Computational Electromagentics and Applications
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Lecture No 43
Mimetic (Finite Difference Method)

We have come to one of the last modules of our 12 weeks long journey in this computational electromagnetic course. The part where we are discussing some of the advance methods like finite volume algebraic topological method and the method that we are going to focus today is called as the mimetic method. And it's one of the last methods that we are going to discuss in this course. So the method is called as mimetic method because of its mimicking character so what it mimics is something we will look into it . So let's look into the today's course content.

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So today's discussion is going to be about mimetic finite difference method.

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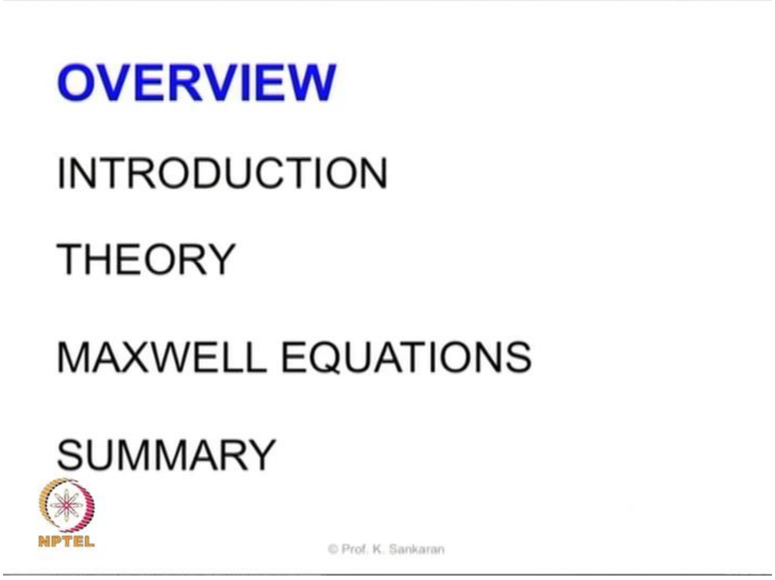

And the overview of today's lecture is going to be the introduction to Mimetic method its theoretical framework the Maxwell equations in the framework of mimetic method and we are going to summarise. As I said this method is quite advanced is not in general period in depth in computational electromagnetic course for various reasons. One of the reasons could be is the formulation itself is quite complex. Their terms and their Technologies are quite different so people don't really use it in teaching computational electromagnetics. But I have promised that I wanted to teach advanced methods as a part of this Course work. So I am going to introduce this method so I am not going to be as vigorous as I was in other methods like finite volume finite element and algebraic topological method . We are going to give general overview and also look at the formulation from a very high level and see how we can model Maxwell equation in the mimetic frame work .

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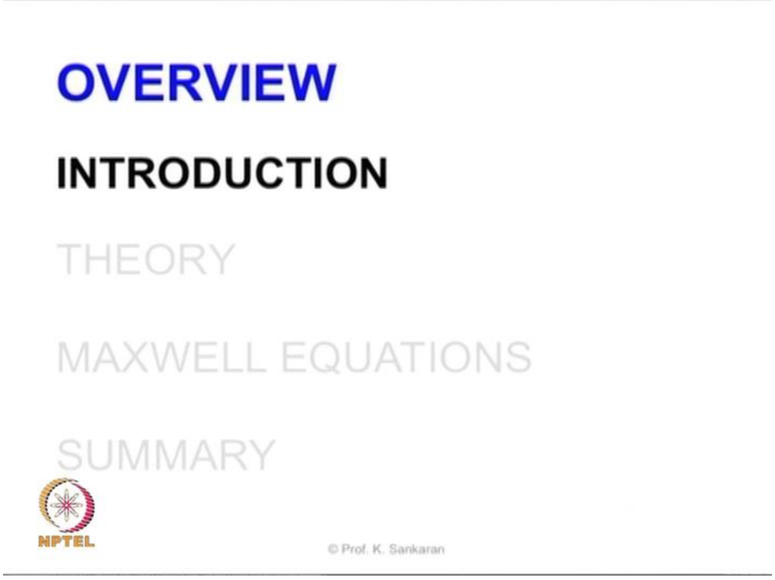

So let's look at the introduction.

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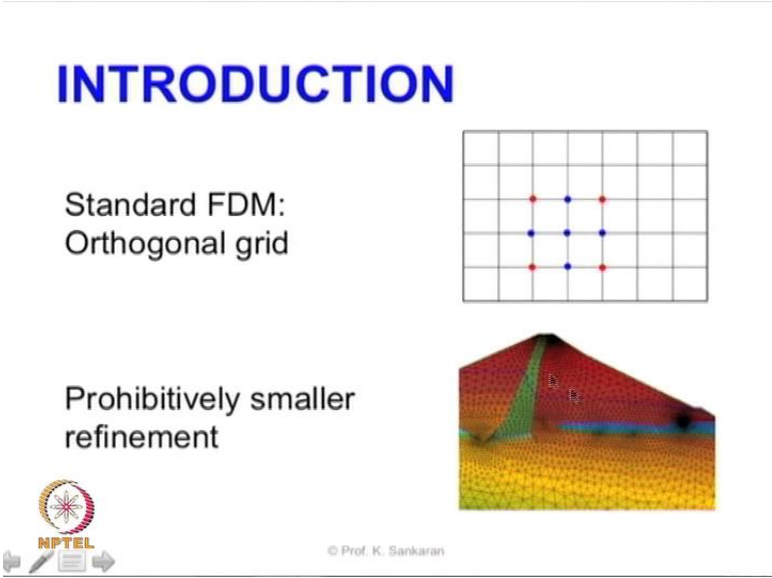

We looked at standard finite difference method as one of the first method that we taught in this course. Where We saw the ease grid where we have staggered Cartesian grid where the e field and h field are separated by space and time and we saw how the structured grid helps in modelling Maxwell equation . But we also said is going to have a lot of other problems like this stair casing error and so on and so forth.

So we discussed about this problem earlier because when you know to capture very very fine details it is going to be prohibitively smaller refinement that one needs when you are working with standard finite difference method because we cannot really go and model such refined areas with standard spatial discreditisation so we have to really refine it to small dx and dy if you are in a 2D and DX DY and DZ if you are in 3D. So that being said there is quite a lot of motivation to really look at finite difference method because it is a elegant simple formulation. But can we use finite difference method for non Cartesian grid? So that means we are still in a orthogonal grid or any kind of cubical grid but can we go for non Cartesian and non orthogonal grid? So that's the main motivation.

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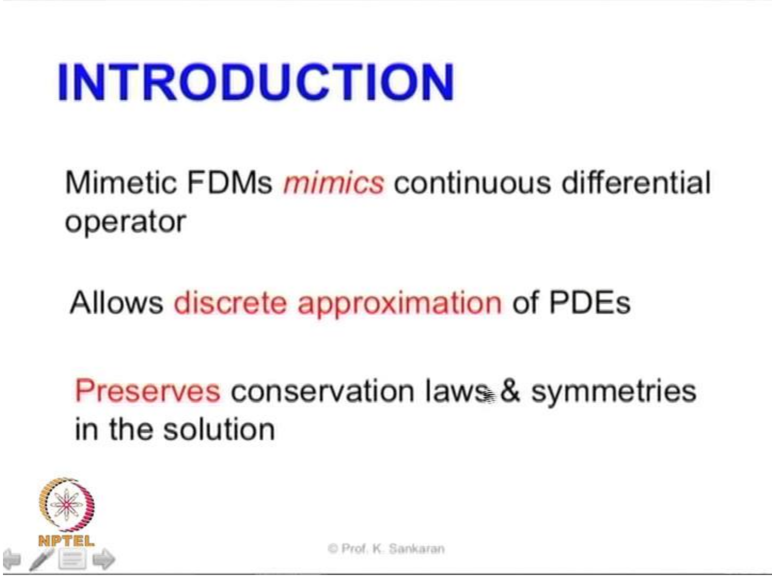

So the idea of limiting finite difference method is that it actually mimics the continuous differential operator and not only that it allows the discrete approximation of PDE if possible and it preserves the conservation laws and symmetry in the solution. So we discussed about conservation laws when we started with finite volume method. It conserves the energy and it also preserves the symmetry of the solution and Maxwell equation is a conservation law so one has to find mathematical models that can preserve the symmettries in the solution. So that being said there is a lot of motivation to go for mimetic methods and one has to really formulate the mimetic method in such a way that it can really capture the aspects of the Maxwell equation in principle and at the same time it can have the simplicity of finite difference method. So that is going to be the aspect that we are going to look into. So let's us now go into the theoretical framework.

