Computational Electromagnetics and Applications
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Lecture 04
Finite Difference Method -02

In the last module we looked at Finite Difference Method. We also showcased some Matlab codes where we saw how the discretization spatial discretization is going to affect the value that we are computing. In todays lecture we are going to look into the concept of accuracy both amplitude wise and also the phase wise which we call accuracy errors in finite difference methods.

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We look into dispersion which is a special type of accuracy issue and we will also look at the stability issues related to finite differencing methods. While doing so we are going to make it a little bit more interesting by simulating some examples towards the end of the module. So with that introduction let us start into the topic of accuracy.

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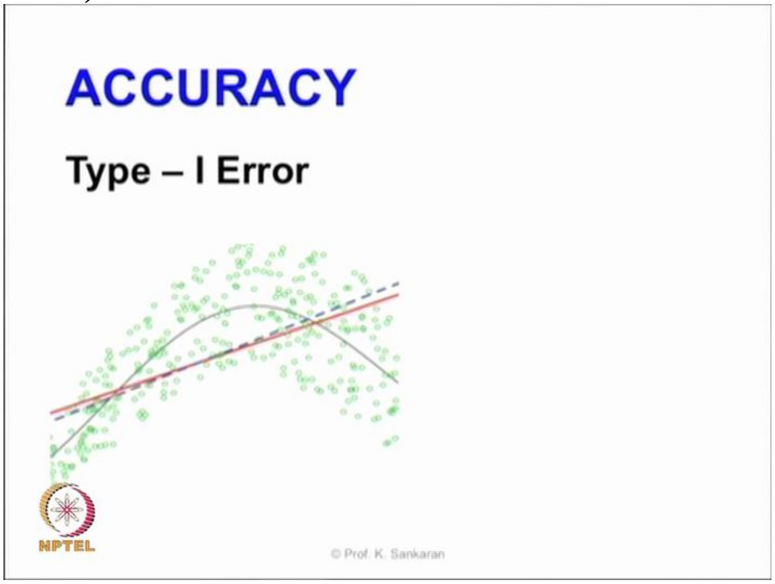

So for any method to be useful we have to talk about what is the accuracy of method when it comes to either numerical versus analytical solution or in the case where we are talking about accuracy sometimes when there is no analytical solution possible people sometime look at experimental data. So always a question is how does the method functions with respect to either an analytical solution or yet say method which we know very good, that works very accurately. So the comparison between the method A versus B or method A versus analytical solutions or method A versus experiments. So accuracy is something that we always compare depending on the case. So when we talk about accuracy there are normally three different types of errors which we address.

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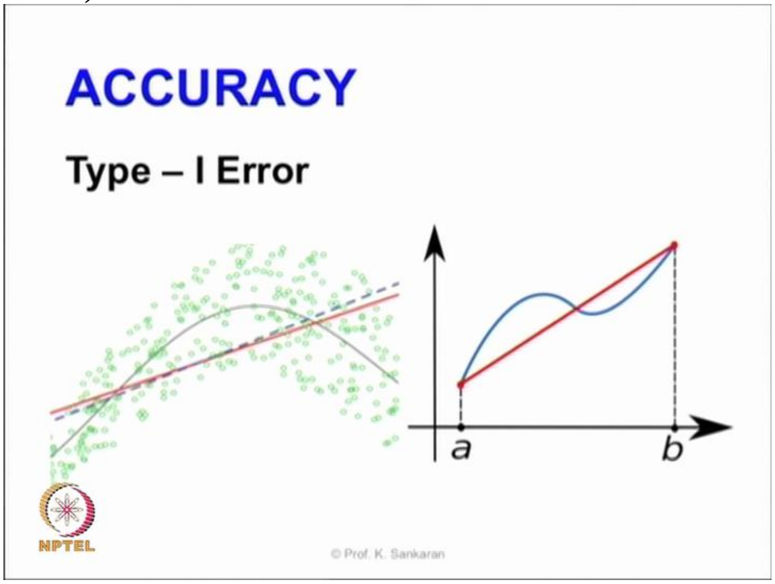

The first type of error is what we have represented here, which is called as linearization error, so for example when the real problem itself is nonlinear and you are trying to you know find certain co relations and then say it is going to behave in a linear manner within a to b.

So that is first of all a kind of an assumption which might be or might not be true. So that might be a source of error which is something that we have to look into.

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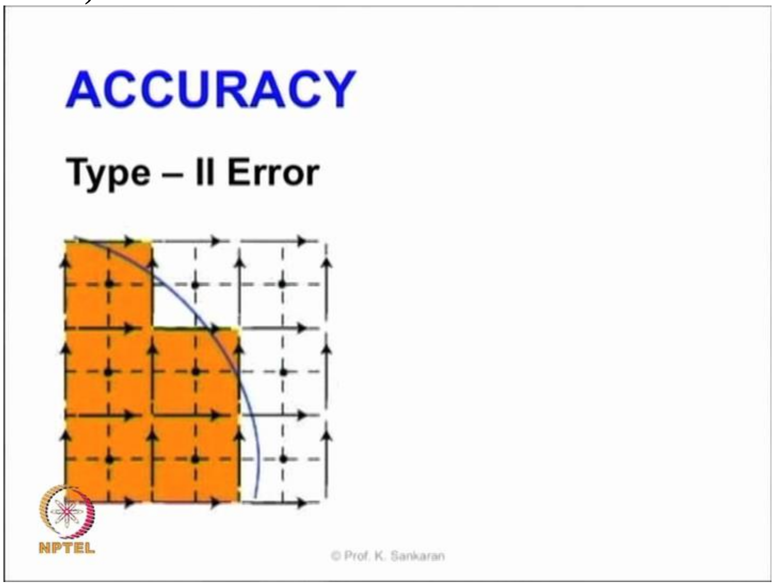

So second type of error is called as discretizationerror.So what i mean by that is let us look at this particular slide. So you have a domain. Let us say a domain which is having boundaries like this.

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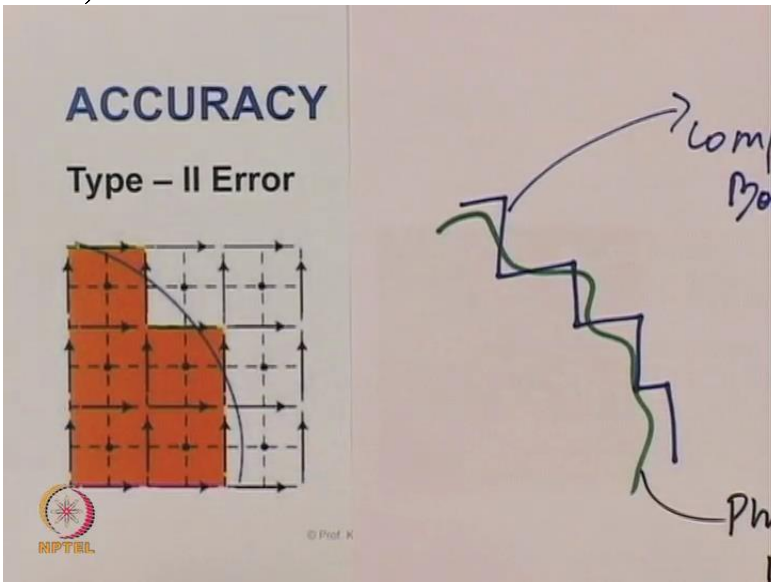

So as you can see in the slide the domain might be discretized by certain way that we call finite differencing or chair casing or depending on however you call it. So as you can see there is a always a kind of error between the domain boundary the actual domain boundary and the computational boundary. The computational boundary here is the blue line which is here. The other one is the real physical boundary. We already acknowledged that for methods that are using finitedifferencing this is a very known well known and well acknowledged source of error. The idea is make these step sizes so small that in a way that you can replicate the real boundary closely.

