Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-2.
Lecture-4.3.
Parseval's Theorem.

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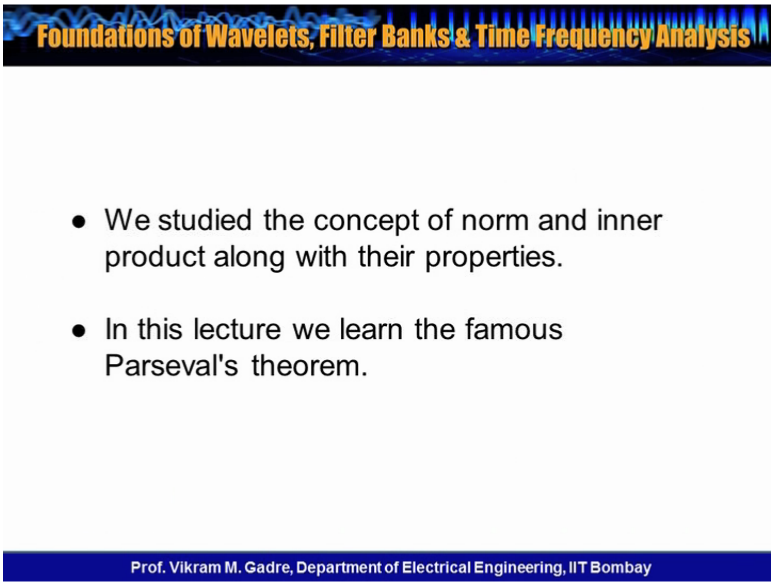

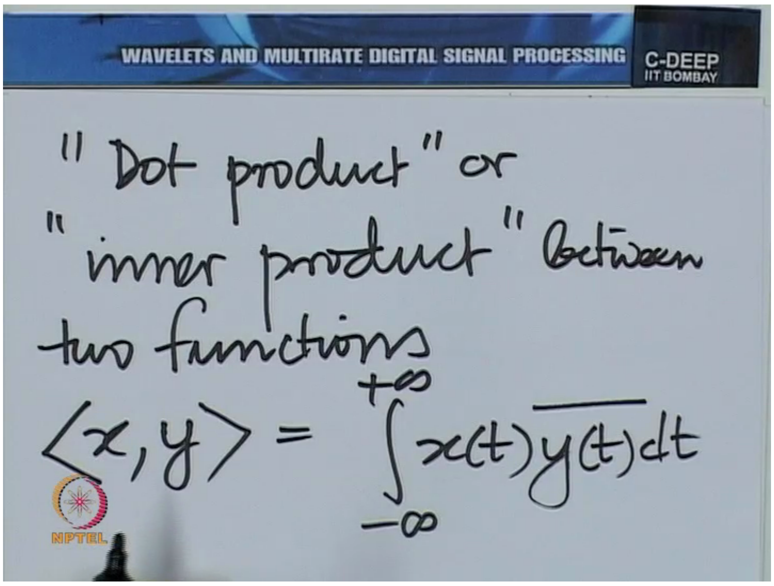

So with that little prelude let us come back to this uncountably infinite dimensional space of functions on the real line, in which case we can generalise. So we can generalise the notion of a dot product or inner product between 2 functions. Essentially, if I take 2 functions X and Y, both on the variable T, the dot product is not going to be a summation anymore but an integral. So XT Y bar T DT. Taking that idea further of multiplying corresponding coordinates and instead of summing, you now integrate.

So the integral replaces the operation of summation here. Now of course it is easy to verify and I leave that as an exercise to you, the properties of linearity and commutativity and so on. So I leave it to you as an exercise here, verify the properties of conjugate commutativity, in other words if I change the order of the arguments, there is a complex conjugation involved. 2^{nd} of linearity in the 1^{st} argument, so if I take a linear combination of 2 vectors or 2 functions in the 1^{st} argument, then the corresponding inner products are also similarly linearly combined and 3^{rd} positive definiteness. So I leave this to you as an exercise.