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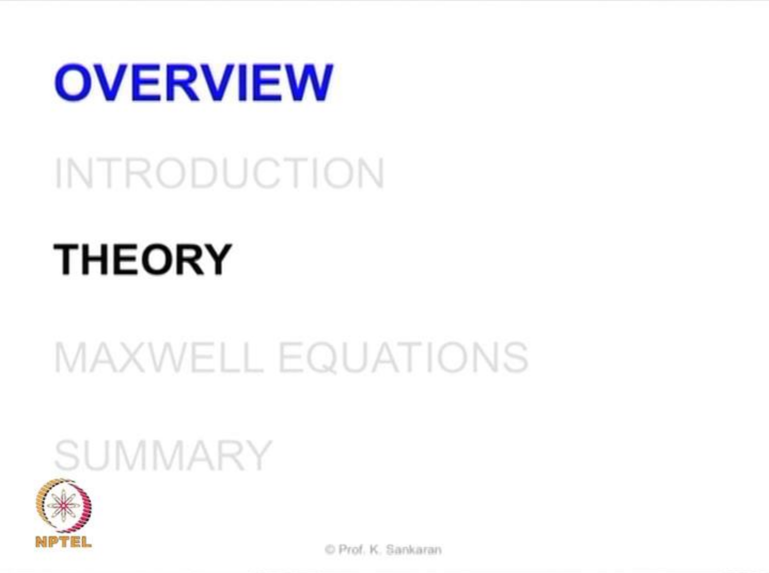

So the theory of finite difference method is something that we know from our first lecture module .

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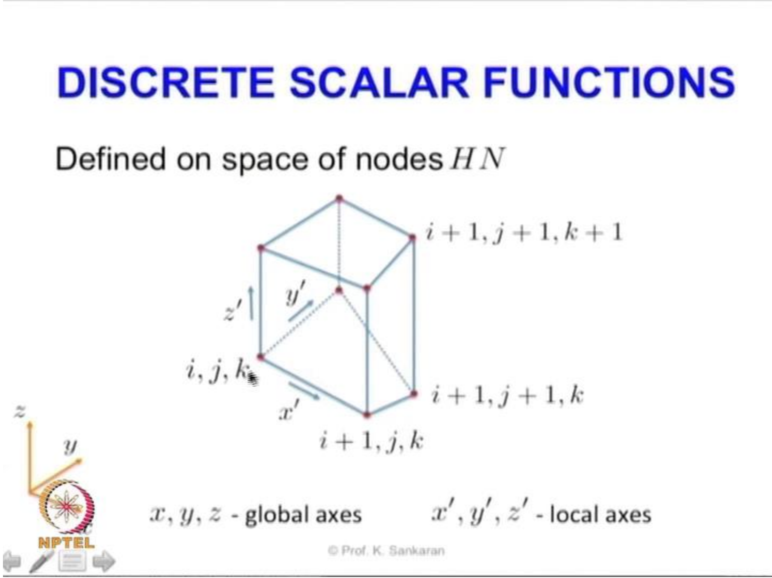

So we are going to use the same theory but we're going to model it on non orthogonal grid. so the discrete scalar functions what we mean by discrete scalar functions are the functions which are having only scalar values and they are defined on the nodes and we're going to call those space as HN. So for now assume that HN is going to be the space where all the scalar quantities are going to be located so those are going to be the nodal values here. And here we say if x y z are the Global Axis locally inside the grid it can be x dash y dash and z dash and as you can see X Y and Z are not orthogonal so we can still consider them in the same manner we consider an orthogonal grid where we can call i j k and the next note in the x dash or x Prime direction is called as Ii plus j k and the one above will will be called as I comma J comma K plus 1 and the exact diagonal node from this node will be called as I plus 1 the one above will be called as i j K plus 1 . And the exact diagonal node from this node will be called as i plus 1 j plus 1 k plus 1 similar to what we have done in the finite difference method.

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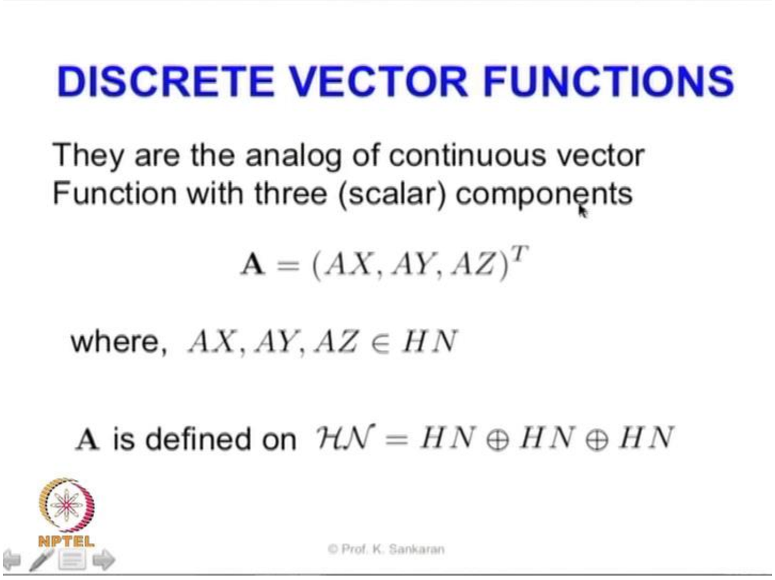

They the discrete vector functions.

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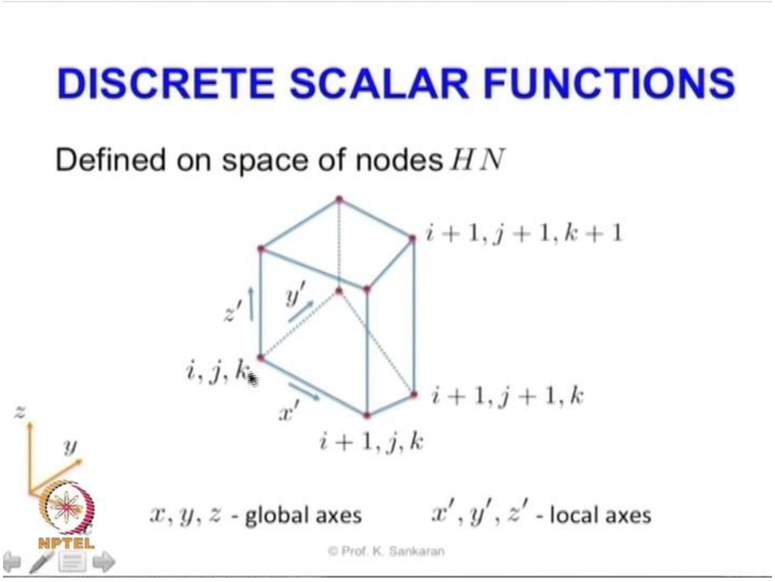

So initially we talked about discrete scalar function

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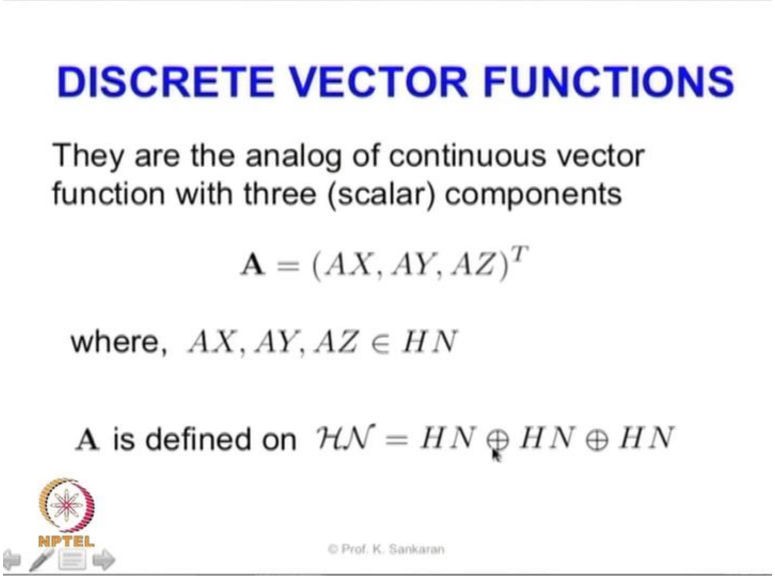

In the same and we can talk about discrete vector function and these discrete vector functions are going to be analog of the continuous vector functions and they have free scalar components . Similar to the discrete scalar function we are going to talk about discrete vector functions and they are going to be analog of the continuous vector function with three scalar components . And These scalar components are Ax   Ay and Az and we call this discrete vector function as A and we are going to define this A on this face which is J N but the Math cal symbol is used. So as you can see the match cal HN is going to be the combination of scalar HN so I am going to call this as vector function and this is as a scalar HN. So the vector HN is going to the space where the discrete vector functions are located. The discrete scalar functions are going to be in the HN.