So you know if you go it so small so you are able to almost get as close as the physical boundary. So this is the type of error which we call as discretization error. And this is a type 2 error. Actually the type 1 type 2 they are just, you know it happened to be we are discussing there is nothing called type 2 error, but the kind of error what we are talking about is important. So this is what we see in this slide

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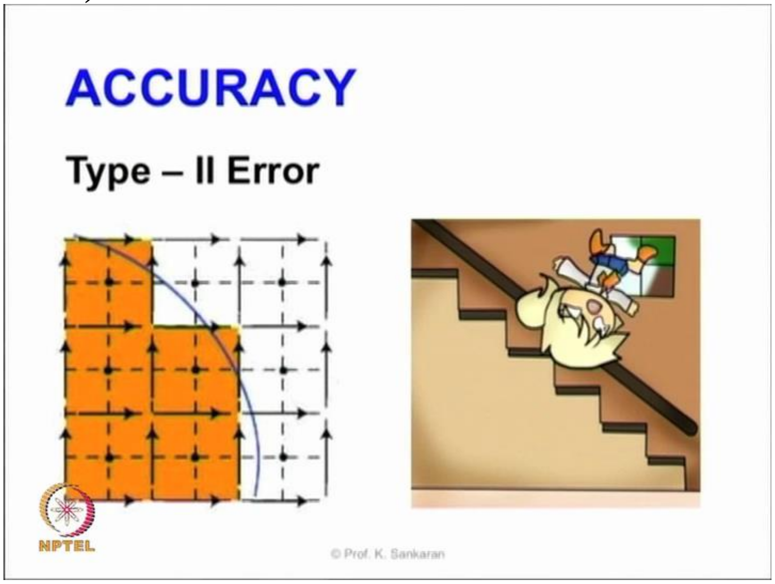

And you can also see that sometimes people make fun of discretization that no you have to be very careful with stepping in space and time because this could easily lead to a lot of error. This is just funny way of putting it.

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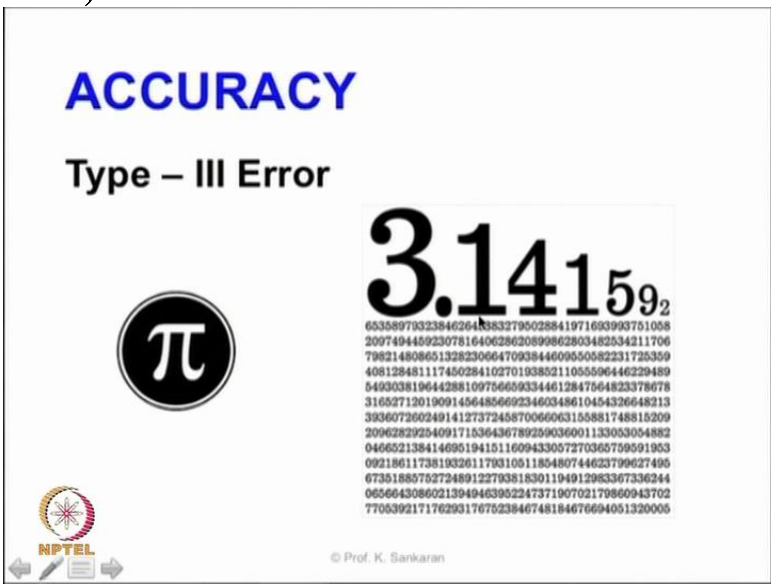

So the third kind of error we are going to talk about is truncation error related to the computational accuracy that one can reach in normal modern day computers. So for example the number pi we know that it has several digits following 3.14159. So the idea is you know the modern day computers can go quite large but still it can be never accurately representing the actual number pi is. So we have to truncate it let us say after two digit, three digit, four digit depending on what is our accuracy level. There is always going to be some error that we are going to bring into. So this is third kind of error.

So with that being said, the modeling error is something that one has to look into because it depends on how we are going to change our delta t and delta x so this is what I wanted to say.

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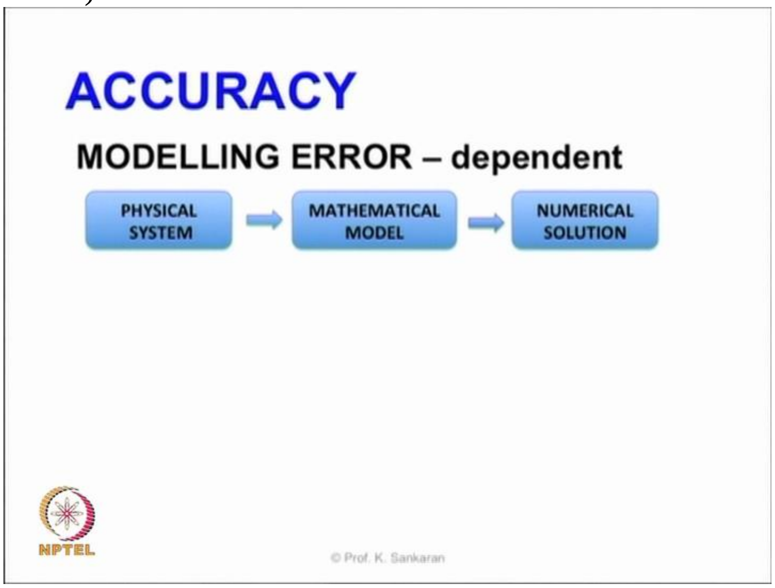

When i say depended on what i wanted to say is it depends on the value what we call delta x and delta t and of course if you are in three dimension you are talking about delta y and delta z also. So this is one of the things so the modelling error can be controlled by having a much more flexibility on these changes.

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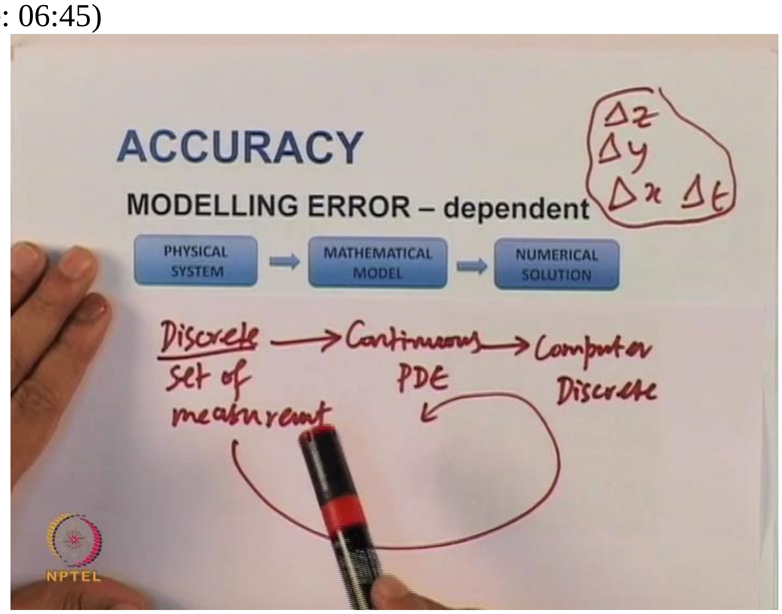

So as you can see in the slide the physical problem or the physical system has to be mathematically modeled. And this mathematical model is going to have a numerical solution. We will see this is very very a round about approach. Why i say it is a round about approach will be clear to you at later stage when algebraic topology the chapters that will come after fourth of fifth chapter which we will be looking into in this course which is a fundamental change of a paradime.

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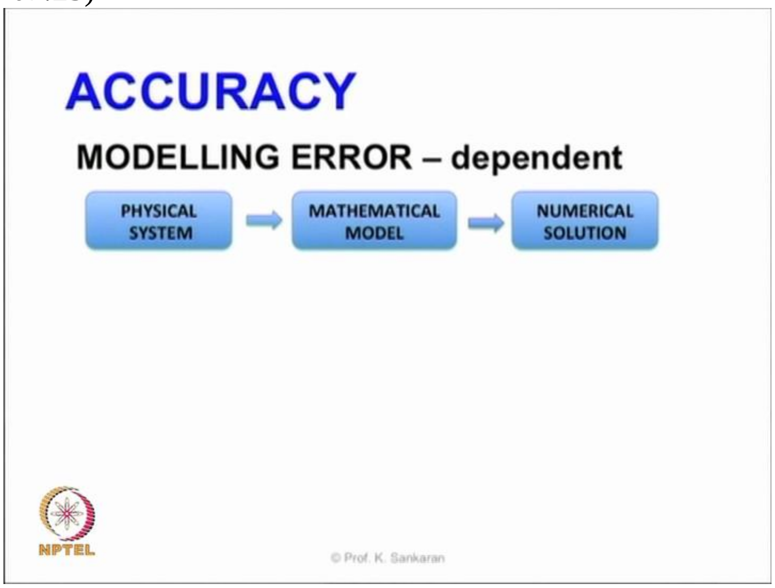

So here the physical system what we call is basically discrete set of measurements. The mathematical model is on the contrary its a continuous. What i mean by continuous is our partial differential equation are in the continuous phase and again when we go to the numerical solution on computer, computer can only handle discrete space so we are going from discrete to continuous to again discrete. Which is going the roundabout way, right? So we will address this issue when we are talking about the method of algebraic topology. For now its enough to know that this discretization what we are getting after going through a continuous form is going to create certain errors.

