**Foundations of Wavelets, Filter Banks and Time Frequency Analysis.**
**Professor Vikram M. Gadre.**
**Department Of Electrical Engineering.**
**Indian Institute of Technology Bombay.**
**Week-2.**
**Lecture -4.2.**
**Properties of Norm.**

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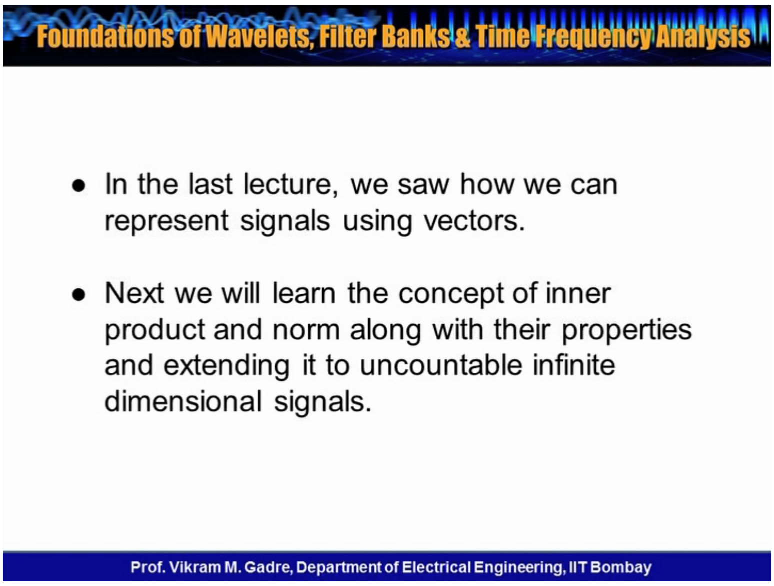


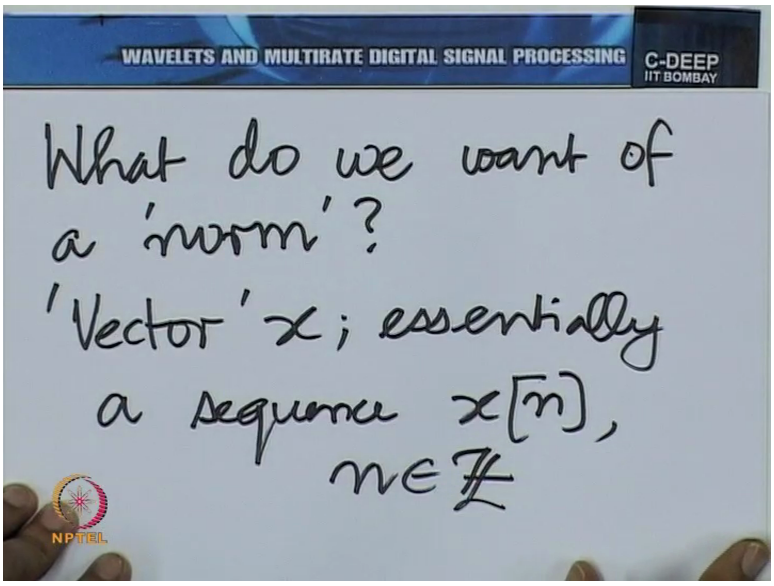

'Norm' of sequence $x$

$= \|x\|$ should be

$\langle x, x \rangle^{1/2}$ and further:



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$\|x\| \geqslant 0$

and $\|x\| = 0$

$\Longleftrightarrow x = 0$.

i.e. $x[n] = 0$

$\forall n \in \mathbb{Z}$



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If $x_1, x_2$ are real

$$\langle x_1, x_2 \rangle = \sum_{n=-\infty}^{+\infty} x_1[n]\, x_2[n]$$

So now the following things that we demand off this concept of norm or magnitude, let us write them down. It is a useful and powerful idea to have around us. So what do we want of a norm? So if I have a vector x, essentially a sequence xn, n over the set of integers, then its norm which we shall denote in the following way. We denote it like this, should be essentially the dot product of x with x square root and further we would want norm of x to be nonnegative and if at all the norm of x is 0, that implies and is implied by the sequence itself being 0 everywhere.