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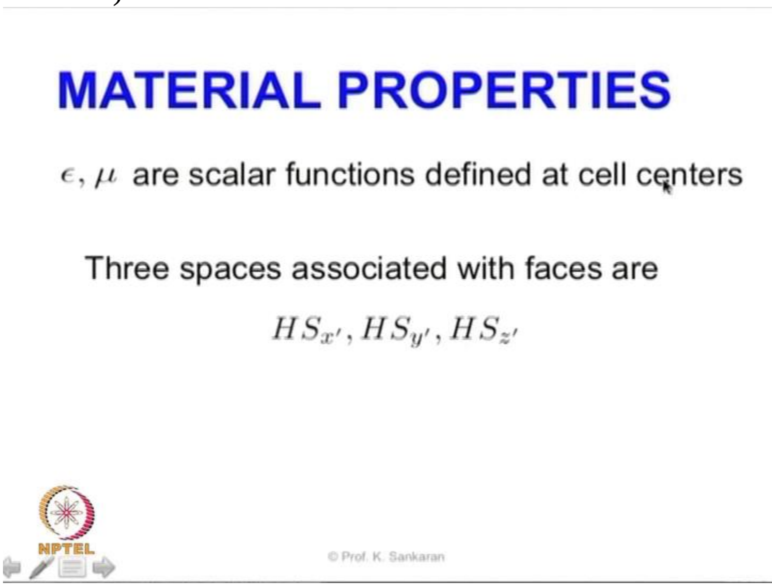

The next symbol and aspect here is the material properties as you have seen before the Epsilon and Mu are the permittivity and permeability that we have been using and these are the scalar functions that are going to be defined at the Cell Centre.

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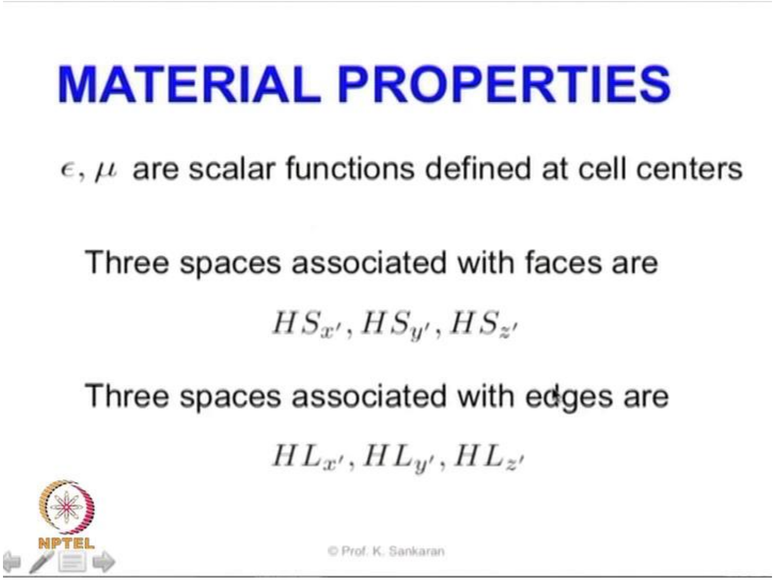

So when I say cell centre these cells could be non orthogonal grid and permittivity and permeability values are placed at the centre of this particular cell. If you are in a 2D you are talking about centre of the surface if you are in 1 D you are talking about centre of the edge. So the material properties are going to be defined at the cell centres and they are going to be the scalar functions and there are going to be 3 spaces that we are going to define and they are going to be associated with the spaces and they are termed as HS x prime H S Y prime and HS Z prime. There are going to be 3 spaces that we are going to assign to the edges so they are called as H L X prime H L Y Prime and HL Z prime . As you can see H stands for faces and L stands for line or edges .

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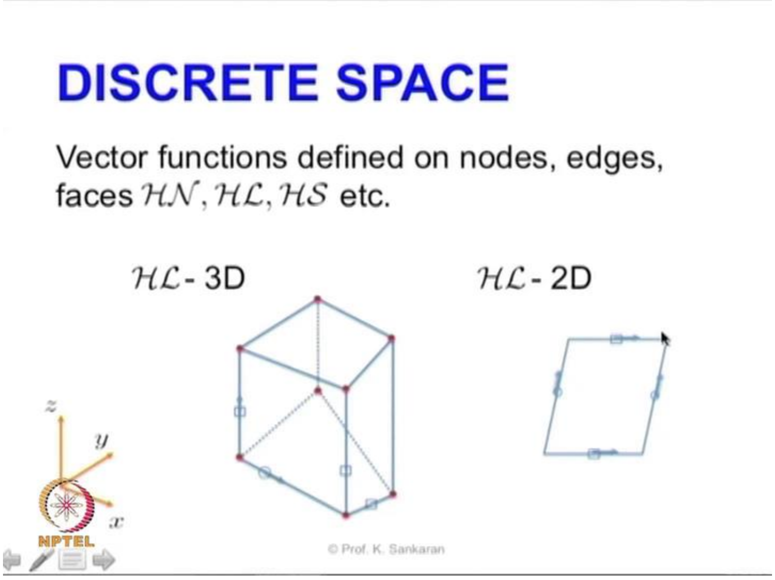

So the vector functions are generally defined on notes or edges or phases. So they are there going to be defined in the space vector HN vector HL and vector HS etc . And this is the 2D counterpart and this is the 3D counterpart. And now when we look at the 3 D counterpart we can define the normal directions accordingly. So the normal directions are going to be the aspects that we are interested in. Similarly in the 2D space we are also having normal to the HS. So this is clear because we saw here so the attrib field are going to be defined on the edges 2D is going to be like this.

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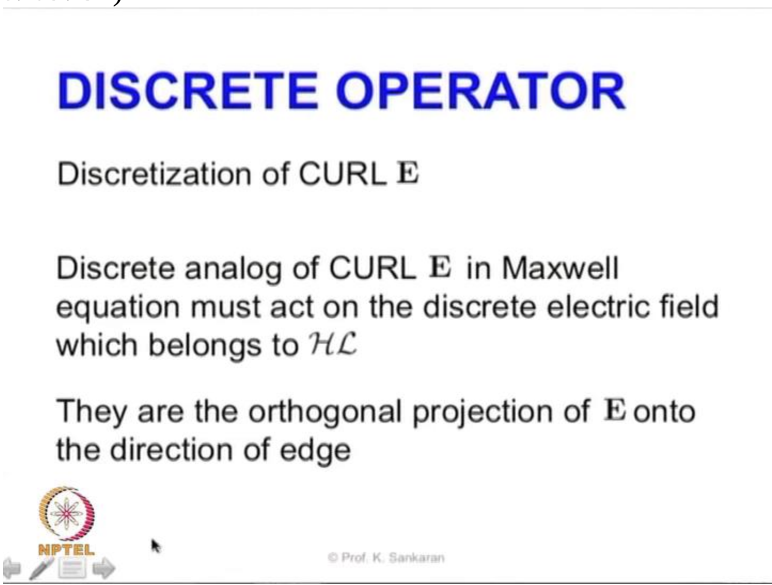

And the discrete analog of Curl E is going to act on that particular discrete electric field Not only that they are going to be the orthogonal projections of E on to the direction of edge itself.

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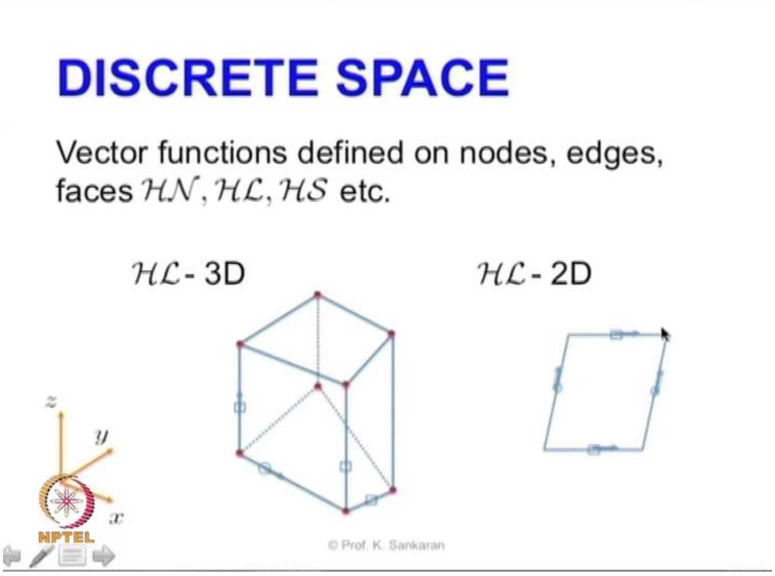

And this we can see here in this figure they are the orthogonal projections of E on to the edges itself.

