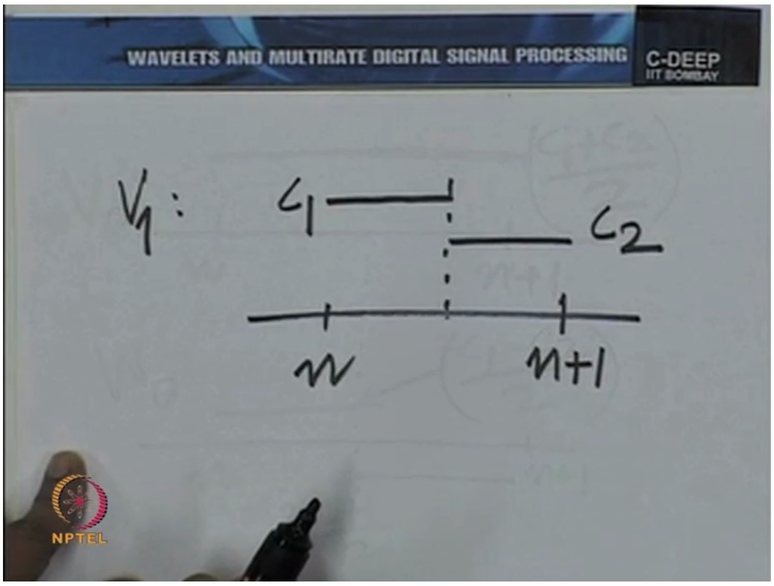

So, if you take this point to be n to n +1 by 2, so to speak, the middle of the interval, and this height here is C1 - C2 by 2. So, this is you see… What I am saying is the function coming from V0 and the function coming from W0, I am just showing one segment of each of these functions, the same thing can be done for each of these intervals from n to n +1.

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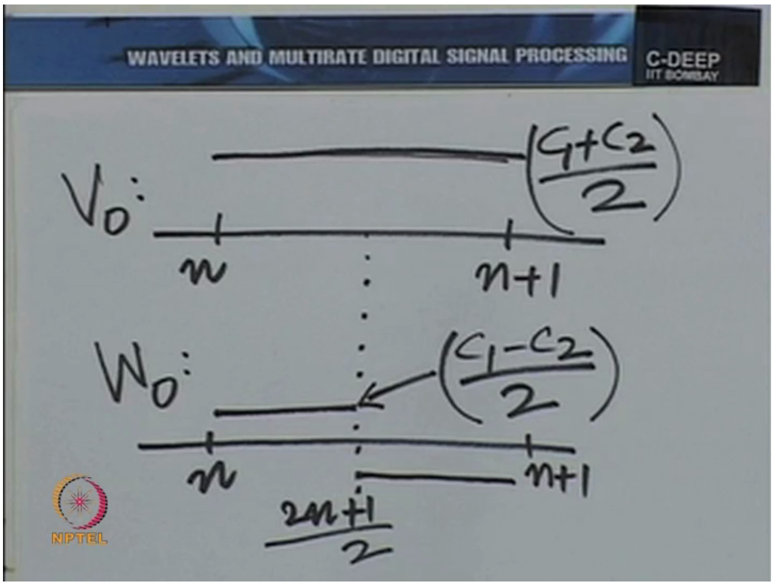

What I am saying is this function whose segment over n to n +1 I have shown here being C1 on the 1st half interval and C2 on the 2nd half interval of course it belongs to V1 is equal to the sum of this function C1 + C2 by 2 on the Entire interval belonging to V0 + this function belonging to W0 which is C1 - C2 by 2 of the 1st of interval and the negative of the same thing on the 2nd half interval. So, it is very easy to see that we can in general decompose a function in V1 into a function in V0 + a function in W0 in a unique way.

And therefore, the orthogonal decomposition of V1 and V0 and W0 is easy to construct. Now, can we also make corresponding construction on the sequences and in fact to some extent, we have already answered the question.