That is x of n is equal to 0 for all n belonging to the set of integers, this is important. So we do not want that norm to be 0 unless the sequence itself is a 0 sequence. A nonzero sequence, even if it is nonzero at one point must have a nonzero norm. And a 0 sequence must have a 0 norm. Does our dot product satisfy this? Well, for real sequence it does. If xn is real, rather if x1 x2 are real and we take the following definition.

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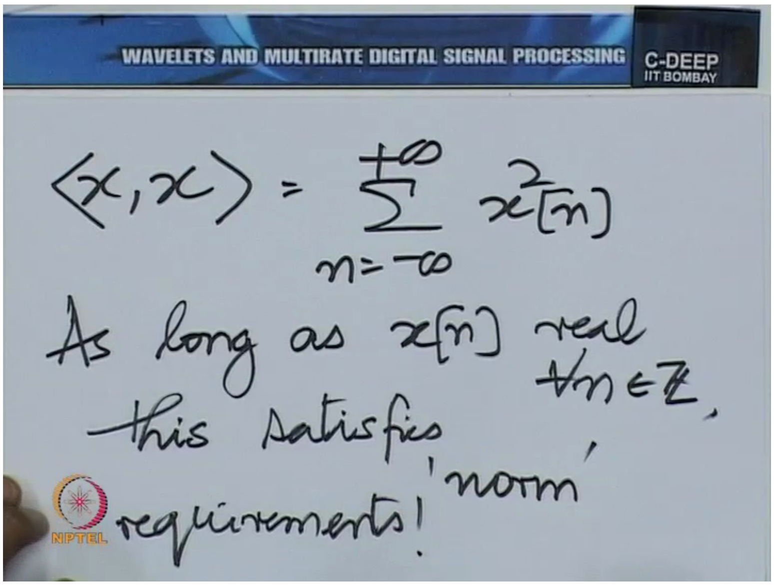


The dot product of x1 and x2 is essentially summation on n going from - to + infinity x1 n x2 n, then the dot product of x with x is essentially summation n running from - to + infinity, x square n. And as long as xn is real for all n belonging to Z, this is non, this satisfies the requirements of norm. It is nonnegative and it is 0 if and only if the sequence is identically 0. But what if this is complex? So we have to allow complex sequences too.

One of the coordinates could be complex and in fact the situation could be such that x squared n could be +1 for one of the coordinates and -1 for some other coordinate in that case because when you square a complex number, nothing guarantees the output is going to be nonnegative. In fact nothing even guarantees the output is going to be real, where is the question of nonnegative? So this definition is not going to work when x1 and x2 are complex sequences in general. And we need to tweak the definition a little.

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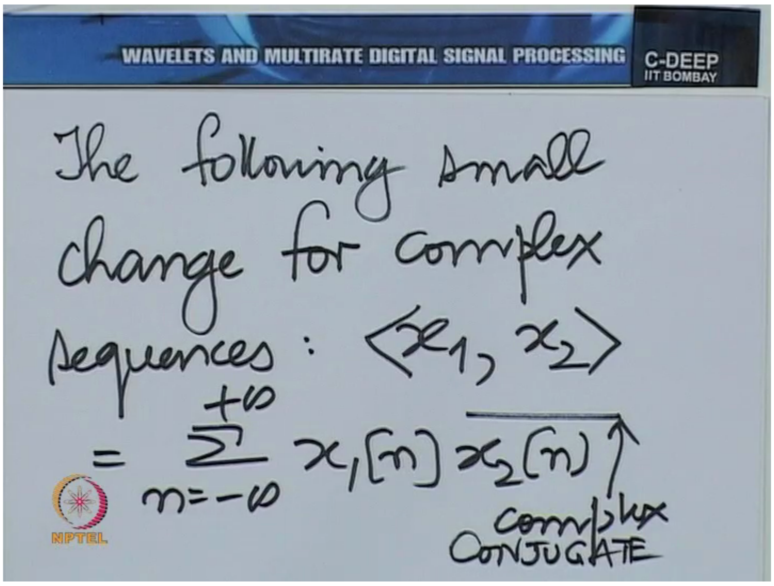