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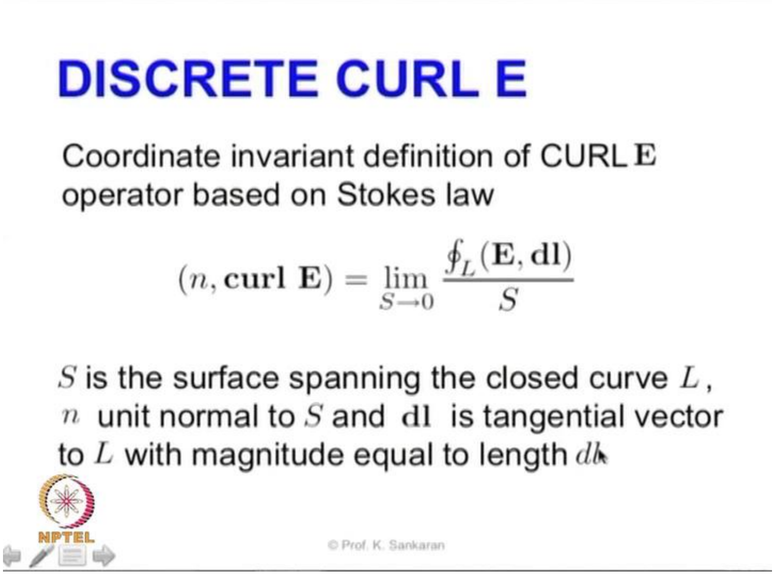

The discrete counterpart is going to have a co-ordinate invariant definition. So the co-ordinate invariant definition of discrete Curl E is defined using the Stokes law as follows . So we have the n curl E which is defined as the limit is goes to zero that is the surface is converging to a very very small area it is going to be the closed integral over the contour which is Defined by L E.dl divided by S. So this is going to be the way we define the co-ordinate in variant definition. So this is the co-ordinate invariant definition of curl E. And as you can see in this expression s is the surface spanning the closed curve L n is a unit normal to S and T L is a tangential vector to L with magnitude equal to the length which is the dl itself .

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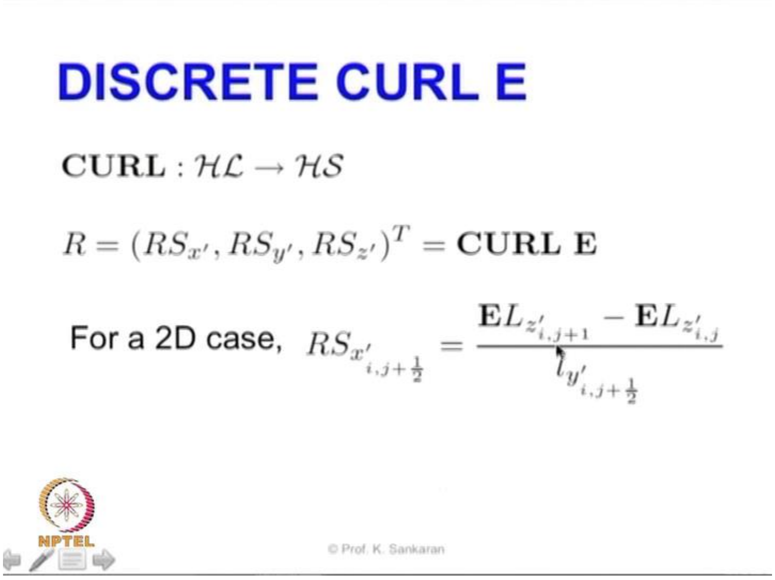

Now we are going to see how this curl operator is going to behave. We know this from our previous lectures on algebraic topology and other methods that is going to take a value that is defined on the vector HL and it's going to take it to the vector HS. And this we have seen also in the algebraic topological method and this is the way we define it and based on that we can find the values the individual components of curl of E as RS x prime RS y prime RS z prime and it is a transpose because we are talking about a column vector and it's going to be equal to the curl of E . For a 2 D case or simple problem that we are trying to address here which is the transverse magnetic field you will have RS x Prime I comma J plus half is equal to as you can you can see here it is J plus half so it has to be y prime EL z prime I comma J plus 1 minus E L z prime I comma J.

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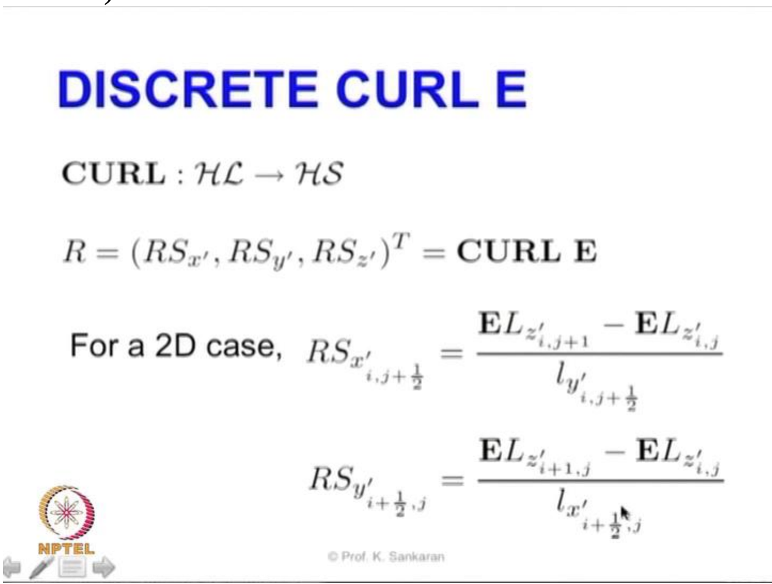

This is the component of the curl and it is very similar to Dow EZ divided by Doe y. So this is the first component we are looking into. The second component will be with respect to X so you see I plus one I Plus half comma J. So it is the Doe E Z divided by Dow X component in 2D. So EL z prime i plus 1 comma J minus ELz Prime i comma J. As you can see what we're doing is forward differencing with respect to y prime and forward differencing with respect to x prime so these are the forward differencing or we're going to do forward differencing for EP.

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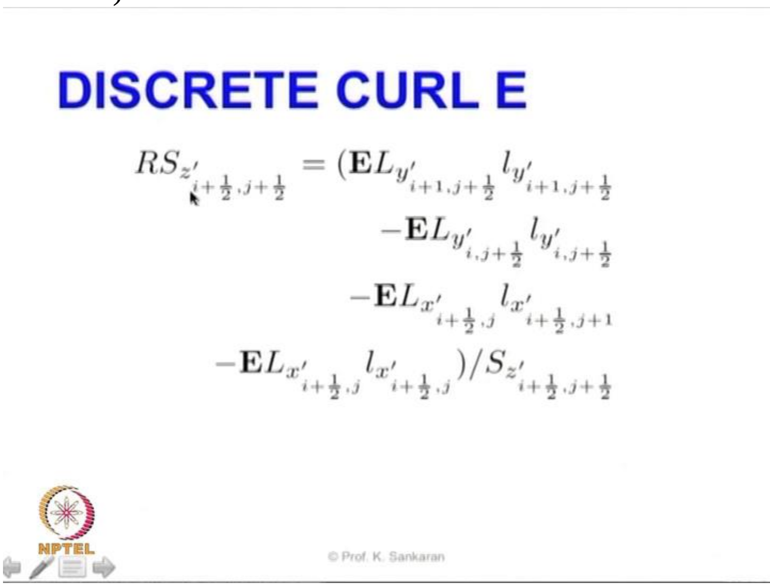

So now we are going to look into the third component which is the r s z prime components and as you can see here we have I plus half comma J plus half so it means it has both the x derivative and the y derivative and what we are seeing here is a very different expression to what we had before. Here in this expression we had always the L component the length component as the denominator whereas we see them here on the numerator.

So what we have done is we have just multiplied the appropriate L terms in the numerator and we did the same thing at the denominator and then you multiply Lx and Ly what you get is a surface area and you see here S of z prime I plus half and J plus half . This is a surface area and the surface area is nothing but the multiplication of the appropriate Lx and Ly term and the third component is this RS z prime component.