But what I wish to emphasise at this point is the famed Parseval's theorem of which we are aware in the context of the Fourier transform. So let me recapitulate that very important theorem in the context of Fourier transform and let us also give an interpretation to it.

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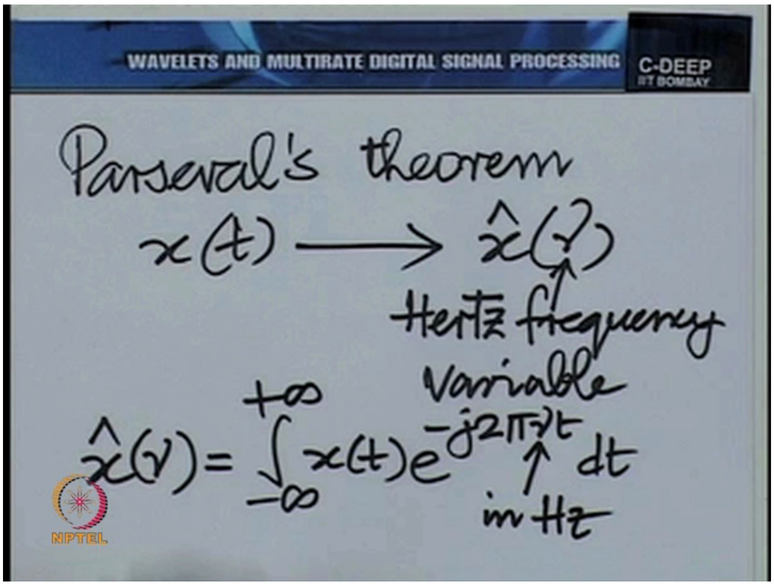

You see that Parseval's theorem as we know it, for continuous function says that if XT has the Fourier transform, now I am going to use the frequency, Hertz frequency variable. So, this is the Hertz frequency variable, nu. In other words, what I mean by that is that the Fourier transform of XT is essentially integral XT e raised to the power J2 pie nu t dt, integrated over all time T. So, this is the Hertz frequency variable in Hertz.

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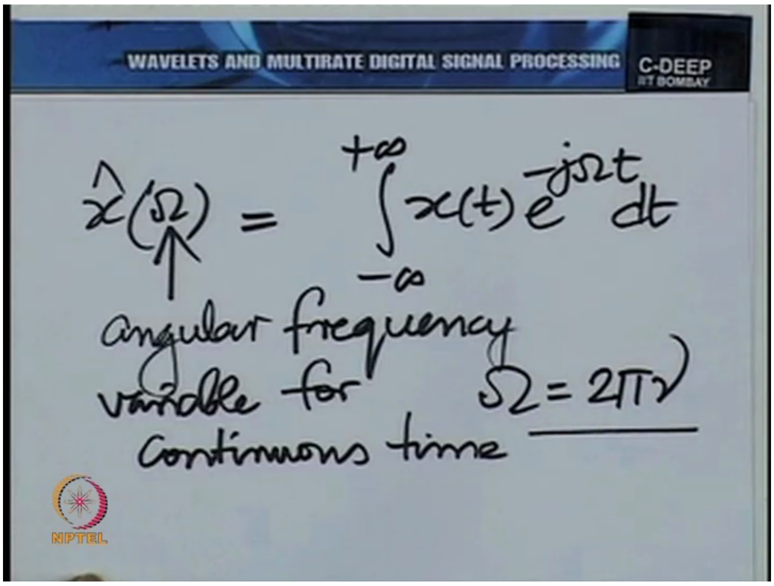

Recall that you can also have an angular frequency variable, so for example, you could write X cap of omega, I use this capital mega when we are talking about continuous time, we are going to follow some notions of different notations for continuous time and discrete time. So, we use this as the angular frequency variable for continuous time. In which case, X cap omega is X of T e raised to the power - J omega T dt. And there is a simple relation between omega and nu, omega is 2 pie nu, Angular frequency and Hertz frequency. Well, simple things, but we should put down all our cards in the beginning, so we do not get confused later.