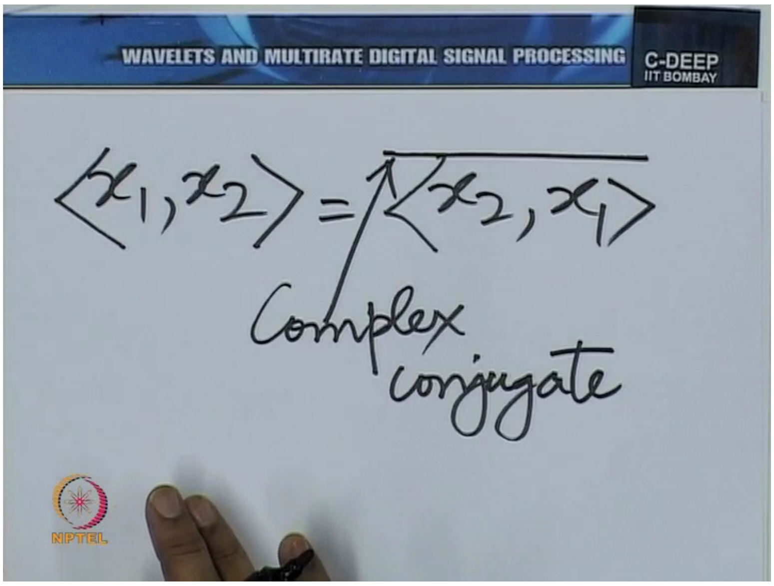


Well, it is not that difficult, after all what we want is that for every coordinate you must get a nonnegative quantity when you take point by point products. So all that we need to do for that purpose is to complex conjugate the $2^{nd}$ argument in that summation. So the small change for complex sequences will do our job. Dot product of x1 with x2 is summation over all n, x1 n, x2 bar n where bar denotes the complex conjugate. Now, one point to note here when we make this little change is that that commutativity property is lost.

So if I take the inner product x1 with x2 and then if I take the inner product x2 with x1, there is a complex conjugate relationship and this is the more general requirement of a dot product. In fact this is the simplest way in which one can define a dot product and sequences. There are many other ways, again infinite number of ways but at this moment we shall not go into the other ways, it will only confuse us. This is what is called the standard inner product, but one can have many other non-standard inner products which obey the following conditions.

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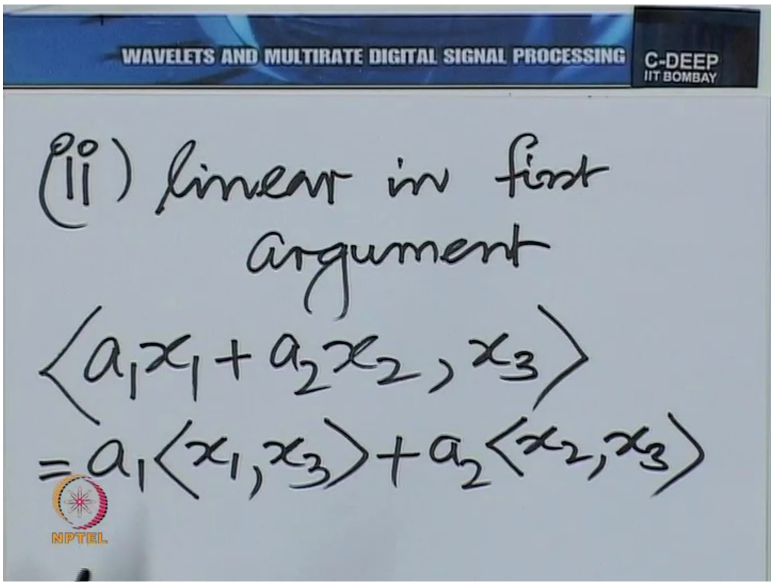