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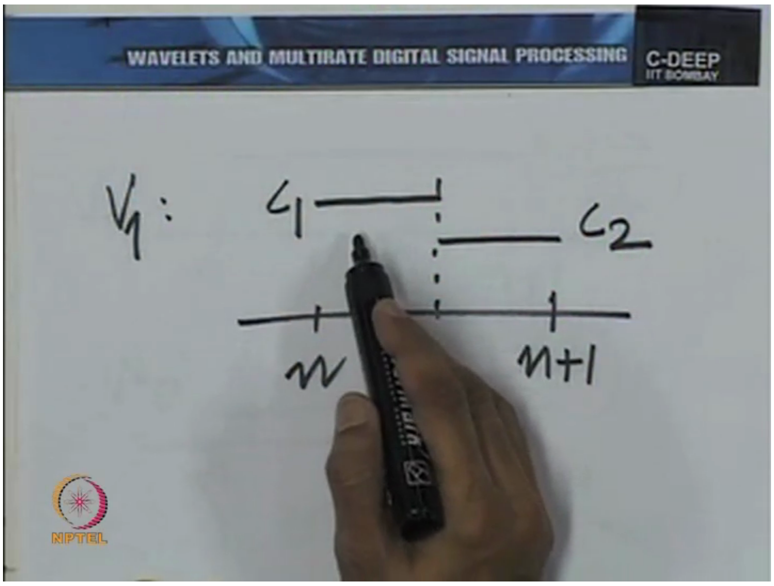

If you look at it, here C1 would be the value of the sequence at 2n, the sequence corresponding with the function in V1 and C2 would be the value of the sequence at 2n+1. Interestingly, the value of the sequence corresponding to the function in V1 at the point 2n and 2n + 1 relate to the values of the sequences corresponding to the functions in V0 and W0 but at the points n and not 2n.

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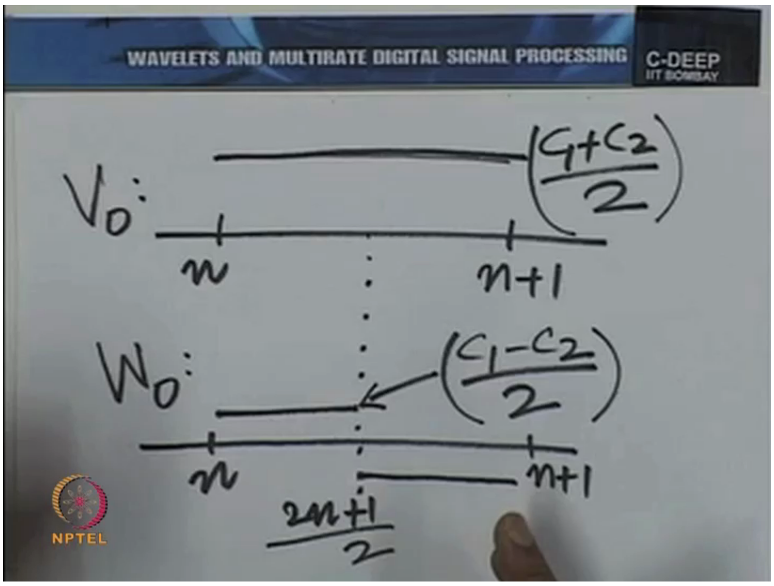

So, you have the value C1 + C2 by 2 for a sequence corresponding to this function, at the point n and not 2n and you have the value C1 - C2 by 2 corresponding to the function in W0 but at the point n and not 2n.

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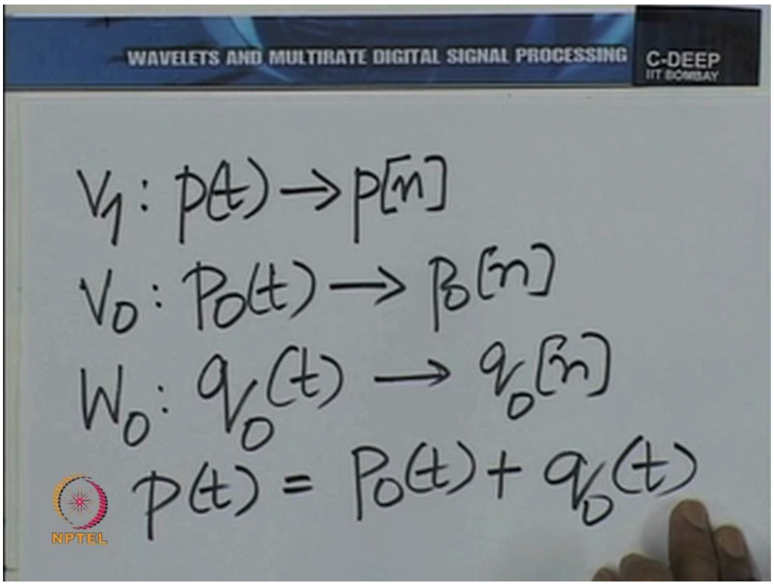

What I am saying is, now if you think in terms of sequences, let us do that, so you had this function in V1, so let us say this function is, now you know for variety, let us use PT belonging to V1 and the corresponding sequence pn, you have the corresponding function p0t let us call it and the corresponding sequence P0n, where p0 is the component in V0 so to speak. And you have Q0 let us say as a component in W0 and the corresponding sequence Q0n.