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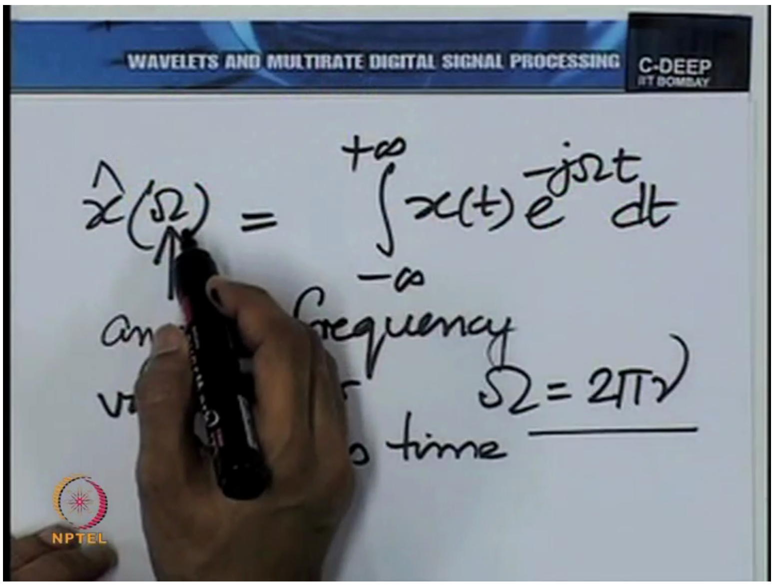

Now again this is a little bit of abuse of notations because I am using X cap of capital omega here and

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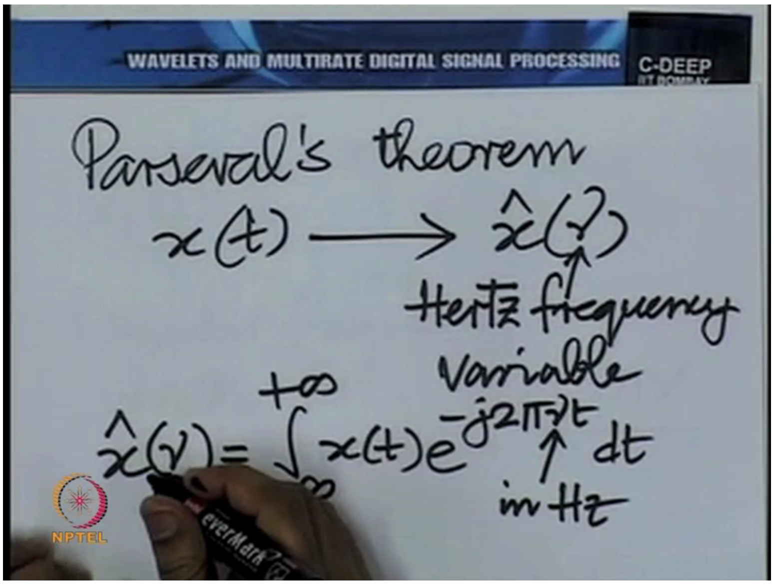

I am using X cap of nu there. And depending on the context, I must interpret either Hertz frequency in the argument or angular frequency in the radians per second in the argument. Normally from the context, it shall be clear. And if there is some confusion likely, we will make it clear by expressive statements. But remember that from the context, we should be clear whether we are dealing with Hertz frequency or angular frequency, radians per second. Anyway, with these details, let us come back to the Parseval's theorem.

