Foundations of Wavelets, Filter Banks and Time Frequency Analysis.
Professor Vikram M. Gadre.
Department Of Electrical Engineering.
Indian Institute of Technology Bombay.
Week-2.
Lecture -4.1.
Vector Representation of Sequences.

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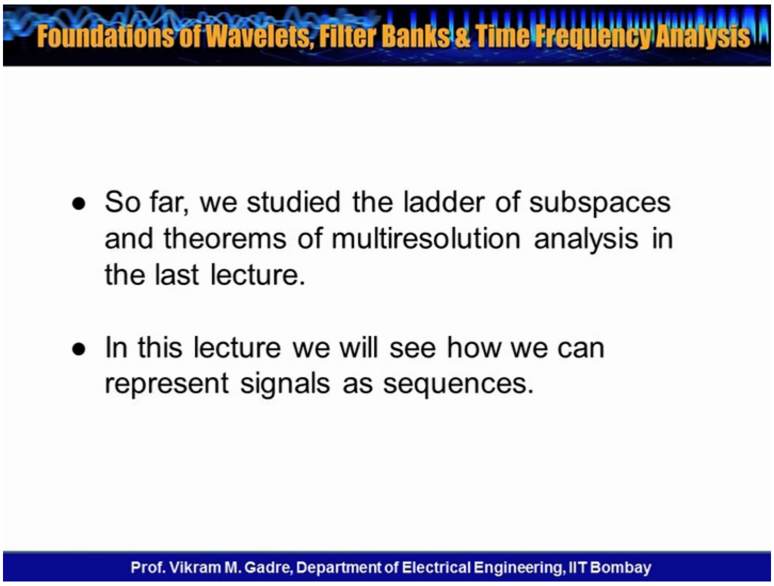

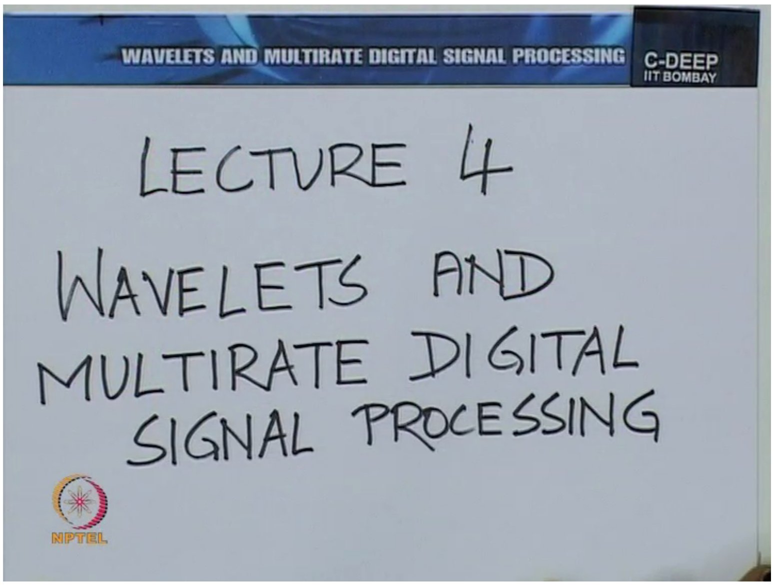

A warm welcome to this lecture on the subject of wavelets and multirate data signal processing in which we intend to build further the connection between signals or functions in L2R and vectors and therefore we wish to build further the idea of thinking of functions as belonging to linear spaces and characterising them in a manner slightly different from what we were doing in the previous lecture. So just to put our discussion in perspective, this is the

4^{th} lecture on the subject of wavelets and digital signal processing and what we intend to discuss in this lecture is the following, let me put down the points one by one.