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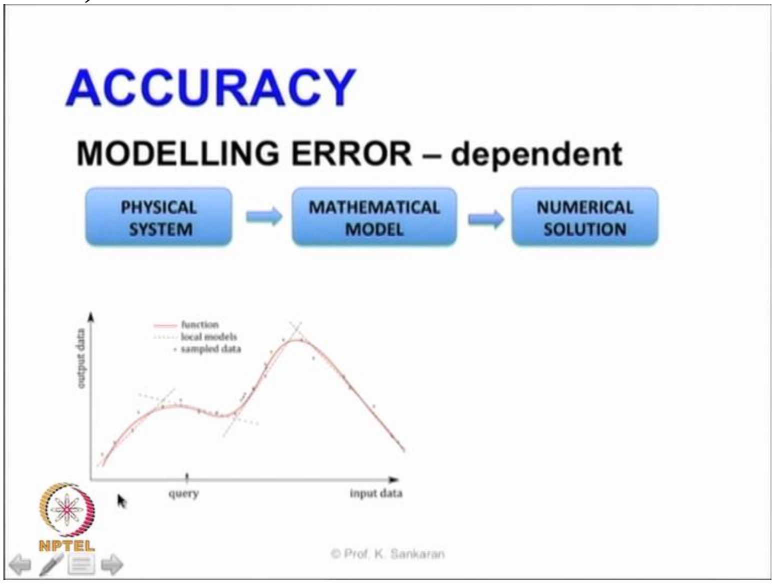

And this we can also see in the case here, where we are approximating a continuous function by set of discrete points and then finding certain piece wise continuous approximation of this discreteness.

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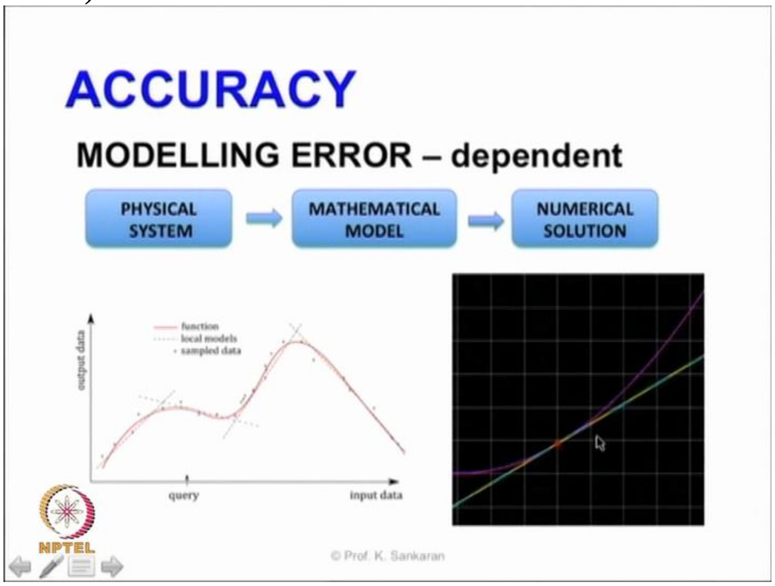

So obviously all these things are going to lead to some kind of error but the idea is when you go from very very forced mesh to very very fine mesh at the limits what you see is the discrete approximation which is measured approximately here by this white curve is going to be as close as the real function which is given here in Magenta color. So basically you see when you are refining the mesh finer and finer, when you are going closer and closer in the spatial and temporal discretization so your mesh size is becoming smaller and smaller at the limit your value whatever you are computing as the discrete value will be equivalent to the value or almost equivalent to the value that you are computing in the continuous space, So that is the idea.

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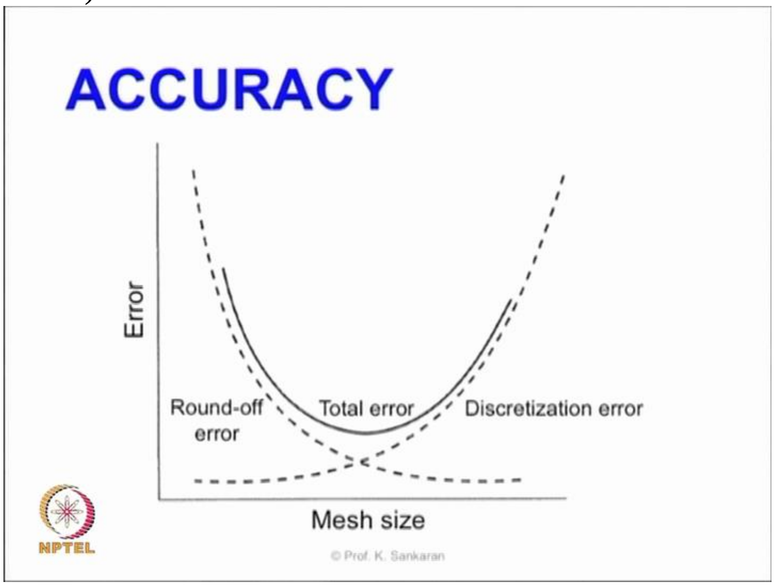

So let us now wind up on the concepts what we have discussed today on the accuracy topic. So what you have is you have the discretization error, you have the error that is coming from the round off. And obviously we did not talk here about the error that is also coming because of the errors of the truncation of the Taylor series. Which is again yet another kind of discretization error, because we are taking only finite number of terms and things on so on and so forth.

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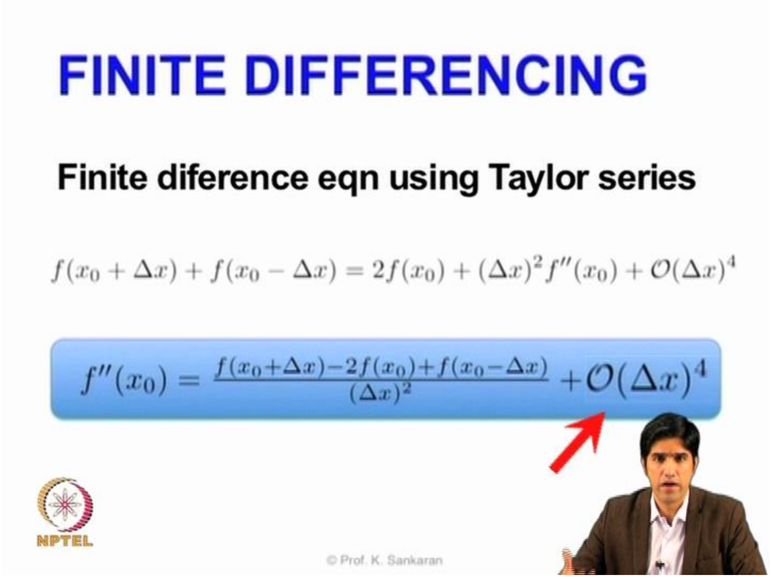

So Taylor series approximation also is a mathematically representing the truncation error.

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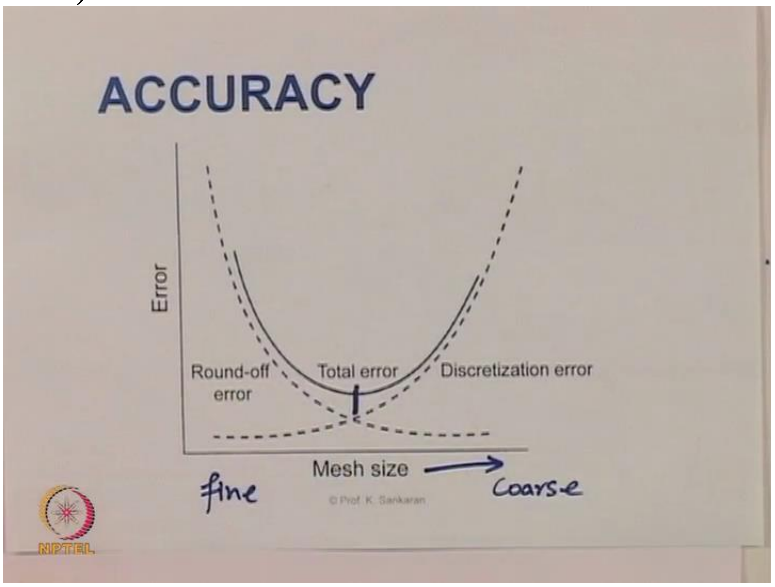

So in this graph here we are going from this is a coarse mesh and this is a fine mesh. So you see when the mesh is becoming finer and finer our discretization error is reducing which is logical and what happens is when the discretization error is reducing the round off error is increasing, it is counter intuitive. The reason why the count off error is increasing is because we have more number of cells and we have to do more number of arithmetics. So as the mesh becomes finer and finer the round off error is increasing because the number of arithmetics is going to increase which is in turn going to give you more round off errors. Whereas the discretization error is going to reduce.