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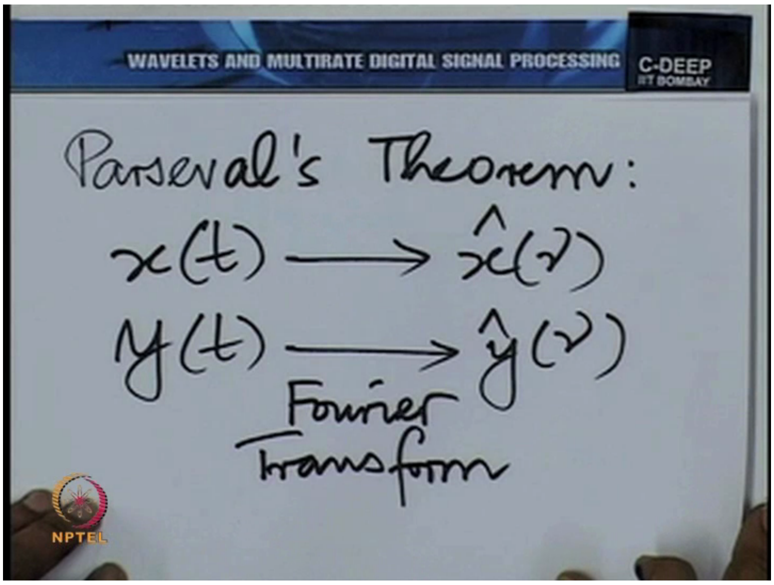

What does the Parseval's theorem say? The Parseval's theorem says the following, if you have the Fourier transforms of X and Y, so if XT has the Fourier transform, let us use the Hertz frequency variable X cap nu and YT has the Fourier transform Y cap nu. This arrow denotes the Fourier transform. Then there is an equivalence of the Fourier transform inner product and the time inner product, that is what the Parseval's theorem says in our language now.

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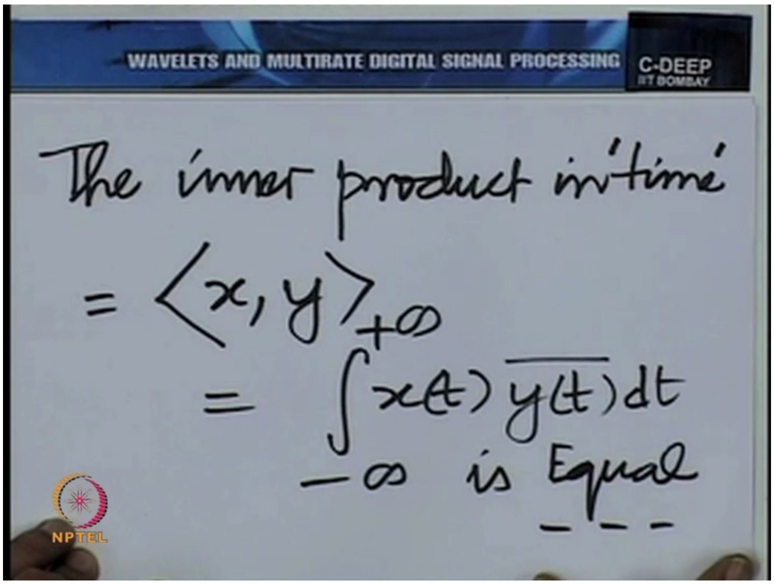

So, the inner product in time, so to speak is equal to the inner product in frequency.

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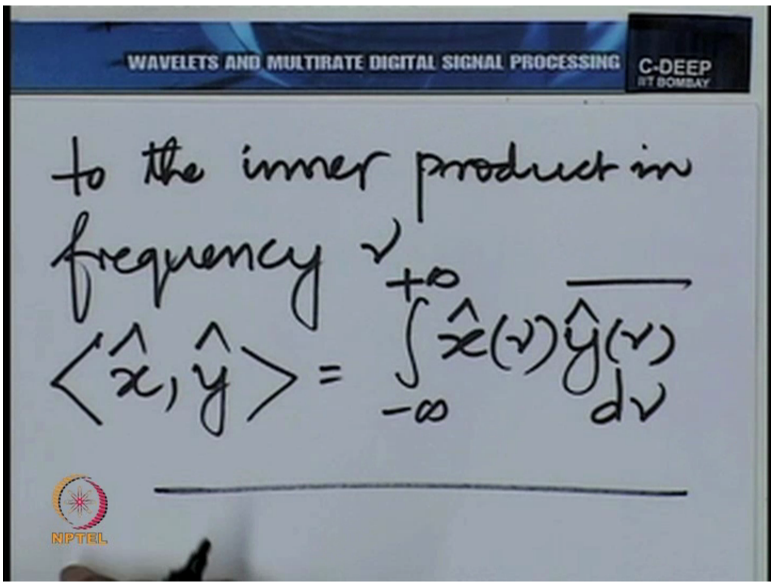