So, what I am saying is PT is of course Equal to p0t + q0t. But Pn is not equal to p0n + Q0n, that is not correct, that is because the orthonormal bases are different. So, now we need to establish a relation between P0n, q0n and pn, that is the next task that we would like to undertake.

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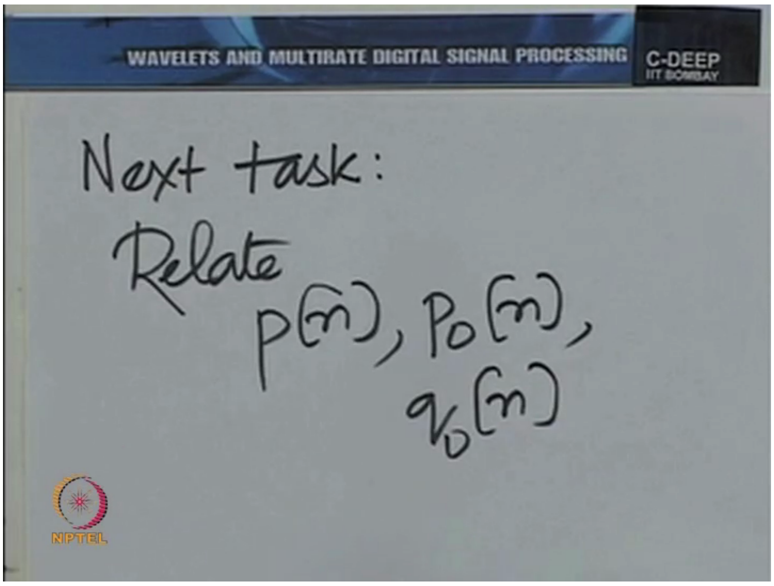

So, our next job is to relate Pn, P0n and Q0n. And in fact, we already have an answer to that question, let me just go back one step here.

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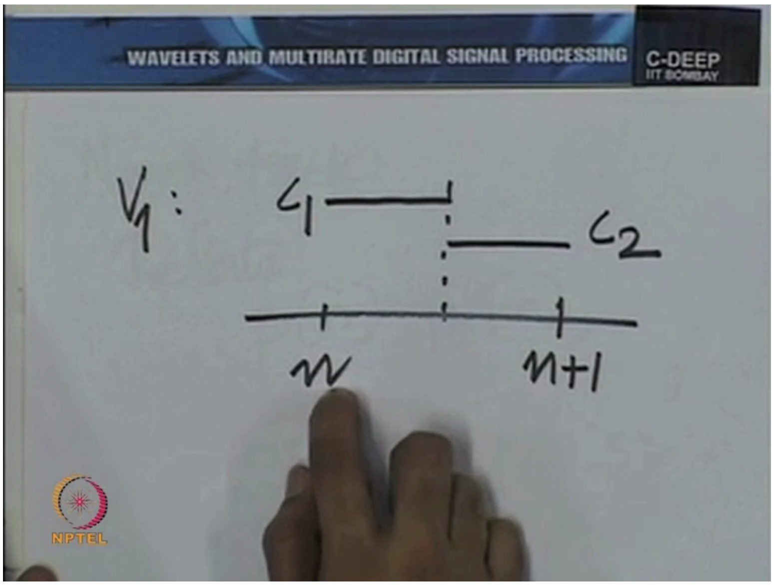

So, we have the answer here.