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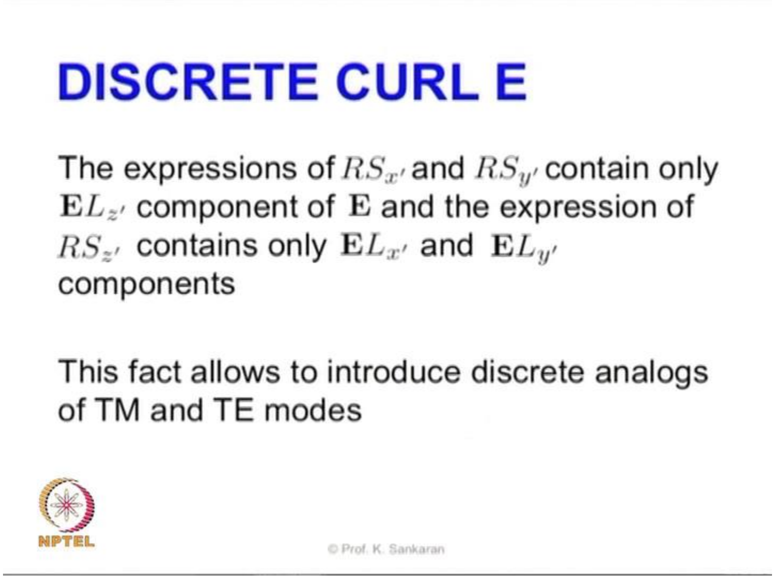

And Now what we can see is the expressions for RS x prime and RS y prime contain only the EL z prime component and that we have seen here in the expression it has only the EL z prime component. Whereas the expression for RS z prime contains both Lx prime EL x prime and EL y prime. So that is something we know this is something we expect from the Maxwell equation itself. And this part is very much allowing us to introduce discrete analogs of the TM and TE modes.

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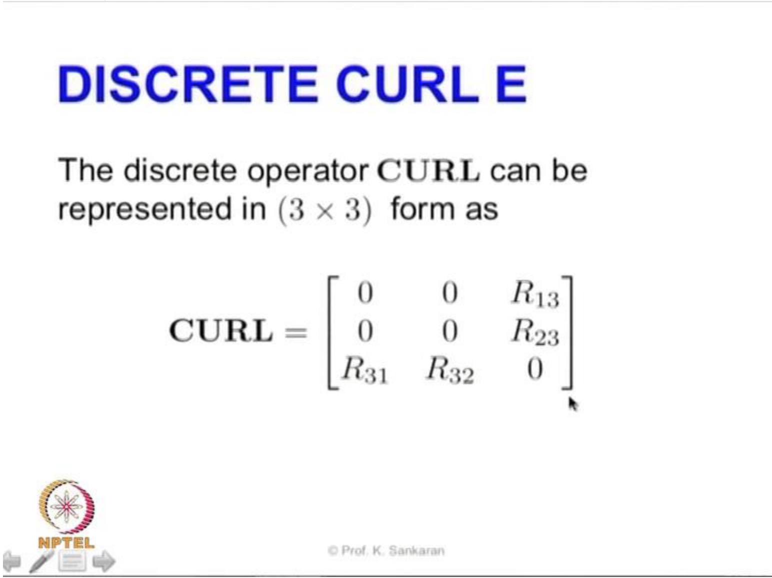

So what we are going to look at is discrete Curl E component and this discrete Curl Operator and we represented in the 3 by 3 form as follows as we can see the first row two columns and the second row two columns they have zero so the elements R 13 and R 31 and elements R 23 and R 32 are symmetrical elements where as the leading terms are zero so this is the way we are going to define the Curl discrete Curl Operator.

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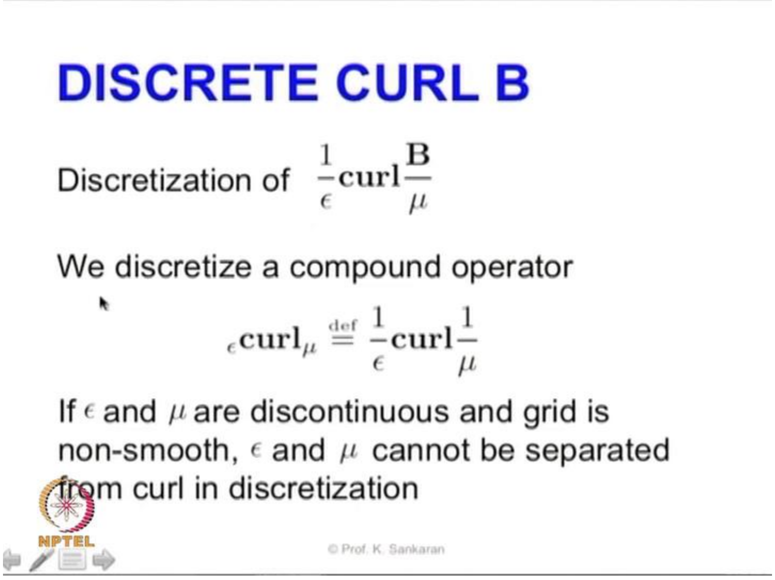

Like the way we did the discrete Curl E operator we going to talk about discrete Curl B operator however here there is a difference we will include the permittivity and permeability as part of the discretization itself. That being said we cannot separate them when the grid is so irregular and discontinuous. So the 1 by Epsilon and so the so the permittivity and permeability will be part of the discretisation itself so we can discretize the compound operator. We call it compound operator because the permittivity and permeability parameters are going to be a part of the curl operator itself and we define it this way. We write this way in a compact notation and we see that Epsilon and mew are discontinuous and the grid is not smooth we cannot separate Epsilon and Mu we cannot separate them from the Curl discretization itself. So that is the reason we keep them together.

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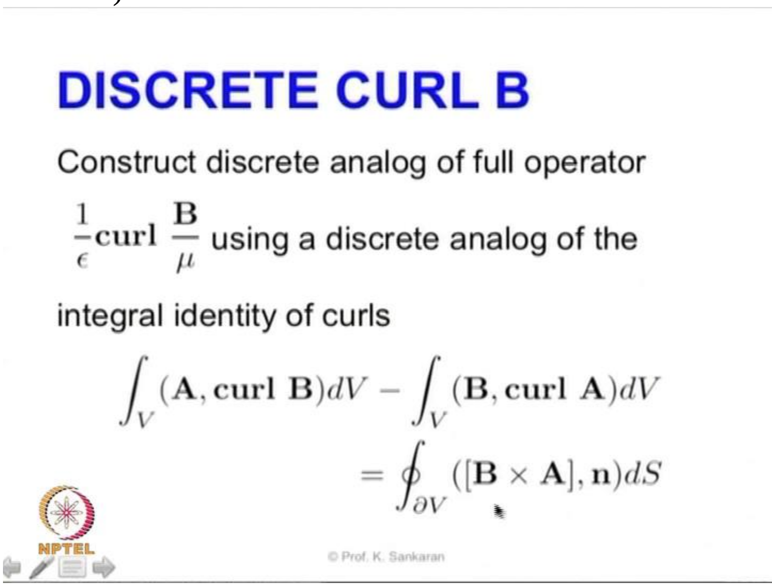

and we can construct the district analogue of full operator the compound operator using a discrete analogue of the integral identity so we start with the integral identity this is from the basic vector calculus when we have any integral over a volume A Curl B integrated over the volume minus integrated over the volume B Curl A DV it is nothing but the closed integral over the closed surface that encloses the volume B Cross A N ds. So we are talking about the normal projection of B cross A and the surface is going to be the surface that is enclosed in this volume.

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DISCRETE CURL B

If $\mathbf{A} = \mathbf{E}$, $\mathbf{B} = \mathbf{B}$

$$\boxed{\int_V \frac{1}{\mu}(\mathbf{curl}\ \mathbf{E}, \mathbf{B})dV = \int_V \epsilon(\mathbf{E}, \frac{1}{\epsilon}\mathbf{curl}\ \frac{\mathbf{B}}{\mu})dV}$$

where, ${}_{\epsilon}\mathbf{curl}_{\mu} = \mathbf{curl}^{*}$

So if we substitute A is equal to E and B is equal to B in this equation what we get is essentially the equation the equation that will give us the discrete Curl B operator and we are going to get them through this equation where we define the compound operator as the Curl star. So this is the compact notation that we can use.

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MAXWELL EQUATIONS

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{curl}\ \mathbf{E}$$

$$\frac{\partial \mathbf{E}}{\partial t} = -{}_{\epsilon}\mathbf{curl}_{\mu}\ \mathbf{B}$$

$$\frac{\mathbf{B}^{n+1} - \mathbf{B}^{n}}{\Delta t} = -\mathbf{curl}\ \mathbf{E}^{\alpha_1}$$

So we are now going to do the Maxwell equations as you can see we have not separated mew and Epsilon out of the equation we are going to keep them as an integral part of the equation itself. So this is done because then Epsilon and mew are distant universe and when the grid is non smooth this is the way we have to do it in order to get the discrete analogue proper. We can do that time stepping as a forward Euler method and we can get the first curl equation as follows Where Alpha One has to be defined. I will define it in the next slide.