In other words, if you take X cap and Y cap and construct their inner product in the same way, treating the frequency as the independent variable or the argument. Now, this is a very beautiful and a very powerful interpretation of Parseval's theorem. When we talk about the inner product perspective, we have a very different way of looking at possible theorem. And in fact, if we really think of it a little more deeply, Parseval's theorem becomes so much more intuitive when we talk in terms of inner products. And let me take a minute now to show you why it is so intuitive.

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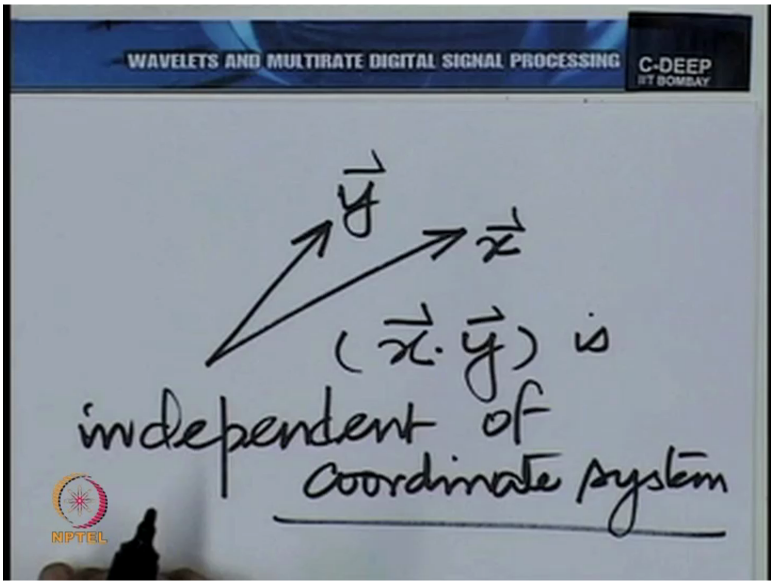

Indeed, what Parseval's theorem says in the language of inner product is this and let us do the same in two dimensions, then it will be absolutely clear. So, I have 2 vectors, let us call them X and Y. Now what Parseval's theorem says is X dot Y is independent of the coordinate system, simple enough. What coordinate system we choose to represent X and Y does not affect the inner product, that is what Parseval's theorem says in a way. And to strengthen, you see, it may not be obvious to you why Parseval's theorem relates to this statement. It is obvious for two-dimensional vectors that the inner product is or the dot product is independent of the coordinate system.

What is not obvious is why is this related to the Parseval's theorem. Well, towards that, we need to go back to what X cap nu really is, in a way. And that will become clear if we write down the inverse Fourier transform.