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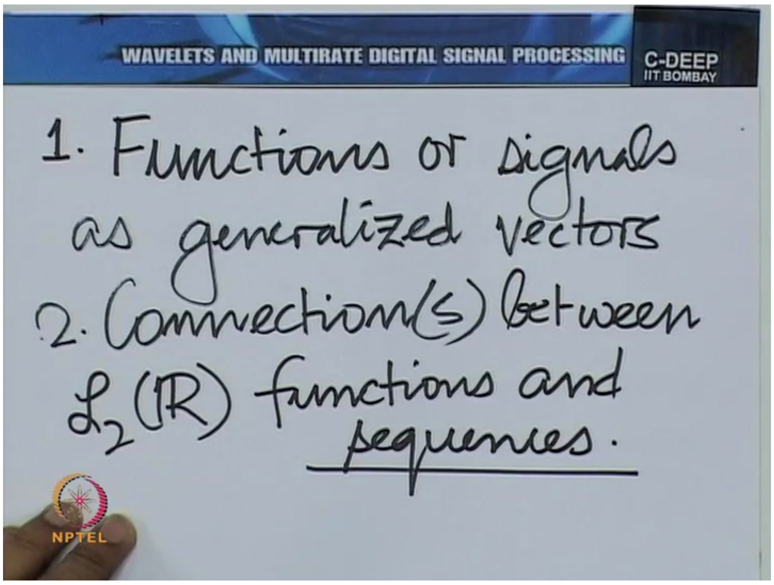

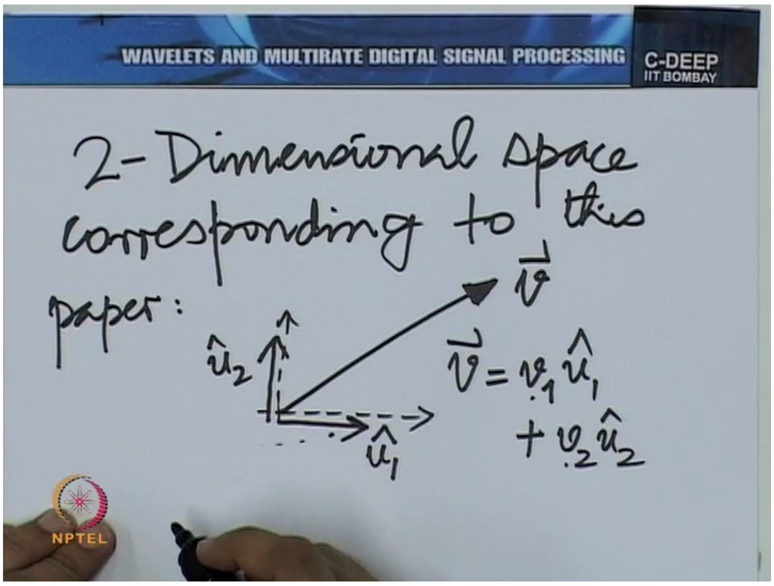

The 1^{st} thing that I wish to talk about today is to think of functions as generalised vectors. This idea is going to be useful to us in many different contexts in this course. So we need to understand this connection between functions or signals and vectors in depth, we shall spend some time on it today. Secondly the connection between L2R functions, connection or connections between L2R functions and sequences, we wish to understand this in greater depth.

So what we are going to show in the latter part of this lecture is that one can intimately relate processing of a function to processing of an equivalence sequence. And whatever we are doing to try and gain information from or modify a function can be done by equivalently processing or modifying that sequence corresponding to the function. Let us then embark on the 1st of these 2 objectives now. You see, let us begin by asking what characterises a vector after all? Let us take a minute and reflect.

What characterises a two-dimensional vector for example? A two-dimensional vector is essentially characterised by 2 coordinates which are independent, we call them perpendicular coordinates. Actually the idea perpendicularity there is also intimately related to the idea of independence. So for example, let me treat the plane of the paper as a two-dimensional space, the two-dimensional space corresponding to this paper. Well, let us take any vector on this two-dimensional space.

Let this vector be V, I am marking this as V. There are many different ways to characterise this vector, in fact notionally an infinite number of ways and one of those ways is to choose the following 2 so-called perpendicular axis. So we choose one axis like this, another axis like this and choose a unit vector along each of them. So I have say a unit vector, let me call it U1 cap unit vector along this axis and another unit vector U2 cap along this axis and then I could write V1, I could write this, sorry, just the vector V uniquely as, say V1 times U1 cap + V2 times U 2 cap.

Whereby V1 and V2 characterise this vector V uniquely in this two-dimensional space with respect to the coordinate system generated by U1 and U2. And there is an infinity of such coordinate systems. In fact one infinity of such coordinate system can be generated simply by rotating this coordinate system of U1 and U2. It is very easy to see that if I take this structure, U1, U2 and rotate it by any angle in this two-dimensional plane, it would give me a new coordinate system.