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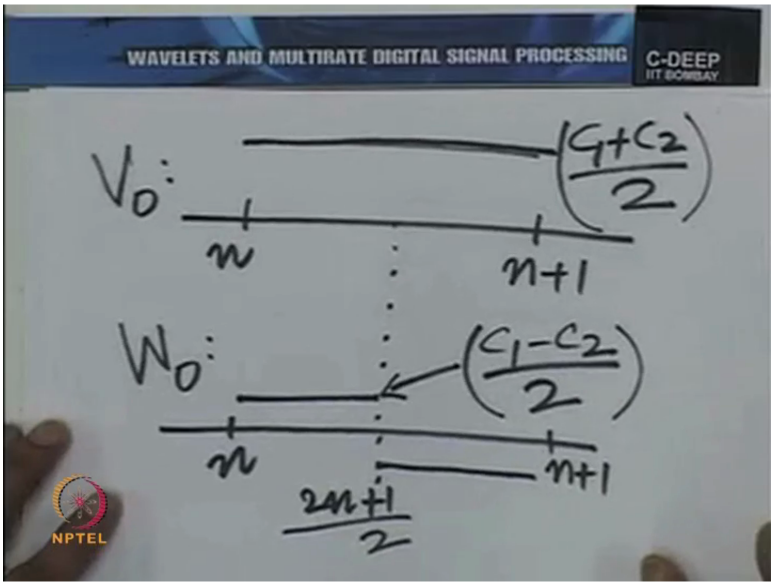

You see, P at the value 2n is C1, P at the value 2n +1 is C2, P0 at the point n is C1 + C2 by 2, Q0 at the point n is C1 - C2 by 2. Let me write down all this formally. So, we have the relationship there, I have almost done my job.

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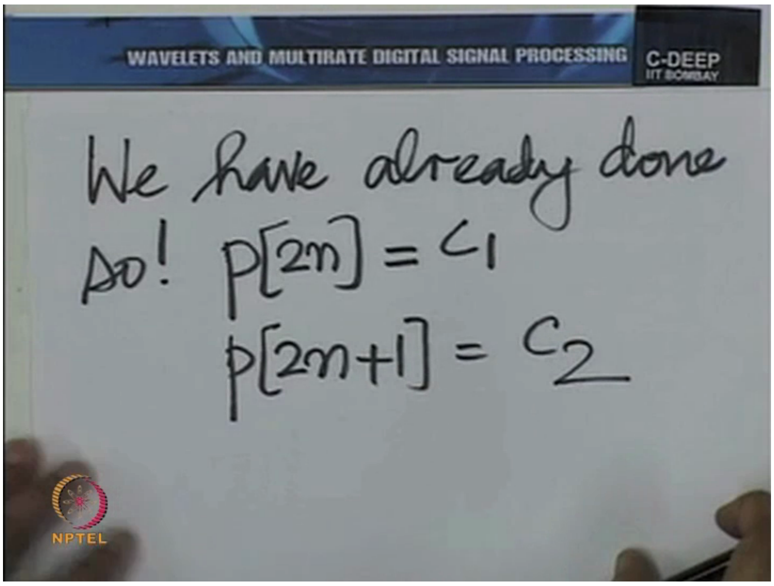

So, what have we said, we said P at 2n is C1, P at 2n +1 is C2, P0 at n is C1 + C2 by 2 and Q0 at n is C1 - C2 by 2.

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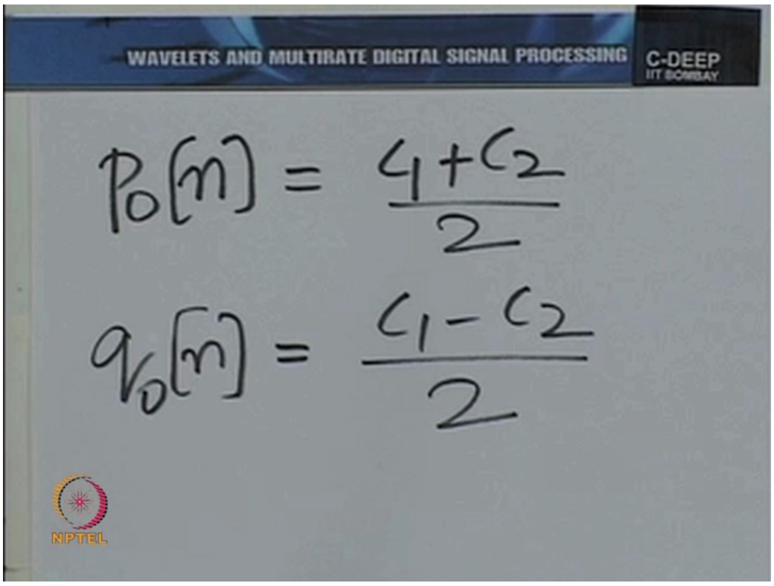

Now let us combine these equations. So, we have P0n is P2n + P2n +1 by 2.