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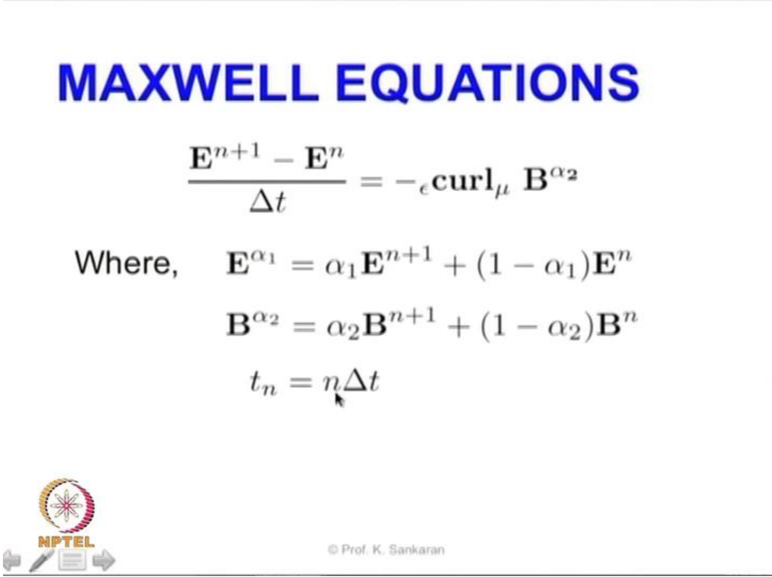

And we can also define the same way the second curl equation and here we have alpha 2 and now we're going to define what is alpha one and alpha 2 in this equation. So E Alpha 1 is equal to Alpha One multiplied by E n plus 1 plus 1 minus Alpha one time En

So it's going to be some kind of an weighted average and the waiting is going to be defined by Alpha one in this case. So the second equation will be Defined by Alpha 2 again there is going to be some amount of the time step in a nd some amount of the most of the time is going to added in order to get the value of E alpha 1 and as you can see the t n is going to be the end time step where delta T is going to be the time discretization.

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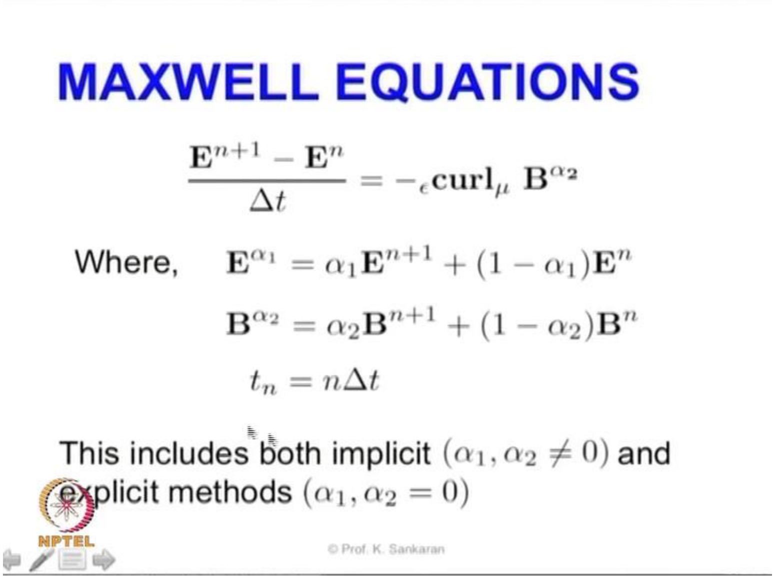

One can see that it includes both the implicit and explicit formulation. If we put Alpha 1 and alpha 2 E is not equal to zero we will have both n plus 1 and n terms having so it will become so both the n plus 1 terms will be part of the update equation for n plus one will be implicit formulation whereas if we put Alpha One alpha 2 is equal to zero both of these first term will go will disappear and will have only this term. And this will be again 1 minus 0 and 1 minus 0 so it will be just E n and B so then it will become an explicit method.

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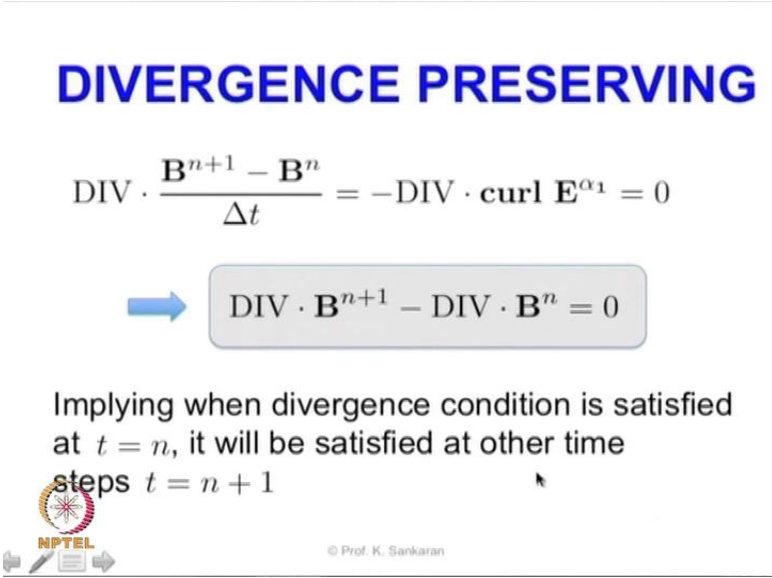

One can also see when we do the dot product of the diversions on both sides what we get is the divergence of B n plus 1 minus divergence of B n is equal to 0 because this is a identity when you have divergence of Curl the divergence of curl will always be 0 this is from vector calculus hence what happens is once the divergence is satisfied for n the time step it will be always satisfied for n plus 1 the time step. And you can go on doing it recursively so once it is satisfied for n plus 1 it will be satisfied for n plus 2 so on and so forth. So for all the time steps it is valid.

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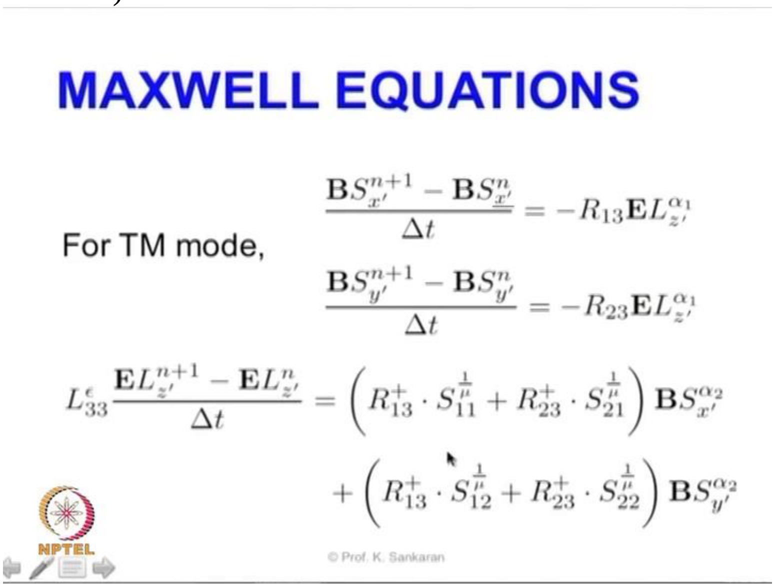

So let us look at the TM mode what we will have is the BS n plus 1 x prime minus BS n x prime this is the time stepping where we have B as n plus 1 x prime minus B as n x prime divided by delta t is equal to minus R 13 EL alpha 1 z prime. And similarly what we get for the second term is BS n plus 1 y prime minus BS n y prime is divided by delta t is equal to minus R23 E L alpha 1 z prime.

So we will get the third equation in a very complicated form because we are going to have the complex operator coming into play and we will get an equation which is given by this complex term. We are not going to go much into the detail of this because we just wanted to introduce this method.

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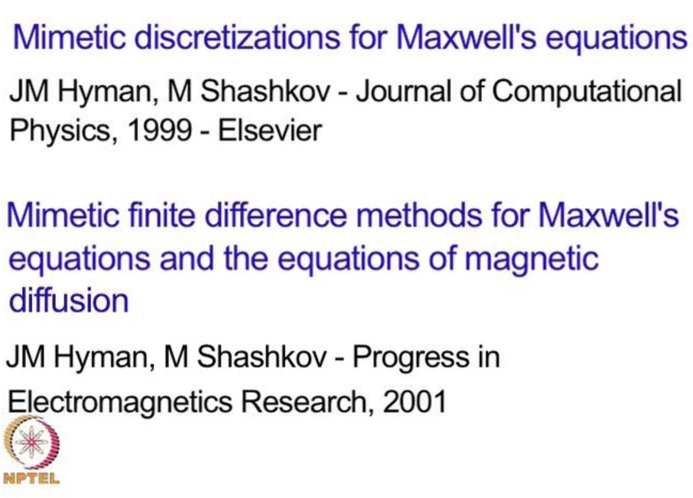

If you want to derive this individual equations step by step we can give references but for now it is enough if you take it at the face value.