So there is an infinity of orthogonal coordinate systems in two-dimensional space and in fact there is also relation between all these infinite orthogonal coordinate systems, simple enough. And orthogonal coordinate systems are not the only kinds of coordinate system for a two-dimensional vector. So for example the same two-dimensional space can be described by the following different coordinate system.

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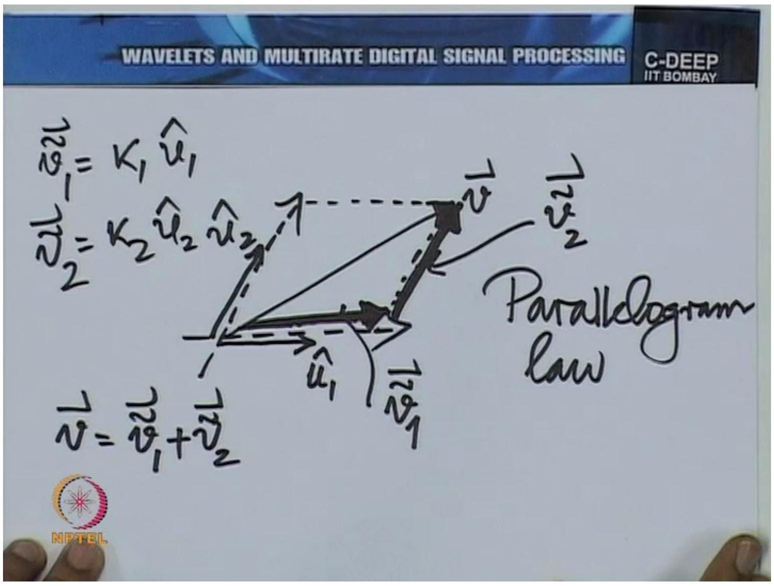