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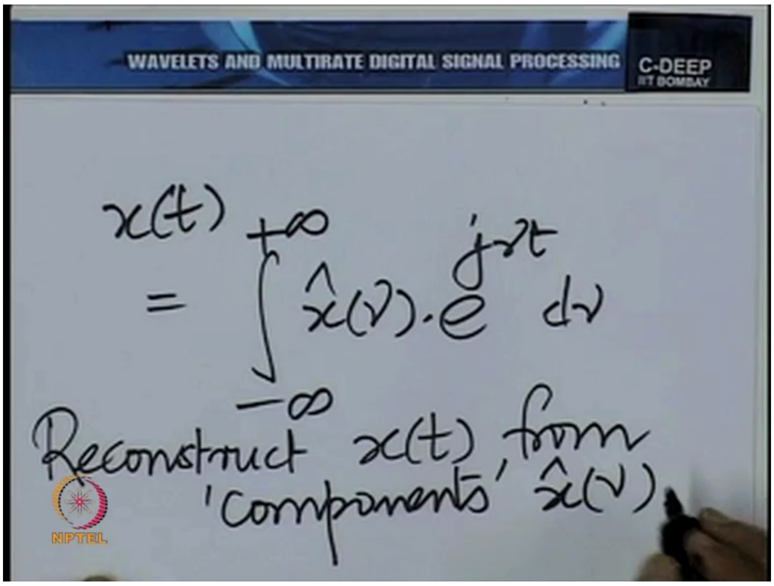

So, we can write down XT incomes of the inverse Fourier transform as X cap nu e raised to the power J nu t dnu, nu is the Hertz frequency variable again. So, in a way what we are seeing is, we are reconstructing XT from its components.

Each of the X cap nu 4 different values of nu is a component here. And this is the way we have reconstructed XT from its components and in reconstruction, we have used these vectors. Each of these e raised to the power J nu t is like a vector, is a function of the real axis. The only catch is e raised to the power J nu t is not an L2R function. So, we have to deviate little bit there from our discussion. But if we choose to ignore that fact, we have essentially taken these coordinates, multiplied them by the corresponding so-called functions along each of the coordinates nu and added them to get the function XT.

So, each of the X cap nu is like a different expression of the same vector X in a different coordinate system. So, what we are saying in Parseval's theorem is that the dot product is independent of the coordinate system. Whether we choose to use the standard coordinate system of time to represent the function or the slightly less obvious coordinate system of frequency to represent the same function, the dot product remains the same.

So, these and some other such interpretations are what are offered when we represent functions in terms of vectors or when we think of functions as generalisations of the ideas of vectors. And now for the last remark in this lecture which we shall build on even greater in-depth in the next, namely, what is the connection between functions and sequences, continuous functions and sequences? Just to initiate the discussion here, without completing it or rather taking it further, we shall do it in the next lecture, but just to initiate the discussion. Let us go back to the idea of piecewise constant approximation.

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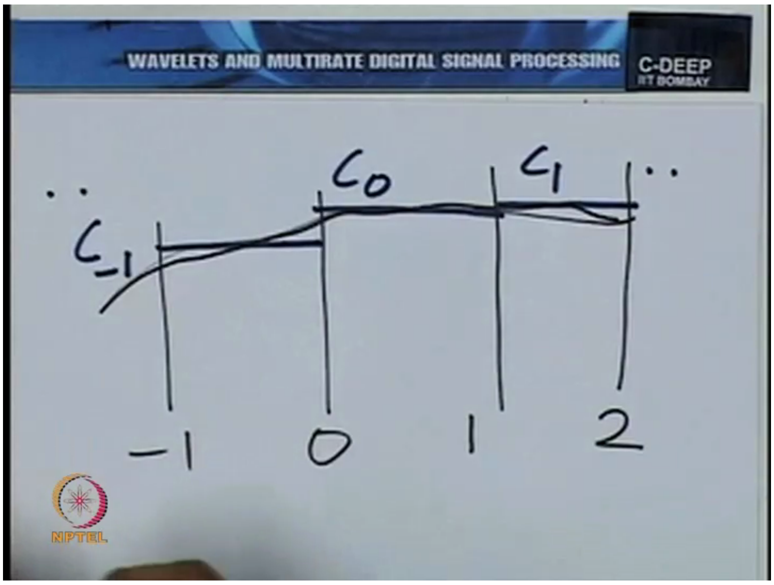

So, suppose we have this piecewise constant approximation of this function on intervals of length 1. So, I take the standard unit intervals. And I make a piecewise constant representation of a function. So, I have this. So, let the values be let us say C -1 here, C0 there, C1 there and so on so forth.