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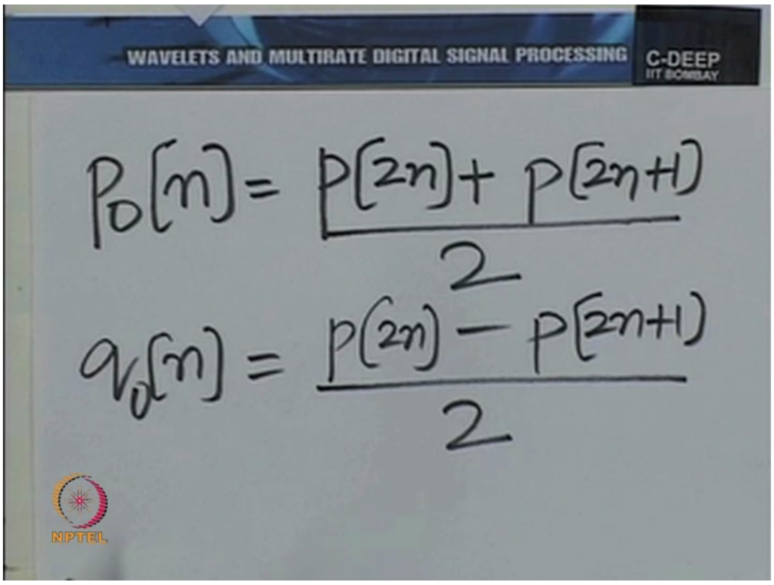

And Q0n is P2n - P2n+1 by 2. And now this brings before us a very beautiful perspective. When we talk about sequences, we can also extend that context to talk about discrete time filters acting on sequences. Can we visualize what we have done here as discrete time filters acting on the sequences? So, suppose you have the following discrete time filter.

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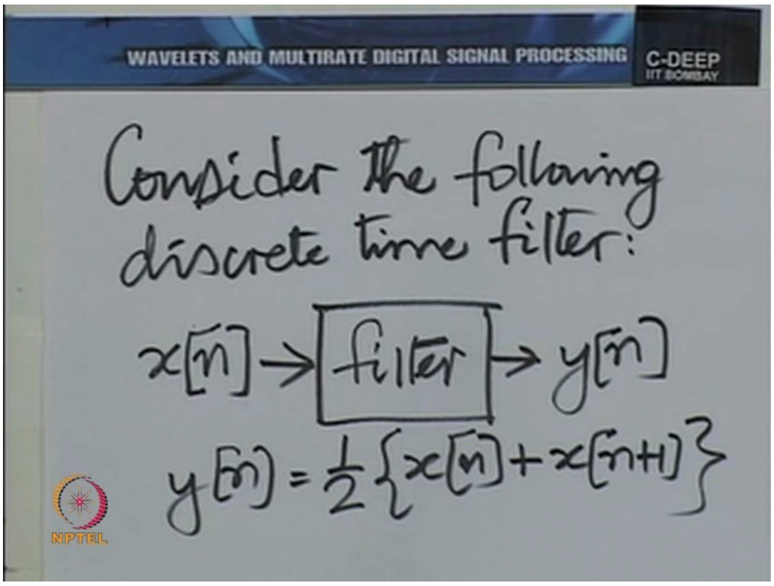

Let the input be x of n and output be y of n and y of n is half xn + x of n +1. You know, it is a non-causal filter, let us not worry too much about it for a moment, let us accept it even if it is non-causal. What have we done here, in this relationship?