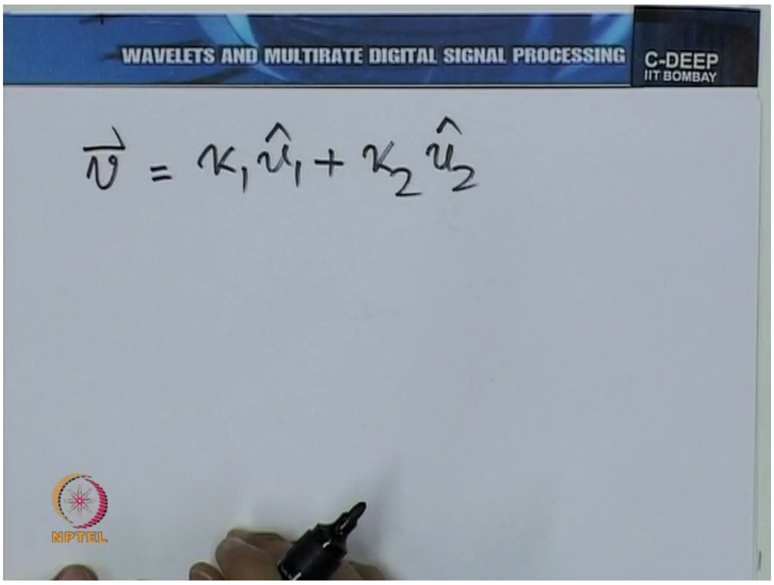

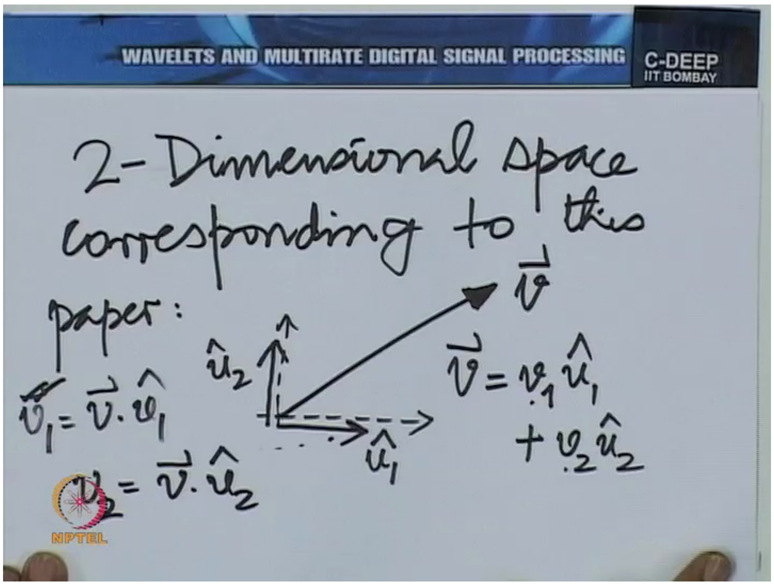

So I will draw the same vector V. And it is perfectly all right to choose a coordinate system something like this, I could choose one coordinate like this and another coordinate like this. And of course I could again have the unit vectors in these 2 directions, U1 cap so to speak, U 2 cap. And I could express V in terms of U1 cap and U 2 cap, indeed I could complete a parallelogram here, so using the parallelogram law I could draw a line parallel from the tip of this vector to this U2, another one parallel to U1 from the tip of the vector and it is very easy to see that this dot dash vector here + this dot dash vector here gives me V.

Let me highlight that dot dash vector, this vector here + this vector here gives me V. Let me call this V1 tilde and it is a vector and let me call this V2 tilde, that is again a vector. Of course we have V is V1 tilde + V2 tilde and it is very easy to see that V1 tilde as a vector is some multiple of U1 and similarly V2 tilde as a vector is some multiple of U2. Thereupon I have V is some multiple of U1 + some other multiple of U2, K1 U1 + K2 U2.

The only catch is determining K1 and K2 is a little more difficult than determining the constants in the previous representation. In fact let me go back to that previous representation. I have this representation previously where V is V1 U1 cap + V2 U2 cap and remember V1 and V2 here of course are constants and very easy to obtain because I cn simply obtain them by taking the dot product of V with U1 cap and V with U2 cap. So in fact in the sense of dot products V1 is indeed V dot U1 cap.

And V2, V1 is a coordinate, not as a vector, V2 is the coordinate is the dot product of V with U2 cap, simple enough. Such a simple relationship does not exist in this context. While we are not hard put to describe the process by which we obtain K1 and K2, it simply says construct a parallelogram, expressing this analytically is a bit of work. So it is definitely very clear from this example that an orthogonal or perpendicular coordinate system has its advantages. It is always nice to have a perpendicular coordinate system in two-dimensional space to represent any two-dimensional vector.

The same idea can of course be extended to 3 dimensions and then one could also conceive of more than 3 dimensions, 4 dimensions, n dimensions and then in principle an infinite number of dimensions too. Now there again when we talk about infinite dimensional situations, we have countably infinite and unaccountably infinite, finer points. But for the moment, infinite is difficult enough. So infinite dimensional vectors in fact lead us to the idea of functions.

Now it is a little difficult to understand infinite dimensional vectors all at once. So to progress towards infinite dimensional vector, it is easier 1st to start from finite dimensional vector of larger and larger dimensions. And all that we need to do is to understand that what characterises the dimension of a vector is really the number of independent coordinates that it has. For example a three-dimensional vector has 3 independent coordinates, a 4 dimensional vector would have 4 and n dimensional will vector will have n.

And a countably infinite dimensional vector would have a countably infinite number of dimensions or countable infinite number of coordinates. By countable we mean we can put the coordinates or the dimensions in one-to-one correspondence with the set of interiors. So we can talk about the 0th coordinate, we can talk about 1th coordinate, we can talk about the -1th, the - 2th coordinate and so on and so forth. What are we talking about here then, if we talk about an infinite dimensional vector, we are in fact talking about sequences.