So there is always an optimal point, the optimal point is where it does not really pay off when you reduce the mesh size. So that is something we wanted to say in this area, because below that point you are not going to gain any more accuracy because the round off error is going to take off.

So with that being said we have covered the most important aspect of numerical error. And this numerical error is mostly related to amplitude. Whatever amplitude what we are computing. So there is obviously another type of numerical error or inaccuracy which is related to the face. So amplitude is one part of the thing what is the face.

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So that is what we call as dispersion. So that is something we will see in the next slide. So will now study what is the meaning of dispersion. And how does it impact our numerical accuracy or numerical result.

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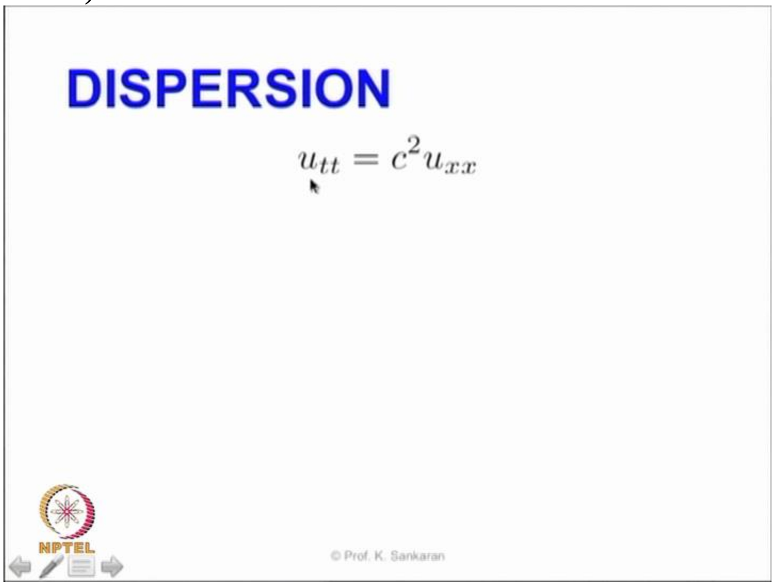

So take the simple equation, this is second order equation and as you can see this could be reduced into a time varying second order equation with second order special derivative here. And of course there is some constant that is sitting here on the right hand side.

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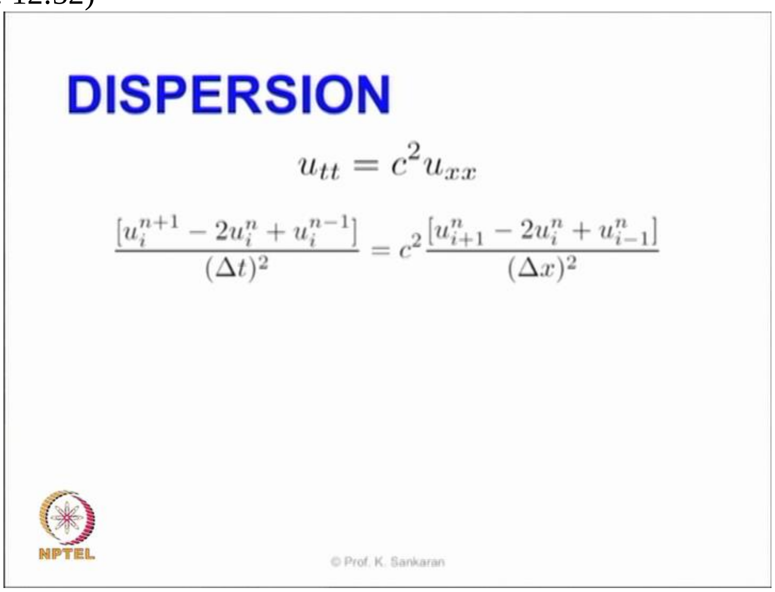

So we know that we can approximate the second order time derivative and second order spatial derivative using the central Differencing scheme.

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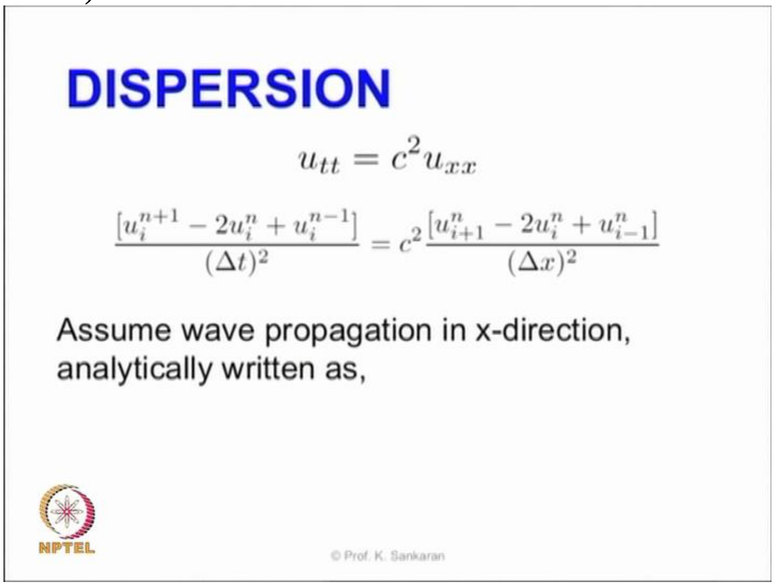

Let us assume that the wave is moving in x direction. So our analysis says that if we have a wave that is propagating in the x direction, we can write the analytical solution.

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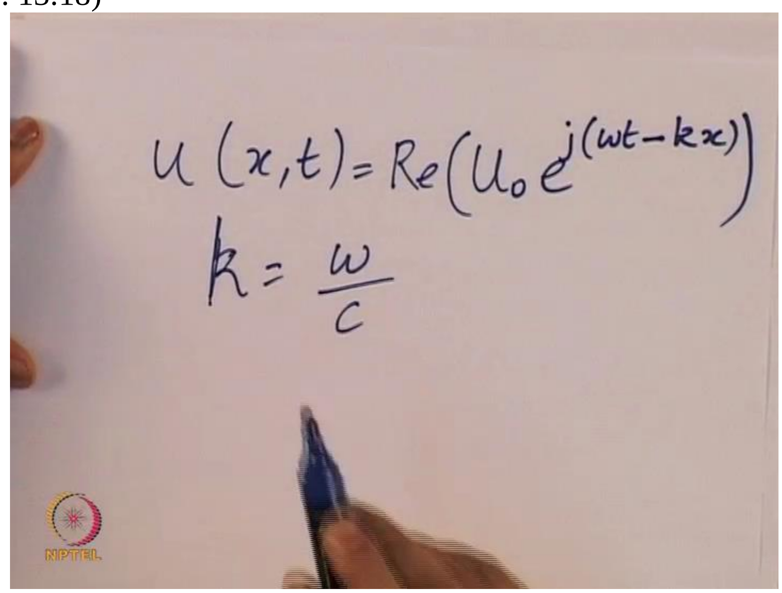

So if u is the analytical solution and u has (x, t) as the independent variable on which this u is going to depend. So we can write the solution of this as the real part of some magnitude, let us say (u 0 which is a magnitude of u times e to the power j (omega t minus kx)). So what it says is there is a its a x propagating wave it has certain time dependency and that is what we have written here. And where k is equal to omega by c. You will see at the later stage there will be a kind of a material dependency so you can represent c as a function of permittivity and permeability of the medium. For now it is enough to know this is the way it is.