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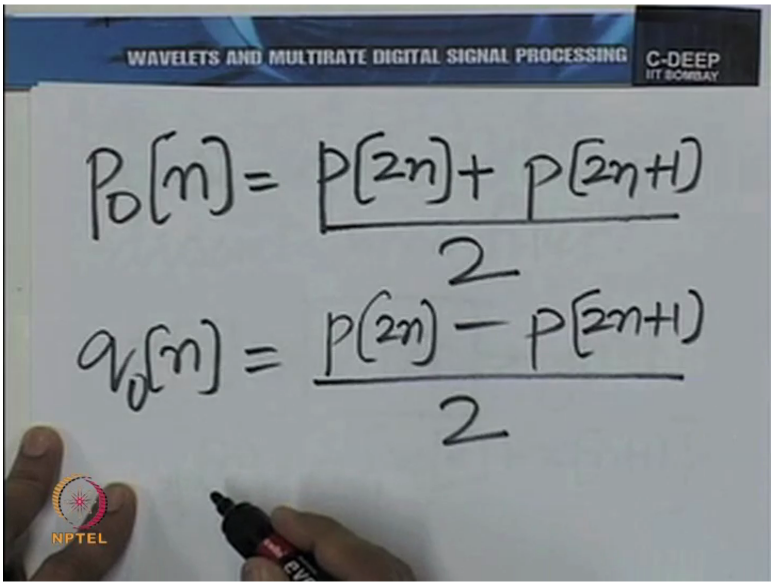

Let us reflect on the connection between for Example this relationship here and the filter we just constructed. If you think of 2n as a variable, let us call it, let us say it is L, so, this is P of L + P of L +1 by 2, then in fact this is essentially the filter acting on the sequence P.

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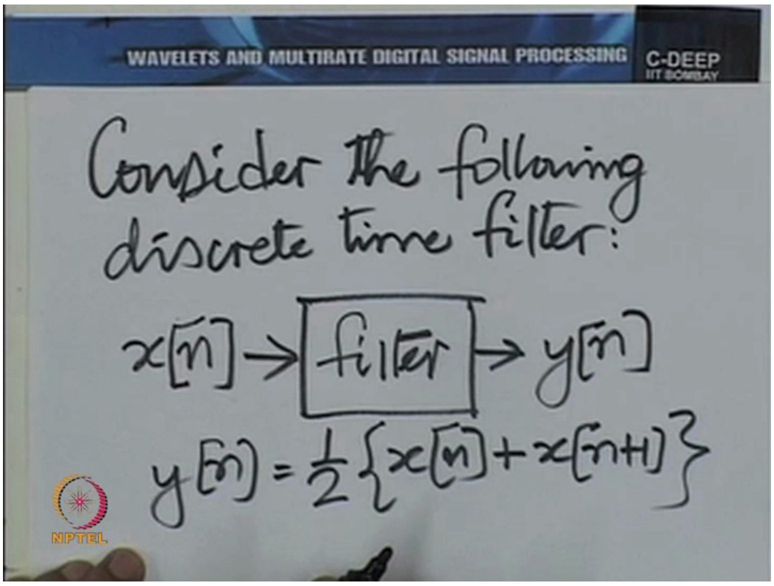

So, what we are saying is use this filter and put Pn here but then a little bit of work needs to be done at this point because if you are putting Pn, let us do that, let us put in Pn.

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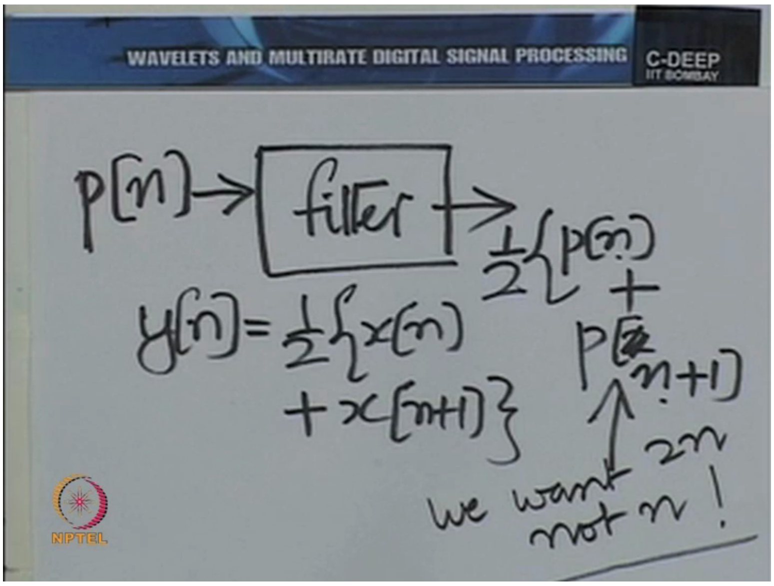

So, if you put in Pn there, if you put into this filter, we just wrote down here, y of n is half xn + x of n +1, what we would get here is half of Pn + Pn +1. I am sorry, p yeah n + 1, that is correct, but we do not want Pn and Pn +1, we want to replace here.