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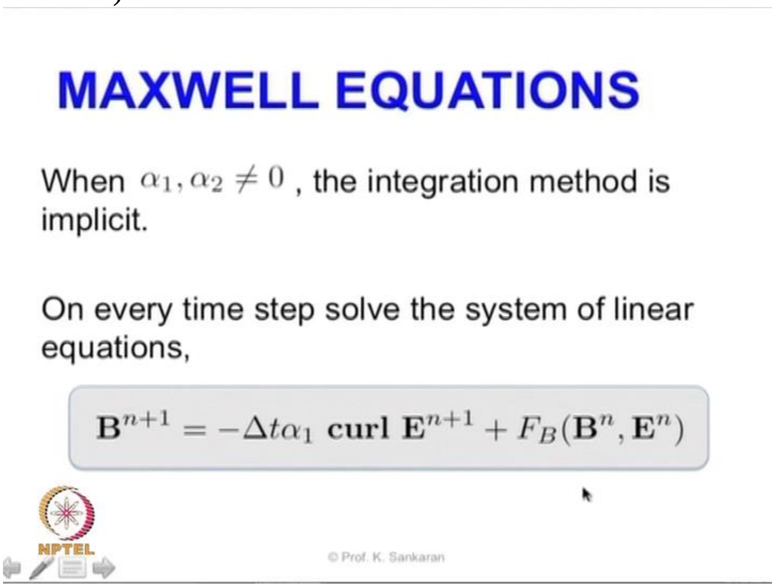

So when we put alpha 1 and alpha 2 not equal to 0 the integration method becomes implicit and every time step will be solving a linear equation which is given by this expression. So what we will get it is B n plus 1 is equal to minus delta t alpha 1 curl E n plus 1 plus F B (B n E n).

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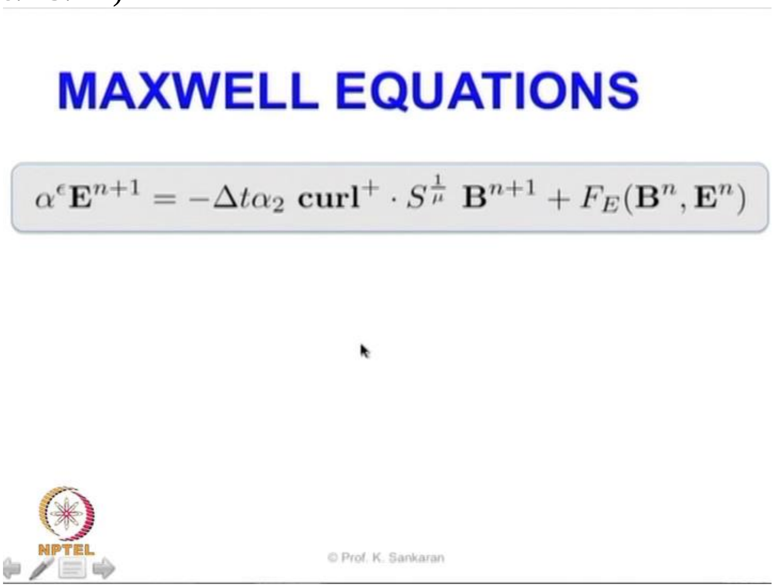

Where we have to define the F E and F B we will define it in the next step. Similarly we get for the second expression the complex curl operator coming into play and the expression is given by this equation. And here we have to define F B.

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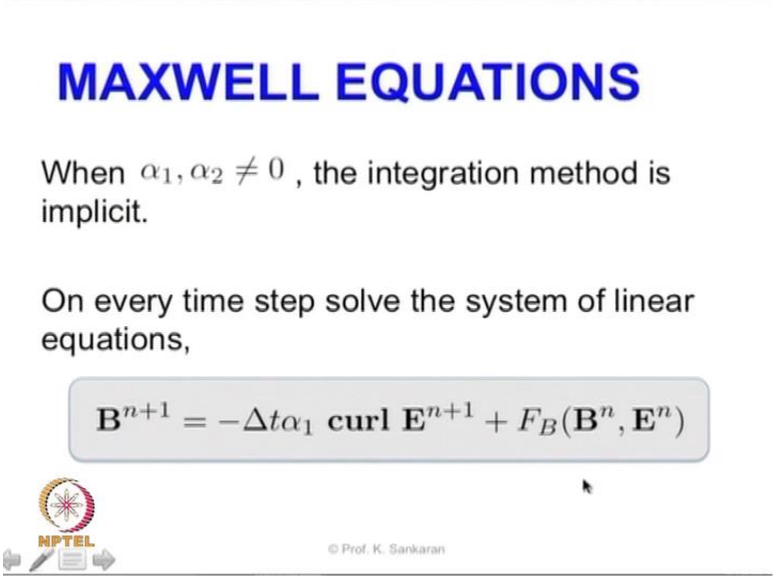

Like we have to define F B here.

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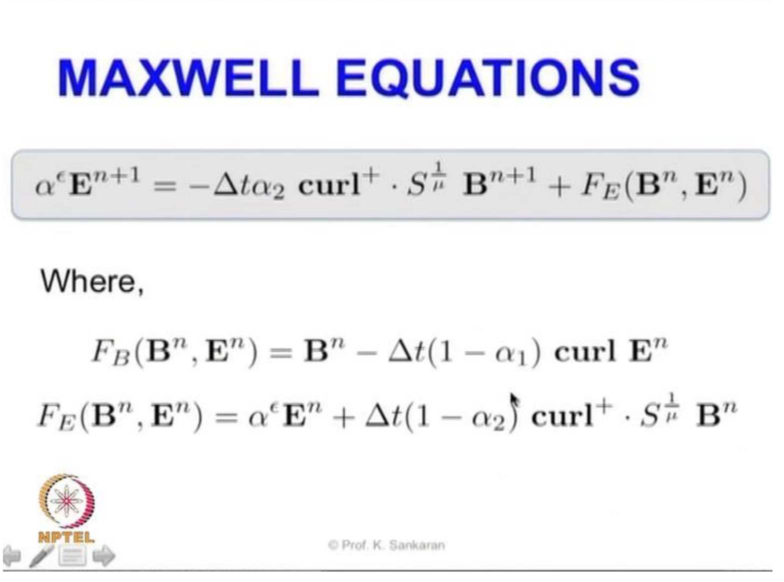

I think this is getting really really crazy here with the expressions and I wanted to be very clear this is a very very advanced method its not something that you can take it at the face value of the expression that we are seeing here. So for now its enough if you understand that there is going to be a term that we call it as F E and F B and these terms are going to be calculated based on the alpha terms that we have defined.

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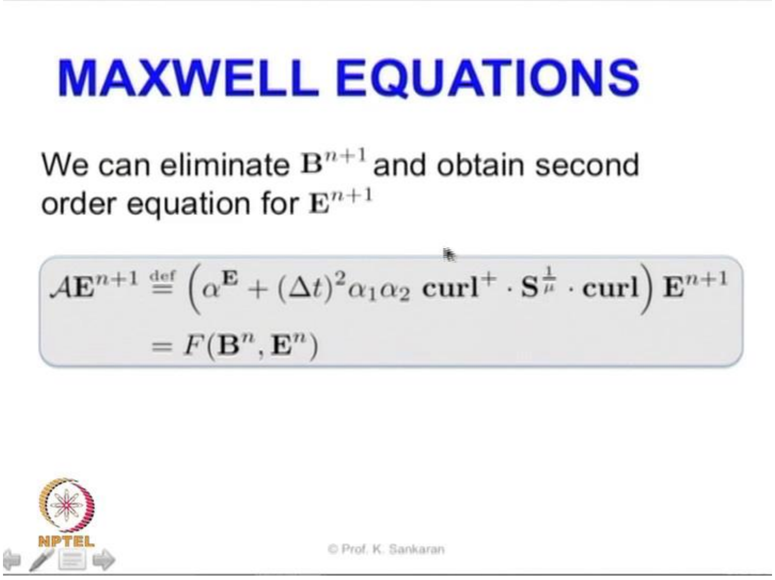

And now you can think here we are talking about two equations and if you are in a simple 2D formulations you will get three equations but basically we can combine these two equations into one equation and that is what we do normally in second order wave equations and that is what we are going to do also in this case. We can eliminate the B n plus 1 term and obtain the second order equation in E n plus 1. And this is something you can intuitively understand it from the wave equation point of view of course the terms here are too complicated what we have here is an A operator operating on the E n plus 1 the value that we need to compute. The moment we defined the A operator we can compute the value of the E n plus 1 using this expression. And as you can see this expression is both having the terms of alpha 1 and alpha 2 and when they are not equal to 0 it is going to be an implicit formulation. When they are going to be equal to 0 this entire term will disappear so the value is going to depend on the alpha 1 and alpha 2 parameter that we have defined earlier. And it is going to depend on the complex curl term that we have defined. And we are having the right hand side like before F(B n En).

So the beauty of this expression is the left hand side is going to be having terms which are E n plus 1 and the A is nothing but complex operator which is given by this expression within bracket and on the right hand side we have the values which are purely dependent on the time step n.