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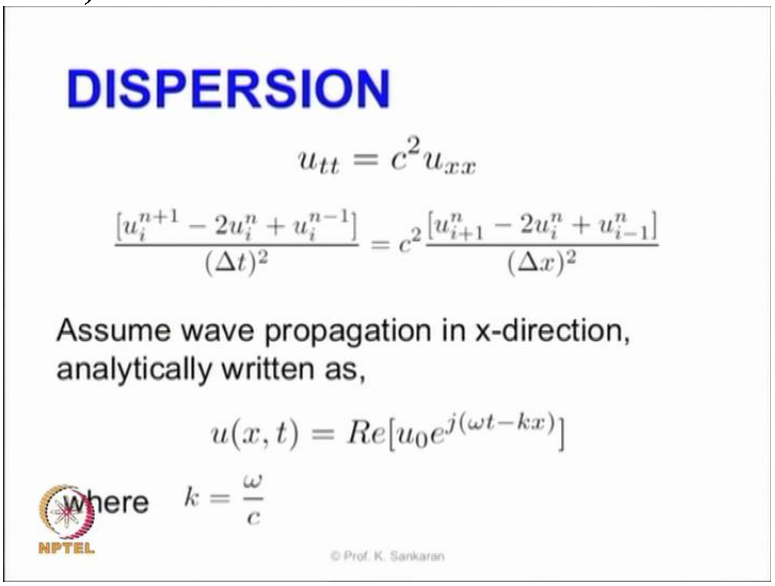

So what we are looking here is we have written the analytical solution for u as a function of x, tas we have described. So now we can write the finite difference approximation.

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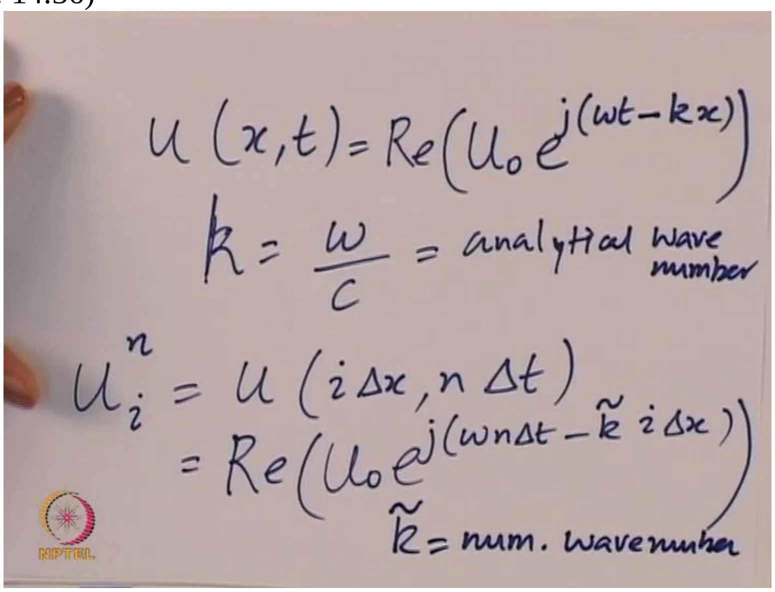

So this is as you can see is continuous in time and in continuous in space. So if you write the same thing as a finite difference approximation so we can write this as u i n which is nothing but u of (i delta x, n delta t). So instead of writing them as delta x delta t, I just denote it by a subscript and a superscript. So this on e will be equal to like in the case of analytical solution you have real part of (u 0 e to the power j omega instead of there you will have n delta t minus k numerical instead of x you will have i delta x)

So this is what we have written and here in this case this is the in the first case the k will be the analytical wave number and in the case of the finite difference or the numerical method we will call it as numerical wave number.

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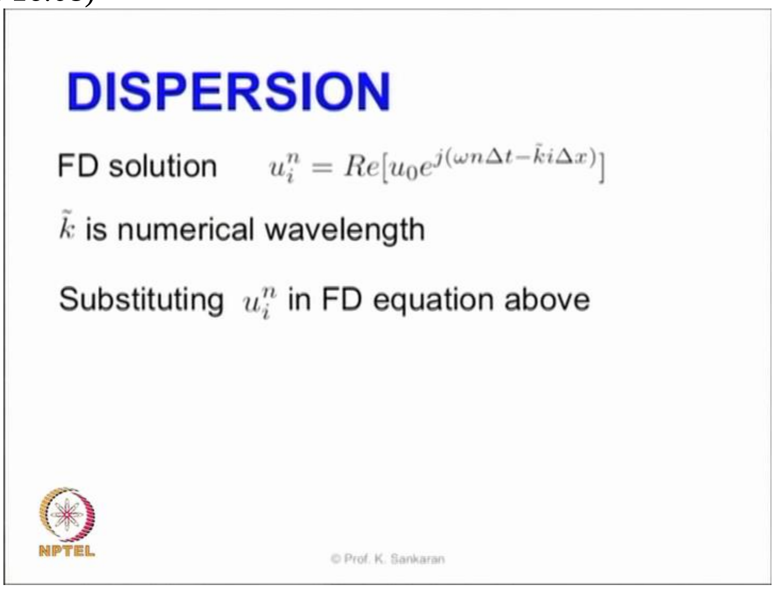

So this is what we have represented in the slide. Now we can substitute the value of u of i, n into the finite difference equation which we had before.

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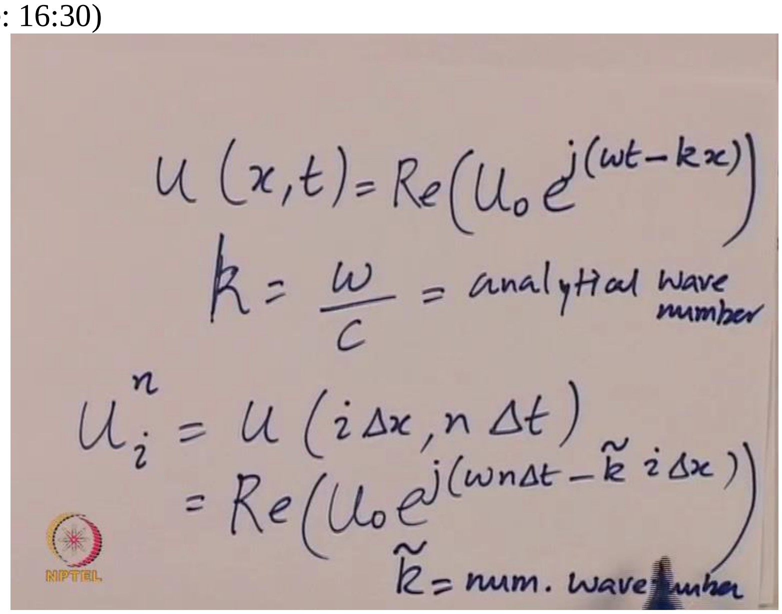

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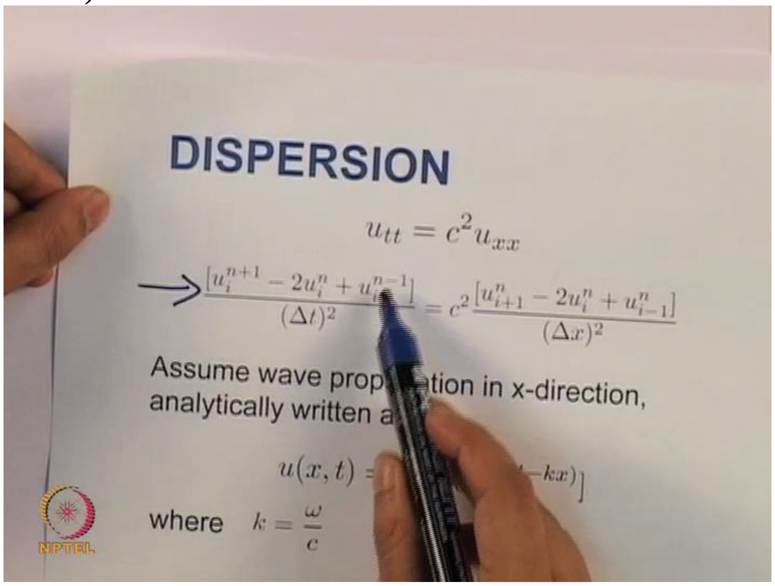

So what i am going to do now is i am going to substitute the value here of u of i, n into this equation. This equation is something that we saw before and obviously the dependence are not just only on i,n there is also i n plus 1, i n minus 1, i plus 1 n and so on and so forth. Obviously the terms will have different n and i depending on what we are having.