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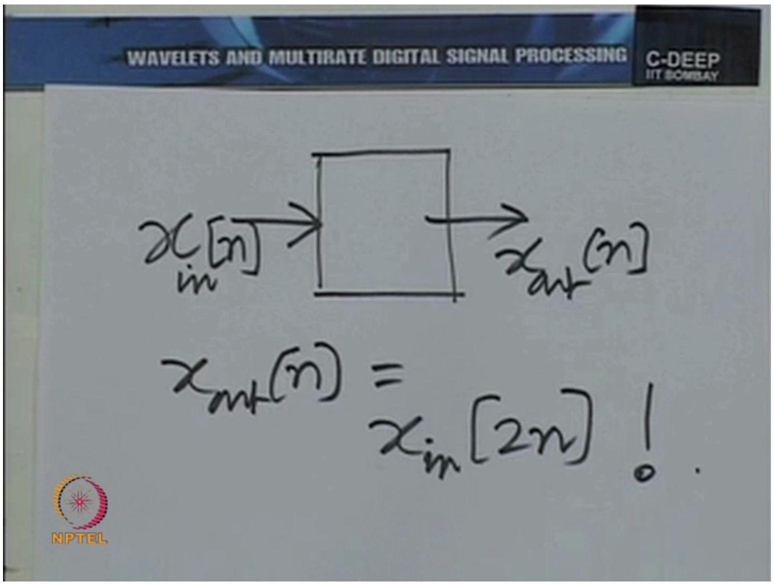

So, what should we do, we should have another system now following this where putting xin and put out xout where xout of n is xin of 2n, we want a system like this.

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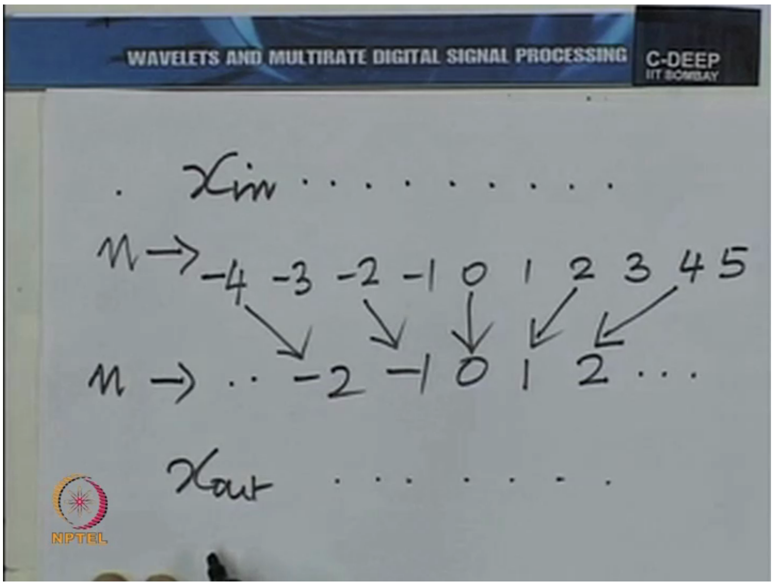

Let us interpret this system. What we are seeing in this system is xin at n, so let us write down n -4, -3, -2, -1, 0, 1, 2, 3, 4 and so on and the sequence xin here, put at these points. So as far as xout goes, you have the index for xout, as far as xout goes, 0 comes from 0, 1 here comes from 2, 2 here comes from 4, -1 comes from -2, -2 comes from -4 and so on. So, in other words, what you are doing, you are retaining the samples at the even locations and throwing away the samples at the odd locations.

NOT only that, after retaining the samples at the even locations, you are putting, putting those samples at half the location number. So, let us summarise this.