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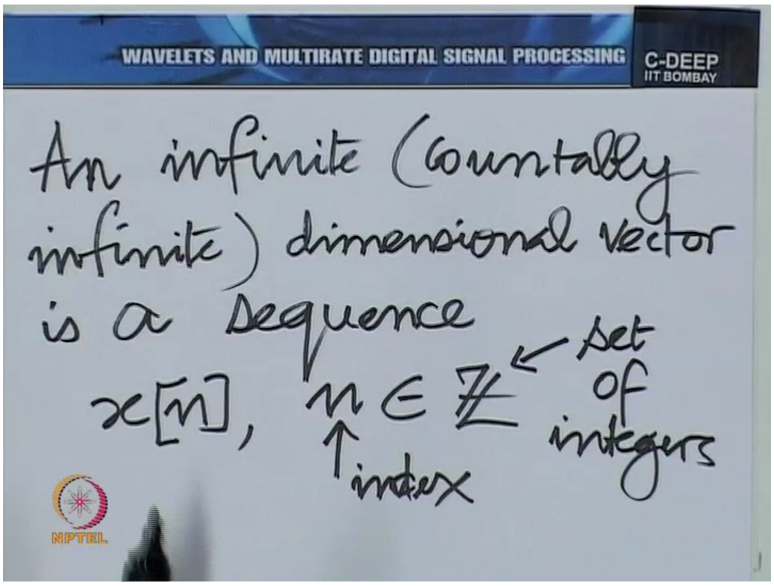

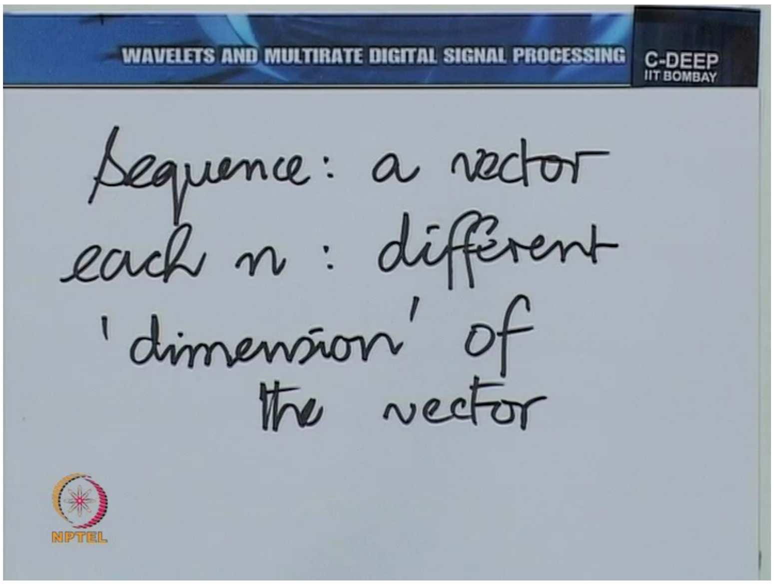

So we build up the idea from there. So here we are, let us make a note of this. An infinite dimensional vector or rather an infinite, countably infinite dimension vector is essentially a sequence. So for example we have a sequence x of n where n belongs to set of integers, over all the integers. Recall this script Z is a representation of the set of integers and this is called the index variable.

So now we have a different interpretation for sequences, sequence is like a vector and each n is a different dimension of that vector. Okay, I think that is important enough for us to write down explicitly. So a sequence is a vector, each n is a different dimension of the vector. And once we have this analogy, then extending other ideas of vectors to this context is not difficult at all. For example, adding 2 vectors, simple, add the sequences point by point. Multiplying a vector by a constant, very simple, multiply each point of that sequence by that constant.

What we would like to do now is to extend some of the other ideas of vectors that we have, some of the geometrical ideas to this context of infinite dimensional vectors. And one of the very useful ideas that we have in the context of vectors is the idea of a dot product. How do we take the dot product of 2 vectors in two-dimensional space? So let us recall.

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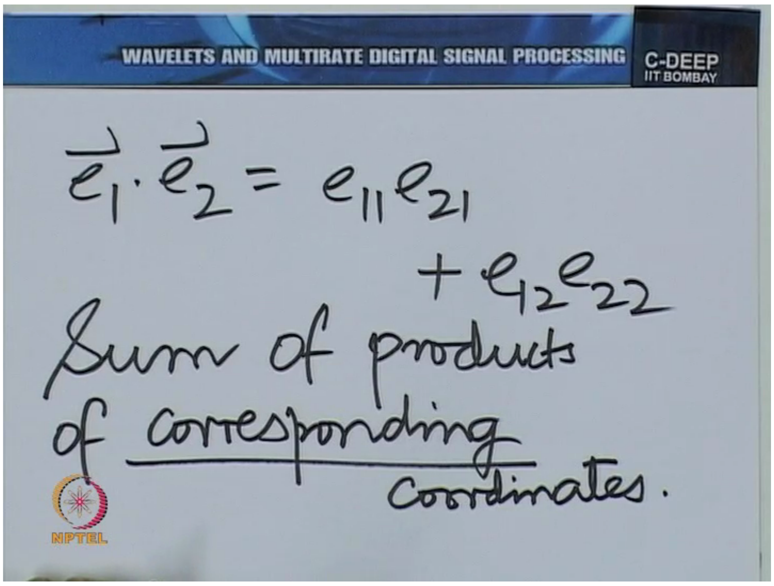