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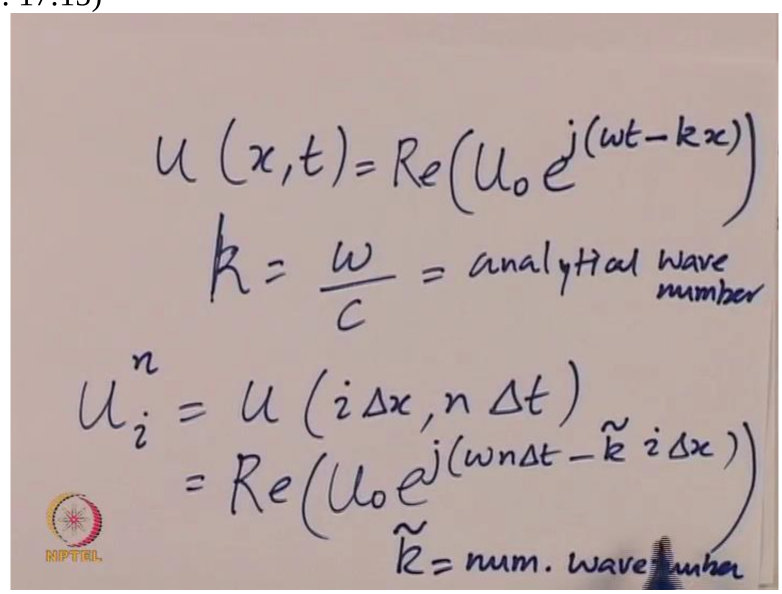

So what we will get essentially is when we substitute this value, what we will get is we will get the finite difference equation to give us and equation basically that is relating the analytical wave number to the wave number of the numerical method.

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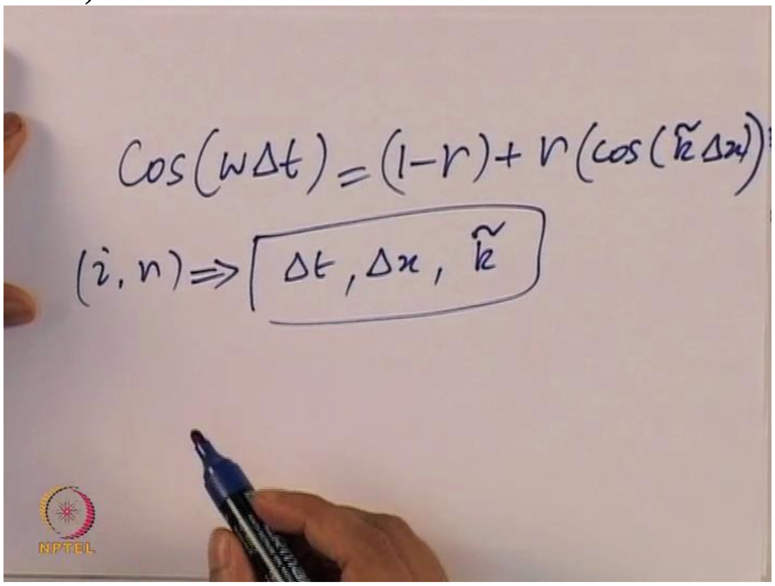

So what we have is cos of (w delta t) is equal to (1 minus r) plus r (cos of (numerical delta x)). So when we club that what we will see is we will get an expression that basically getting rid of all the i components and n components. All the i components and n components will disappear so you will get an expression which is only in terms of delta t, delta x and things. And that is what we have got here.

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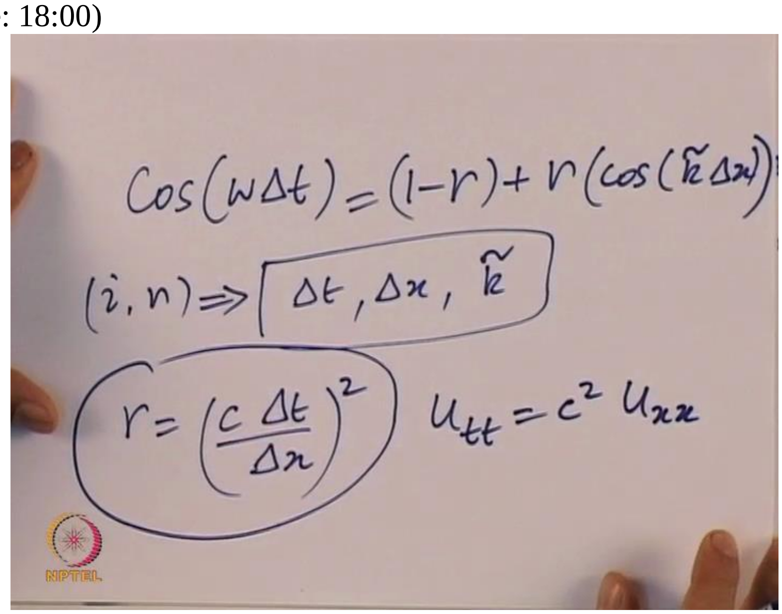

And do not forget the value here what we have in the case of r is the same r that we saw before. It is nothing but c delta t by delta x. We saw this is in the earlier examples, for this particular problem where we have u tt is equal to c square u xx. The value of r is given by this expression.

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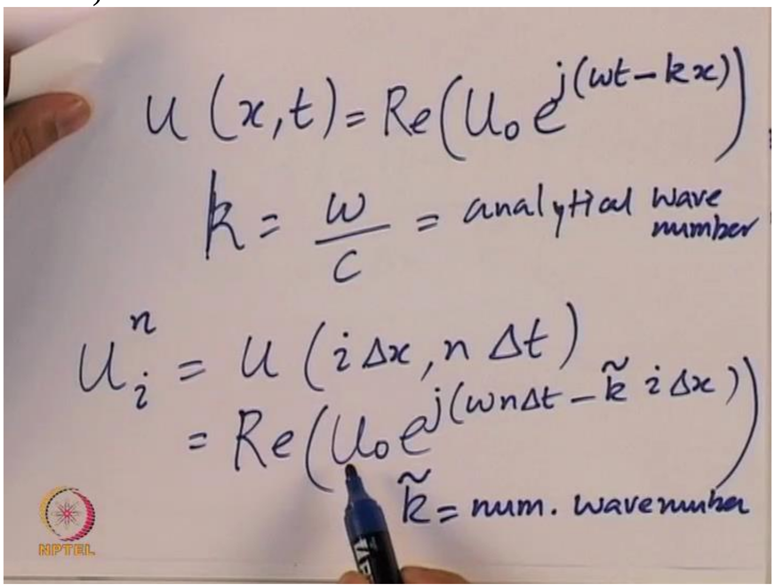
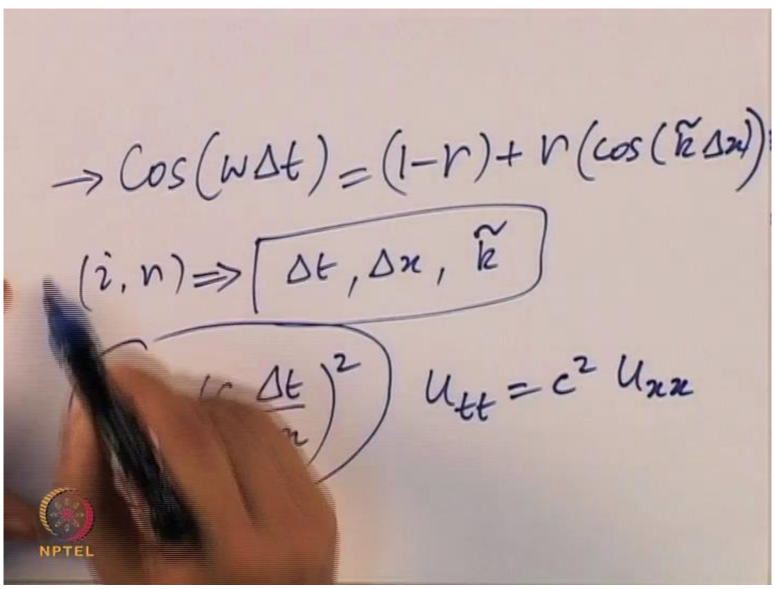


So substituting the numerical value of u of i this value into the finite difference equation.



What we are essentially ending up is an expression given here. So this expression is basically relating various things that are important for us it relates the delta t r k tilde and delta x.