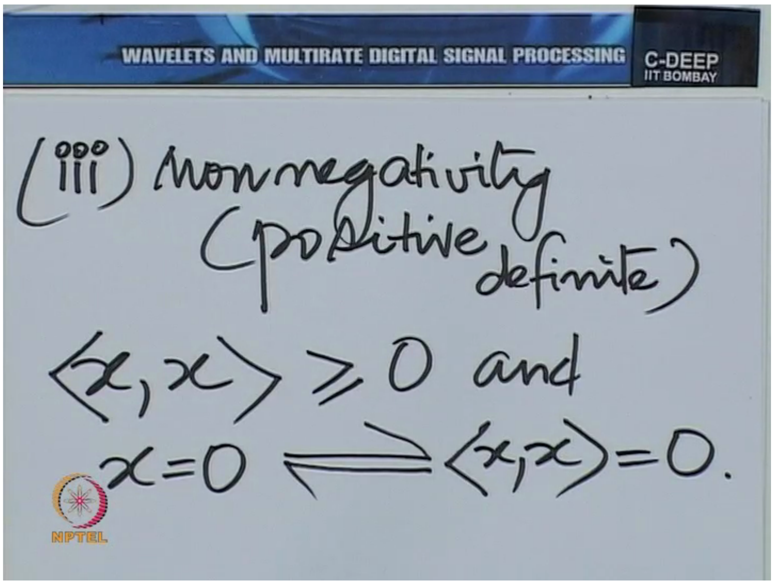


The 1st condition is this that we write down here, the inner product of x1 with x2 is the complex conjugate of the inner product of x2 with x1. Secondly, the inner product is linear in the 1st argument, in other words if I take A1 x1 + A2 x 2 where in general A1 and A2 could be complex and take the inner product with x3, it is essentially A1 times inner product of x1 with x3 + A2 times the inner product of x2 with x3. This is the 2nd requirement of an inner product, linearity in the 1st argument.

The 3rd requirement of the inner product is what we have been building towards all this while, namely what is called the positivity or non-negativity. In fact positivity is more appropriate, positive definiteness. namely, the inner product of x with x is always greater than equal to 0 and x equal to 0 implies and is implied by the inner product of x with x being 0. In fact any operation between 2 sequences x1 and x2 which obeys these 3 conditions is called an inner product.

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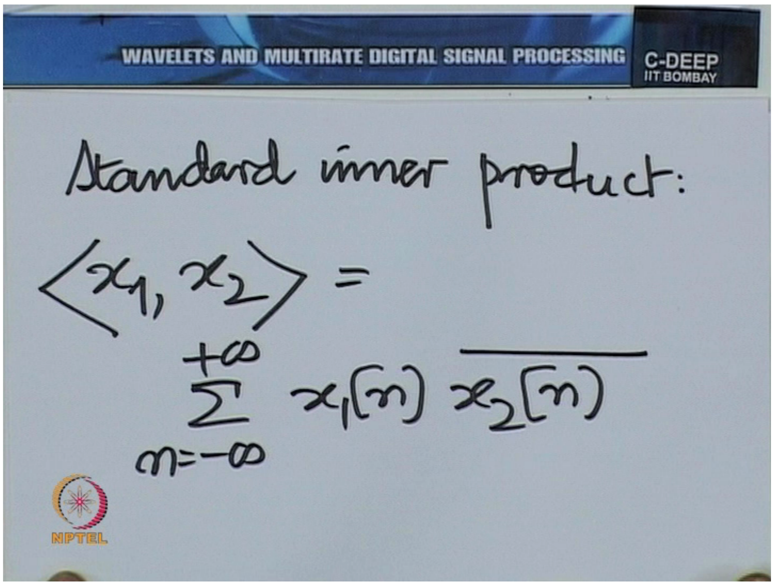