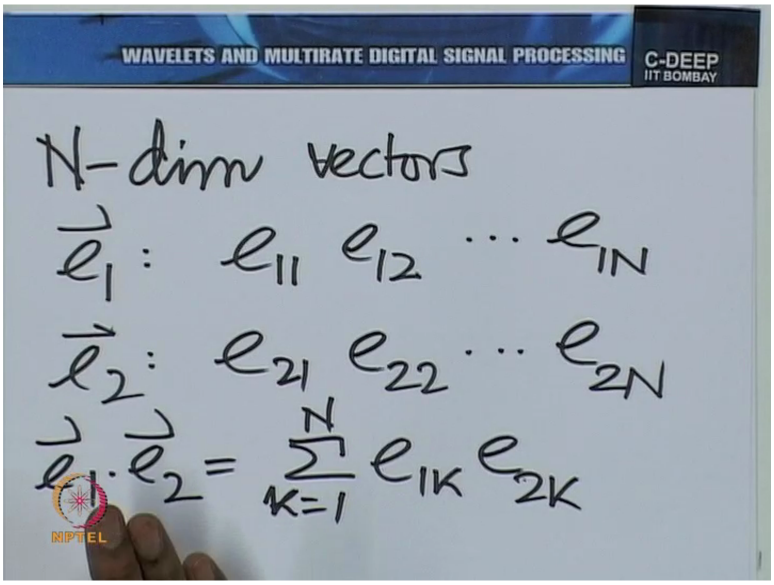

So suppose for example we choose a pair of orthogonal coordinates, so we have U1 cap and U2 cap as we did some time ago orthogonal to one another, perpendicular to one another and we have 2 vectors, let us call them e1 which had the coordinates e11 and e 12. So e1 is e11 U cap, let U1 cap + e 12 U2 cap. And similarly e 2 as a vector has coordinates e21 U1 cap + e 22 U2 cap. Then the dot product of e1 and e 2, e1 dot e2 as we write it is essentially e11 e 21 + e12 e22.

So it is the sum of products of corresponding coordinates. 2 dimensions is easy enough to understand, 3 dimensions, easy to extend, in fact n dimensions, equally easy to extend. Suppose we had 2 n dimensions vectors characterised by coordinates, say e11 to e1n. So you have several n dimensions will vector. e1 characterised by coordinates e11, e12 up to e1n.

And similarly e2 characterised by the coordinates e21 e22 up to e2n. Then of course e1 dot e2 is easy to express if we generalise this.

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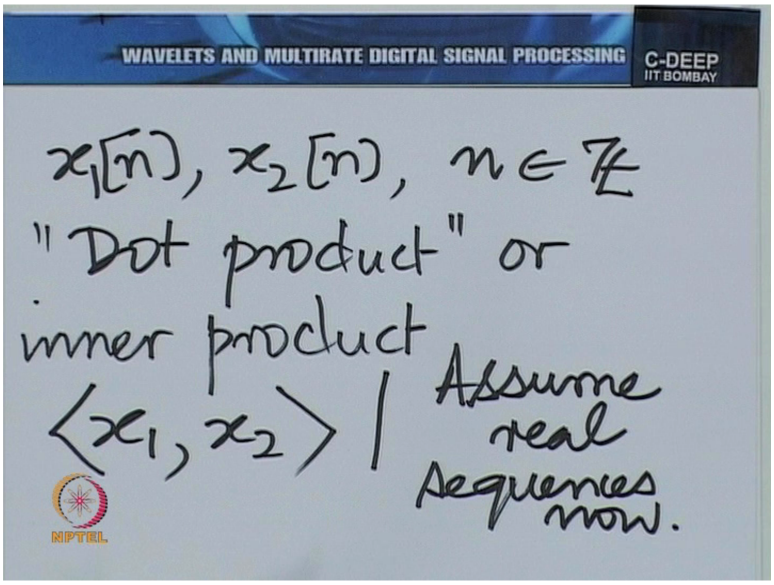