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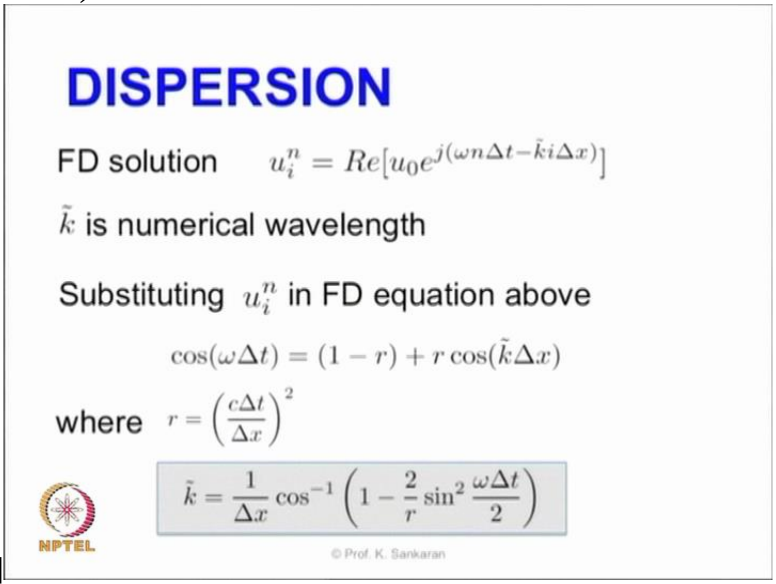

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So what you see in this slide is you get k tilde which is equal to 1 by delta x.

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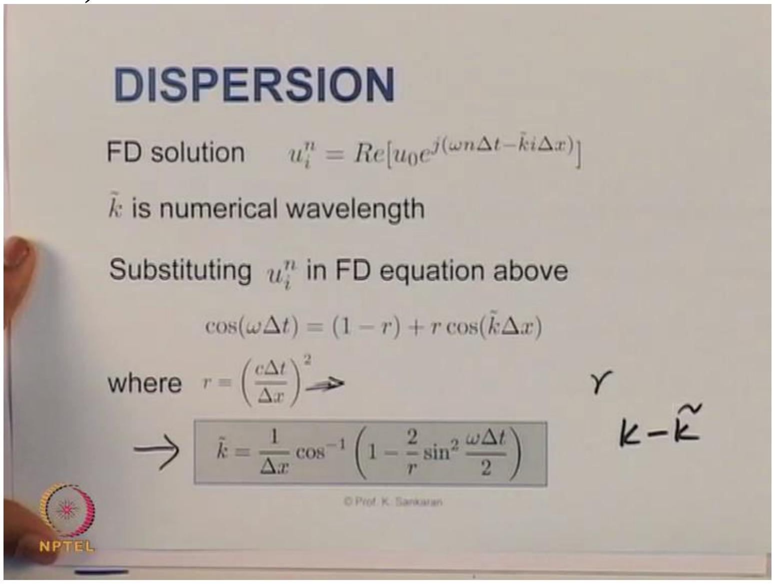

Basically you are getting this 1 by delta x for this expression here and cos inverse of (1 minus 2 by r sin square omega t by 2). What we have done is we have of course expanded this cos omega t in in its see expansion series and we have taken only the first few terms and if you expand this in the way we have done so far so what you will get is you will get an expression for k tilde as a function of delta t, delta x and r. And do not forget r is the one which we have represented here in this equation. So this is a very good starting step because with this k tilde we can compare it with k already that we know so we can compare k minus k tilde to get an understanding of what is the dispersion.

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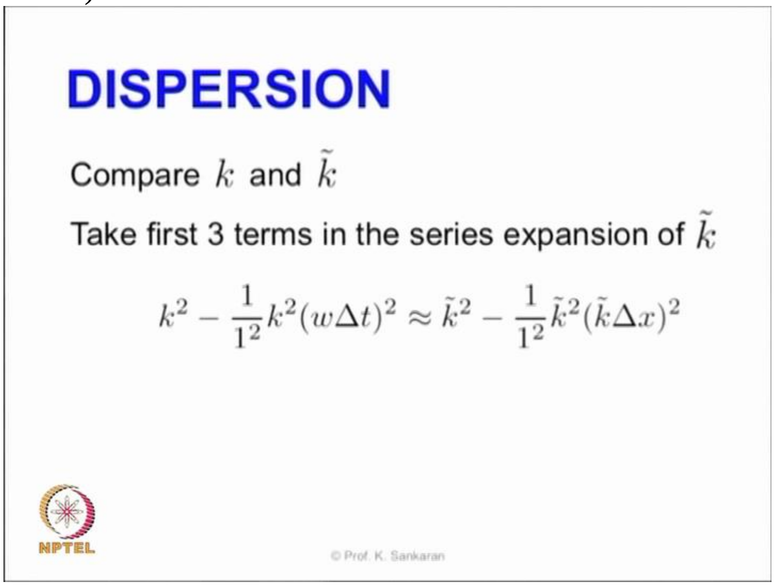

So what we are essentially doing is we are taking only the first three terms in the expansion series and we are expanding the k terms to get an expression which is represented in this slide. So you have on the left hand side the analytical value on the right hand side you have the numerical wave number value.

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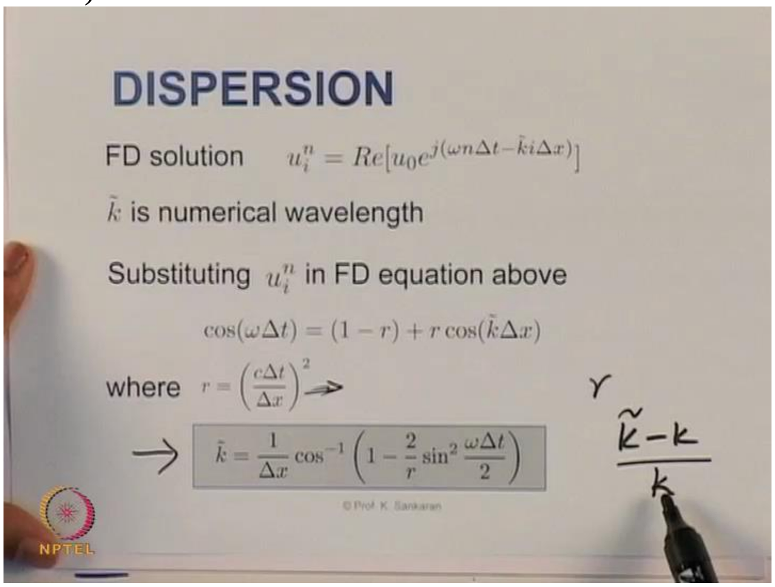

And basically using that we can compute the value k tilde minus k divided by k. What you are doing here is you are trying to see what is the ratio of the error in the wave number with respect to the original wave number. That is what you are trying to compute.

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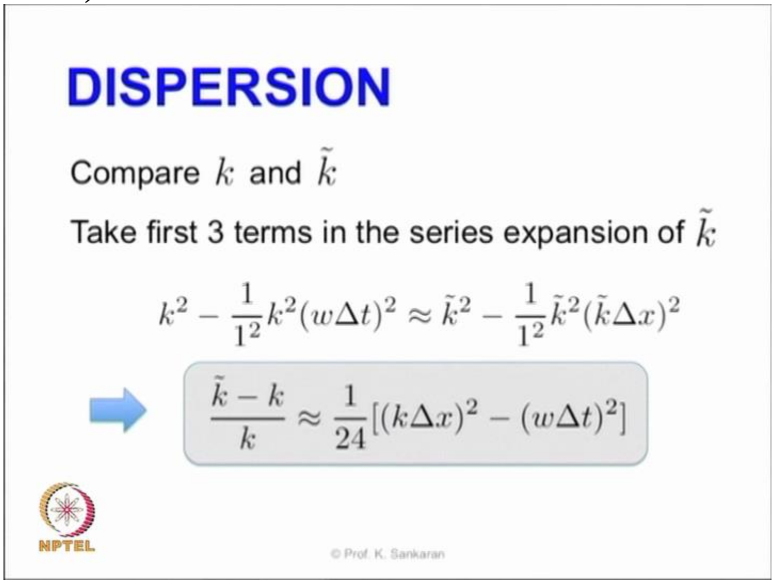

And essentially what you will get is an expression for this example which is given in this particular equation. So what you have is you have a dependency that is both on the spatial and temporal discretization as you can see in the slide here.