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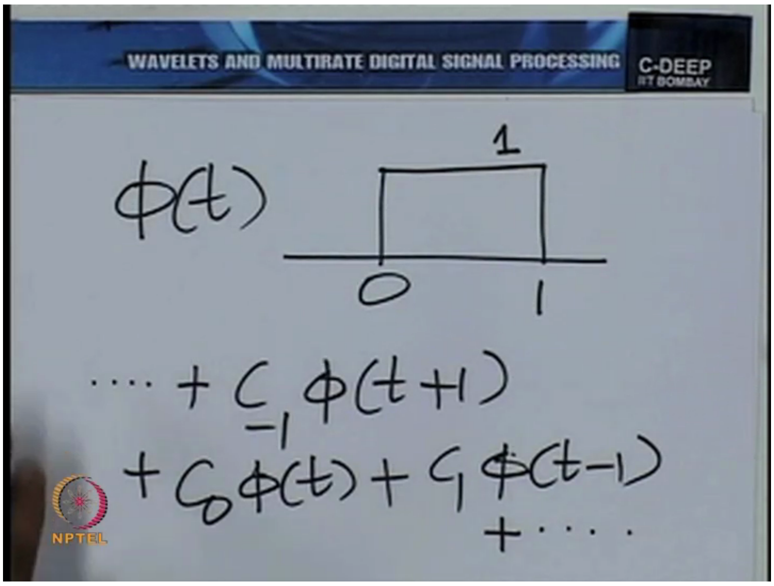

Now it is very easy to see that if I take the basic function Phi T described this way, 1 between 0 and 1 and 0 elsewhere, then this piecewise constant representation can be written as C -1 among other terms, Phi t +1+ C0 Phi T + C1 Phi T -1 and what have you, afterwards.

So, to conclude just this introduction of this correspondence, we can note that equivalent to this piecewise constant representation that I had here,

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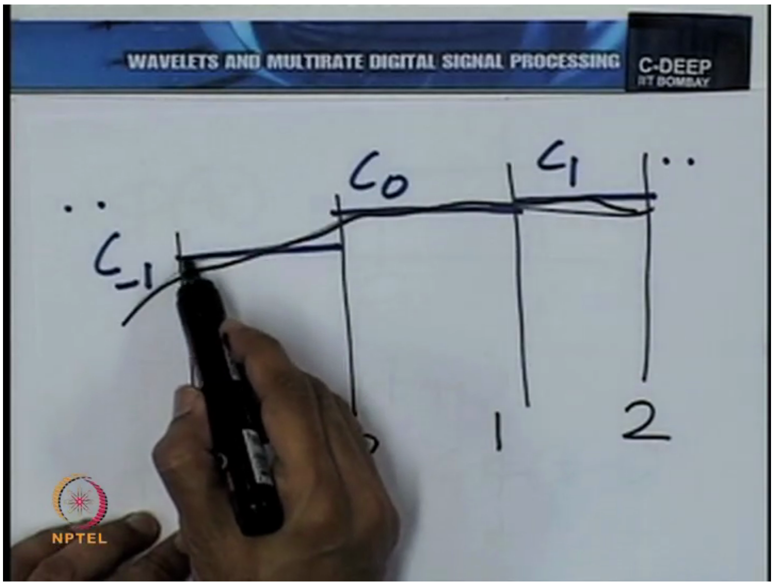

this function in V0 that we talked about last time, equivalent to that function is a set of values C -1, C0, C1 and so on. So, the sequence CN, N over all the integers is equivalent to that piecewise constant function in V0. Any of them can be constructed from the other, from that piecewise constant function, we can construct the sequence, from the sequence we can construct the piecewise constant function given Phi T. Now this equivalence is what we shall take further and delve into deeper in the next lecture. And in the next lecture, which shall also build further these ideas of vectors, functions and sequences. Thank you.