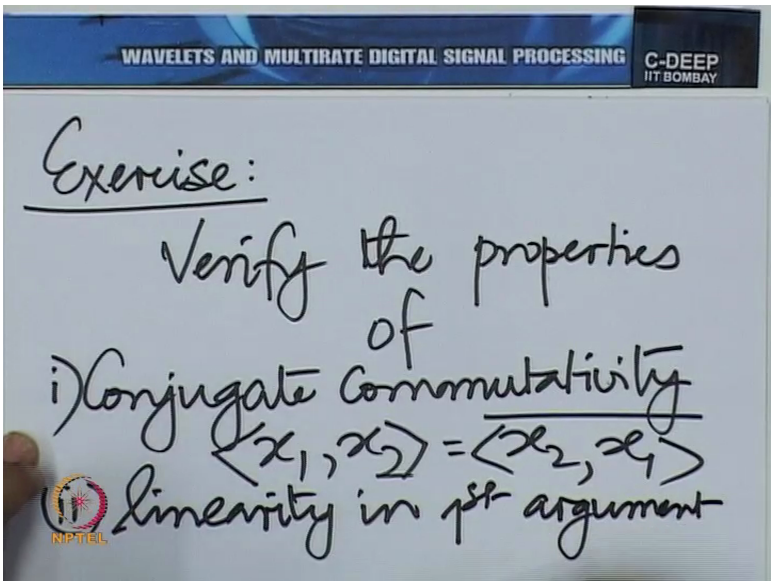

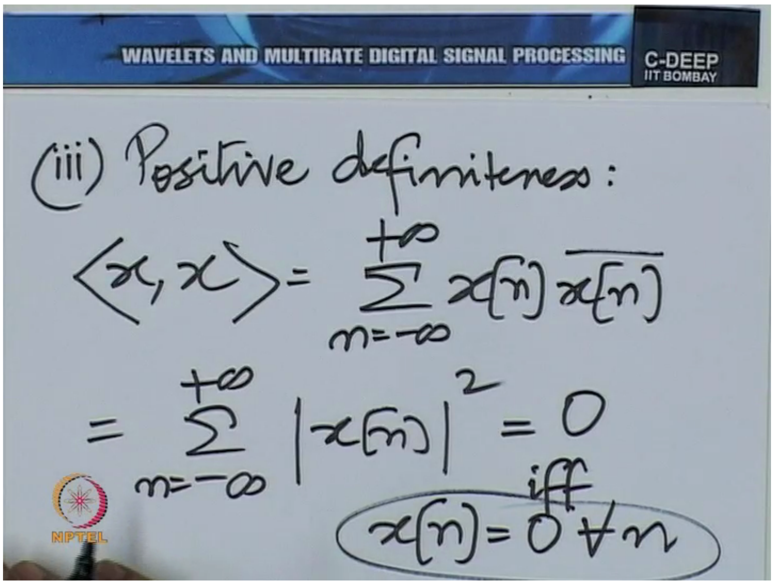


And the standard inner product that we have just described is one such which we shall use very frequently. So in the discussions henceforth, when we say inner product of sequences, we mean the standard inner product unless otherwise specified. Alright, so let us just verify this, for completeness let us verify this for the standard inner product. The inner product of 2 sequences x1 and x2 is essentially the sum n going from - to + infinity x1 n, x2 bar n, definition.

The 1st property as we said is complex conjugate, easy to verify, so in fact I leave it to you as an exercise, verify the properties of what is called conjugate commutativity, the 1st property and linearity, linearity in the 1st argument. I leave it as an exercise, easy enough to do. But we shall, because it is so important, verify the 3rd property, the positive definiteness. Indeed if we take the dot product of x with x, it is summation n going from - to + infinity, xn, xn bar, which is summation n going from - to + infinity mod xn squared.

And it is very easy to see that this is equal to 0 if and only if xn equal to 0 for all n. Even if one of the coordinates is nonzero, that particular model xn squared is going to be nonzero and it is going to contribute a positive term and of course it is very easy to see that each term for every n I mean is strictly positive if xn is nonzero. So far so good. So now we have build up the idea of inner product or dot product between 2 sequences which is going to be useful to us. So we move from two-dimensional to three-dimensional to n dimensional, n is finite and then to countably infinite dimensions.