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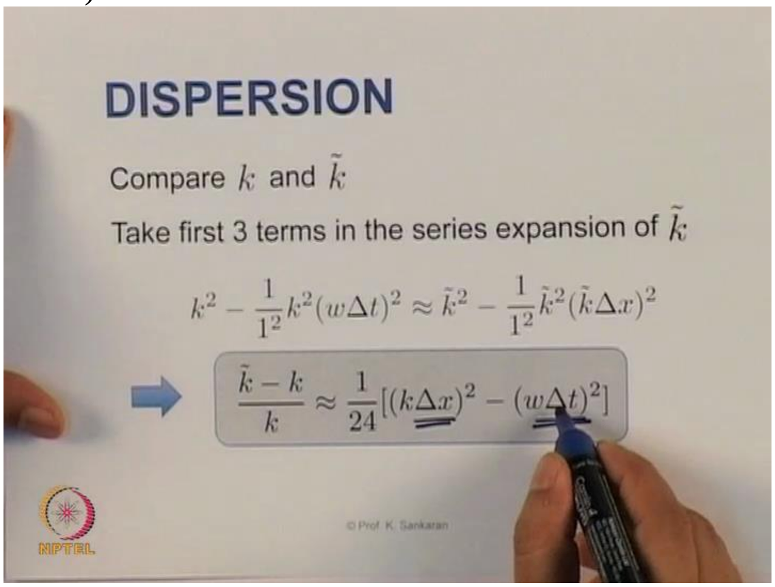

And you also have dependency that is telling that once i fix the value of delta t based on certain delta x i am already going to get certain value of numerical error in k.

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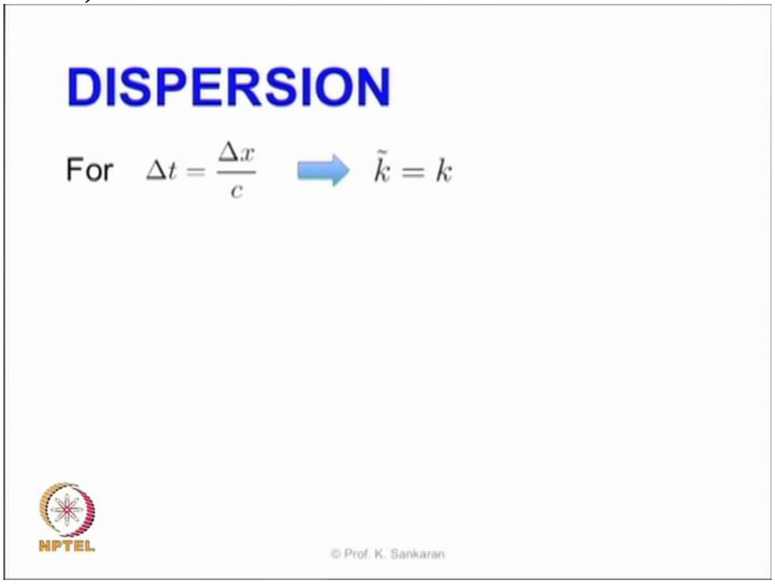

So what we can see if we choose delta t is equal to delta x by c. What you will get essentially is the analytical solution and the numerical solution will be correct. But obviously we cannot do this because of numerical stability issues. So we cannot go at the exact delta x by time stepping. So we have to reduce the value of time stepping by certain fraction that of the delta x by c. So what we are doing here is we are seeing how much of that is going to be good enough.

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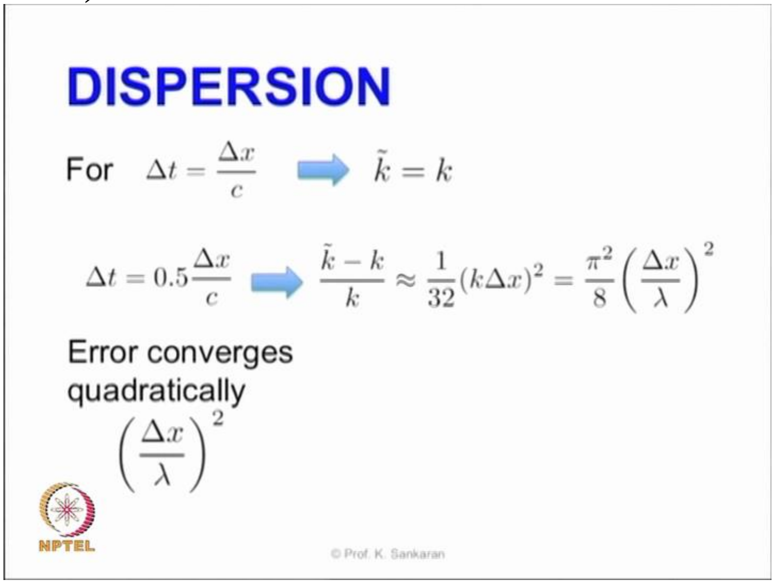

in other words let us say half the value of delta x by c what is going to happen is i am going to get an expression that says my error is going to converge quadratic ally. If I am going to go in this time stepping my error will converge quadratic ally as a function of delta x.

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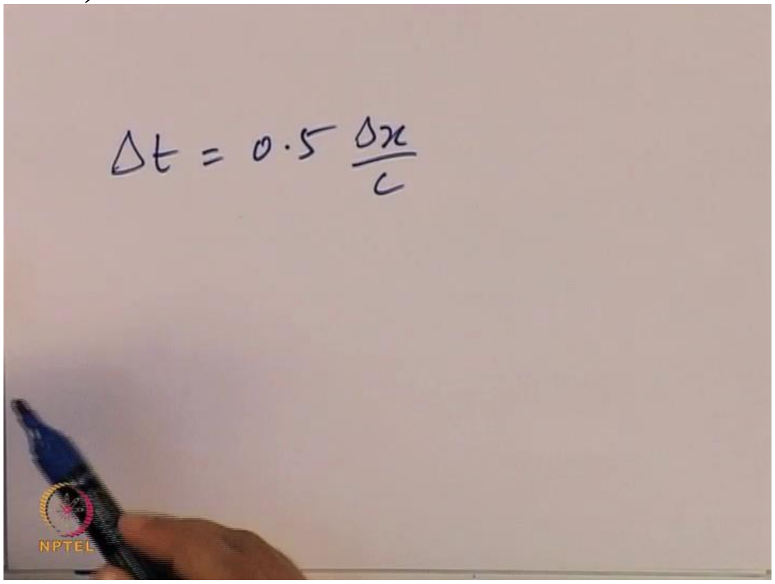

So if i say delta t is equal to 0.5 delta x by c, then i can reduce the error if i reduce the value of delta x step by step. So then the error will converge in a quadratic manner. But there is something that we need to pay attention to.

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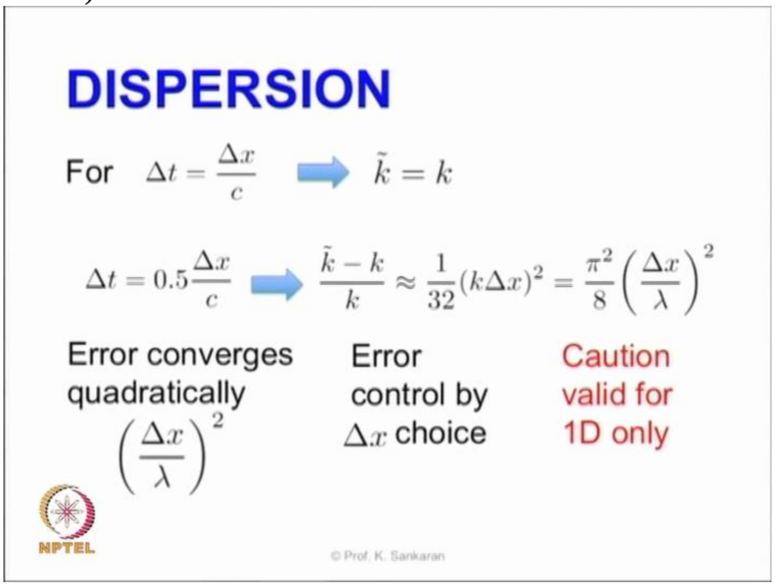

This particular problem is only valid for one dimensional case because if i go in a two dimensional case my wave propagation direction is no longer in the x direction alone its going to depend on it can propagate pretty much in any direction. So in three dimensional case it could happen my propagation direction is x or y or z direction so we cannot fix one particular direction. In the case of one dimensional problem we can pretty much control the error because we can fix the direction of propagation as delta x and based on that we can vary the value of delta t. So as to reduce the error pretty much we want. But in the case of multi dimensional problem this is not possible and obviously we are limited by certain choices and we will see what those choices are.

So with that being said we have covered pretty much the main ideas about amplitude error and also error related to the phase, which we call it as numerical accuracy issue and dispersion issue.

In the next module we will look into the concept of stability which is very much connected to the concept of numerical accuracy.

Thank you!