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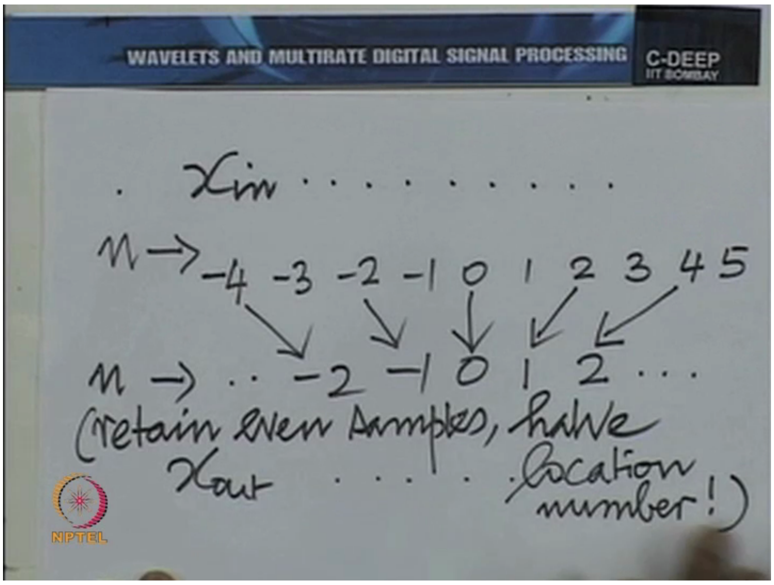

Retain even samples and halve the location number. Now, this system is a new system as far as a basic course on discrete time signal processing is concerned, we need to christen it, we need to give it a name, in fact let us go back to that system.

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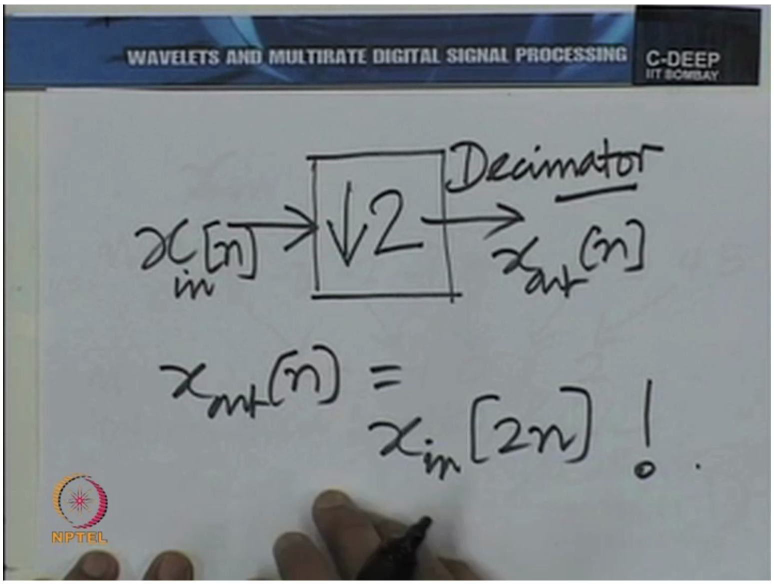

And let us give it both a symbol and a name. The symbol that we shall give it is a down arrow followed by 2. And we shall call it a decimator.

You know the word decimate actually has a very cruel meaning. I am told that in the days of wars, during the Roman Empire, a very cruel thing that warriors used to do was to kill 1 out of 10 or maybe 9 out of 10 and that was what was called decimation. Take 10 of them and eliminate from each group of 10, that was a cruel way to deal with people. But the word decimate has also percolated down to the literature on digital signal processing. Here decimation means retaining one out of so many samples.

So, in this case, decimation by 2 means retaining one out of 2 samples. In fact the 1st of each pair of 2 samples. Out of 0 and 1, you retain 0, out of 2 and 3, you retain 2. Not only that, after you retain only one out of two, compress so that the Sample number is halved or if you retain one out of three samples, then compress, so the Sample number is multiplied by one third. So, if you are decimating by a factor of 3 for example, the 0 Sample will go to 0, the 3 Sample will come to 1, the 6 Sample will go to 2, the -3 Sample will come to -1 and so on. If you are decimating by a factor of 2, then 0 Sample will come to 0, the 2 sample will go to 1, the 4 Sample will go to 2, the -2 Sample to -1 and so on as we have just shown.

So, what do we have here, we have a filter followed by a decimator and that together helps us construct the sequence P0n from the sequence pn. Now, we shall see us in the next lecture that we can similarly construct the sequence q0n from the sequence pn by using another filter and decimator and we shall build up further from there to do something to reconstruct Pn from p0n and q0n and all this shall together lead us to a totally different structure in discrete time signal processing which we shall call a 2 band filter bank. With this little trailer for the next lecture, let us conclude the present lecture, thank you.