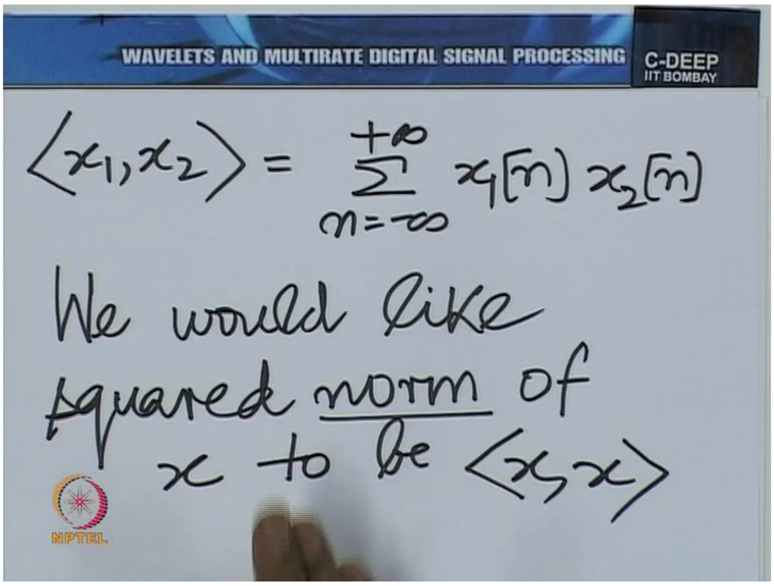

It is essentially summation K from 1 to n e1 k times e2 K. So dot product generalised to n dimensions, of course we assume these are orthogonal coordinates. Now we can even take this to infinite dimensions. So we can think of the dot product of 2 sequences, let us say x1 and x2. So we have here for example 2 sequences, x1n and x2n defined over the set of integers n over all the interiors. Their so-called dot product or inner product as the formal name is, so if we instead of dot product now, we would like to use the term inner products to generalise.

And we denote the inner product this way. For the moment, let us assume these are real sequences, for the moment. In that case if we generalise, it is easy to see that the inner product, inner product of x1 and x2 is simply summation on n, n running from all the way from - to + infinity, x1n times x2n. And of course it is clear that the dot product or the inner product as we are going to call it in this generalised situation is commutative. That means if I interchange the roles of x1 and x2, the result does not change.

However, we would like this inner product or dot product notion to give us some of the powers and some of the conveniences that the dot product offers in the context of vectors. One so-called convenience, or one so-called interpretation or meaning that we derive from the dot product is the notion of magnitude. In fact one could think of the notion of magnitude as induced from a dot product if one desires. Or in other words one could calculate the magnitude of a vector by using the notion of dot product as one path towards the calculation of magnitude.

Incidentally, the word magnitude of vectors is used for small dimensional vectors like 2 and 3 dimensions. But when we go to these generalised situations of n dimensional vector or countably infinite dimension of vectors, we replace the word magnitude by the word norm. So we say that we would like the squared norm of x to be the dot product of x with x, as is the case with vectors. So if you recall A dot A, where A is a vector in 2 or 3 dimensions for that matter, is the magnitude square of A.

The same should hold good here. When we take the dot product of a sequence with itself, it should give us the squared norm of that sequence where the norm is the more general word for the magnitude. In fact, in L2R the norm is representative of the energy but at this moment we are not talking about L2R because we have not yet come to the situation where we are dealing with functions of continuous variables.

So we will postpone that interpretation for a minute, not very far away from now, and once again come back to sequences. Even for sequences, when we take the dot product of a real sequence with itself, we indeed get something that we liken to the energy of a sequence. So it is not uncommon to refer to the dot product of a real sequence, or for that matter sequence with itself as the energy in that sequence.

Anyway, I kept emphasising real for a good reason. When we talk about the magnitude of a vector or for that matter there is more generalised word norm, what is it that we expect of a magnitude? We want the magnitude or the norm to be a nonnegative number, in fact strictly positive if that vector is nonzero.