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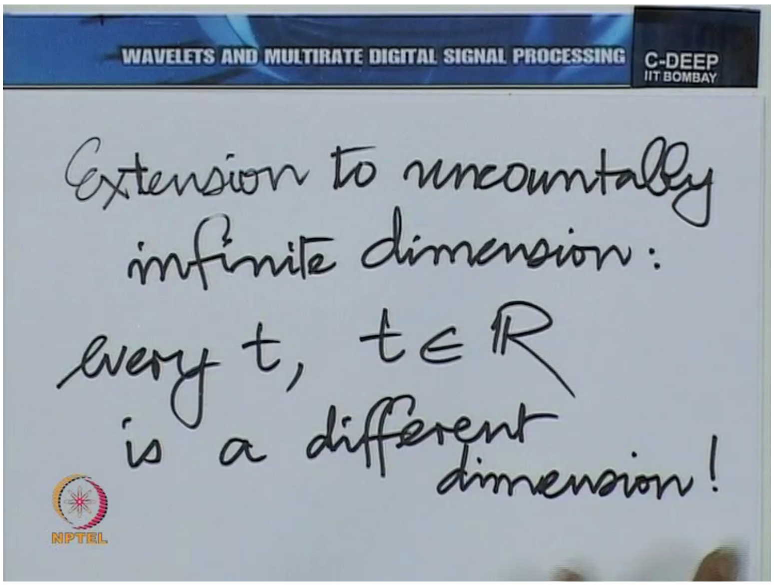


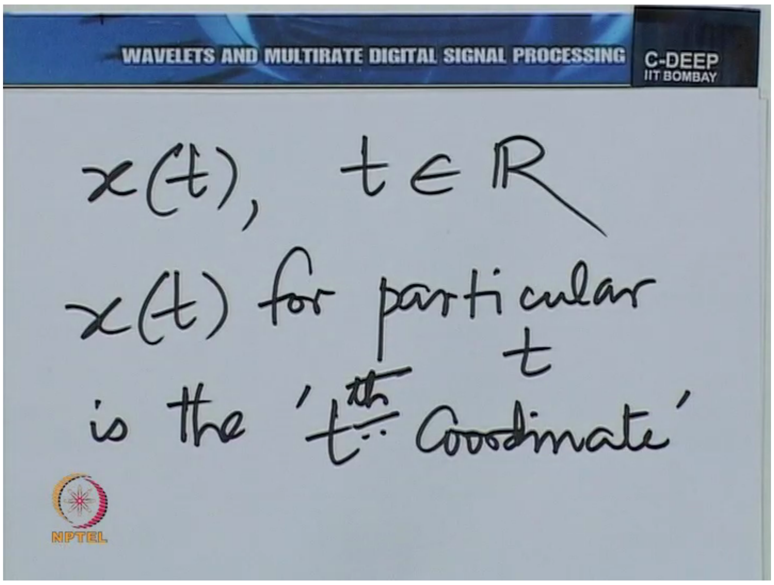


Now let us move to unaccountably infinite dimensions. So suppose I take a function of the continuous variable t, how can I extend these notions? So extension to unaccountably infinite dimensions. While this is going to be very difficult in general, but very easy in particular if we simply accept that every t for real t is a different dimension, simple. If you have a function x of t, t over the real numbers, x of t for a particular t is the th coordinates so to speak and there is an unaccountably infinite number of such coordinates indexed by the real numbers.

So in principle, in a given function you have complete liberty to put down the value of xt at every different point t. The only catch is we have agreed that we would like to make the function square integrable. So that does put some restrictions on xt but not a very serious one even so. Now, you know dealing with infinite dimensional spaces, if we wish to do it very rigorously and very very carefully and you know to satisfy the fastidious mathematician is a difficult job and we do not really intend to do that all the way in this course.

If some of us do wish to take that puritanical perspective, one of course would benefit from it in some ways and one could look up a book on functional analysis but what we wish to do is rather to give intuitive understanding of some of the concepts at different places. The intuitive understanding will not be different from a more rigourous understanding for those specific situations. But it might not quite be complete. Even so will not suffer too much in our study of wavelets, in our applications of wavelets if we take this intuitive path to some extent, not not all the time. I mean to some extent in the context of dealing with infinite dimensional spaces.