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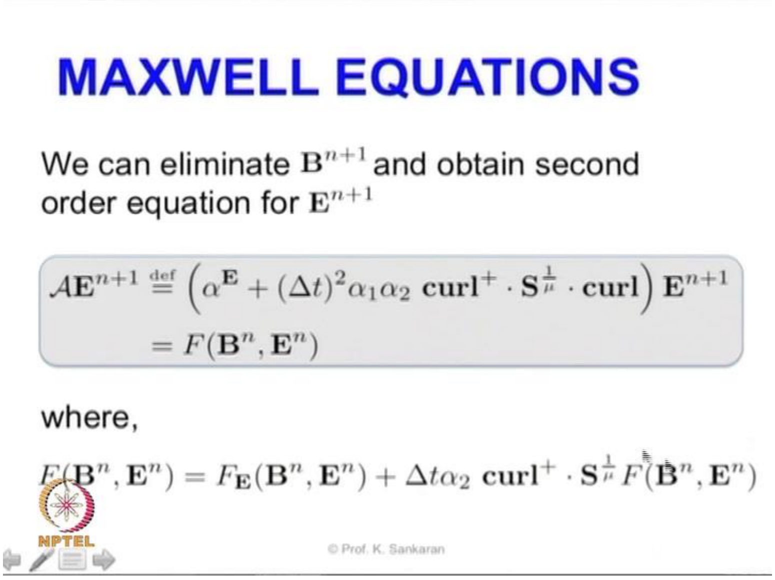

Here the F(B n E n) is going to be given by this value.

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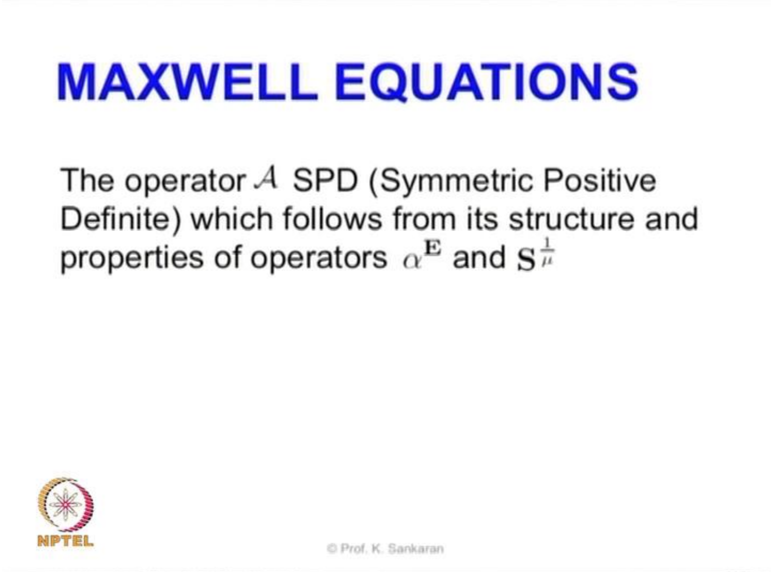

A is the SPD operator and we discussed about positive definite operator when we discussed finite element method and on the top of it when the operator is also symmetric it is called as Symmetric Positive Definite. And A operator is the symmetric positive definite operator which follows from a structure and properties of the parameters that we have defined earlier. The reason for introducing mimetic method towards the end of our course module is it is really an advanced method and one has to spend quite a lot of time really looking into the derivation of each and every parameters. We are going to give you references for this for advance learners this is not for any course work this is something only out of your curiosity that you wanted to learn this method. You are welcome to really look into those course work. You are not going to be judged on your ability to understand all the individual derivatives of the mimetic method. However I would appreciate if you understand the over arching aspects of mimetic method and that is going to be the part of the discussion that we are going to summarize in the next slides.

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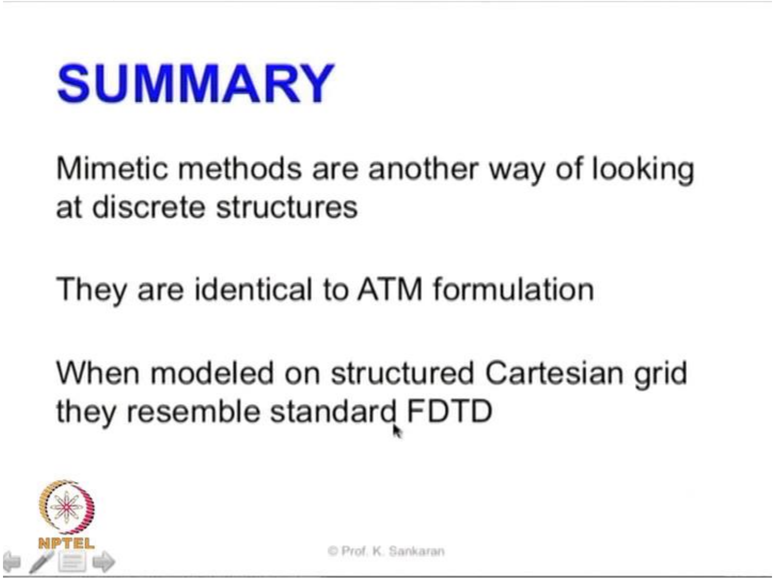

So the mimetic method is an another way of looking at the discrete structure this is something you have to know it is going to be another way of looking at discrete structure because it is going to give as a similar formulation to algebraic topological formulation. In that sense there is going to be a lot of connection to algebraic topology. So what I want you to know is mimetic methods are very similar to the formulation of algebraic topology and they are another way of looking at discrete structures. What I mean by discrete structures are nothing but discrete formulation of Maxwell equation. And then we modelled them on a structured cartesian grid they resemble the standard finite difference method. So those are the three important points that you should know. Apart from that you should also know unlike finite element method the operator that we are going to define are going to be easily obtainable even on a very very coarse grid.

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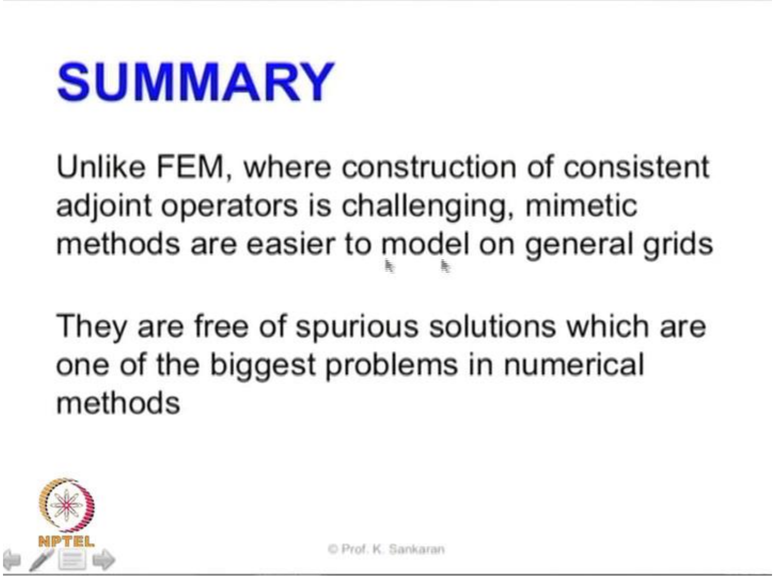

Because in a finite element method we are forced to find or forced to construct the consistent adjoin operator and it is going to be very challenging. Whereas in a mimetic method it is very easy to do on any general grid. And due to that reason the mimetic methods are going to give us an exact discrete counterpart of the partial differential equation we are modelling.

So because of that they are going to be free of the spurious solutions that are common to finite element methods. For example in the case of finite element method based on the nodal element methods is going to have a spurious solutions. However you can avoid that by going into edge element method. But still somehow some of the edge element methods can still lead to spurious solutions. Whereas the mimetic method is completely free of spurious solutions or spurious modes. So that way it is going to be a better solution than finite element method. So those are the important aspects that I wanted to leave you with. You do not have to really know every detail of the mimetic method in order to go through this course work. It is mportant for you to know there is something called as mimetic method. The discrete aspects which are going to mimic the continuous PDE that we are going to model. Secondly we can take into account the discontinuous nature of the material parameters even on a rough grid and also because of these properties they are going to free of spurious solutions or spurious modes.

So in that sense these are the important aspect of the mimetic method based finite difference method. They can be very similar to the finite difference method when you are modelling them on a standard structure cartesian grid and they will have similarity to algebraic topological method which dealt in a much more detail in the previous modules. I wanted to end this module with this overarching summary that are the aspects of the mimetic method that I wish every person who is taking this course should know.

So I thank you for being part of this entire 12 weeks journey. We have learnt a lot during this entire 12 weeks.

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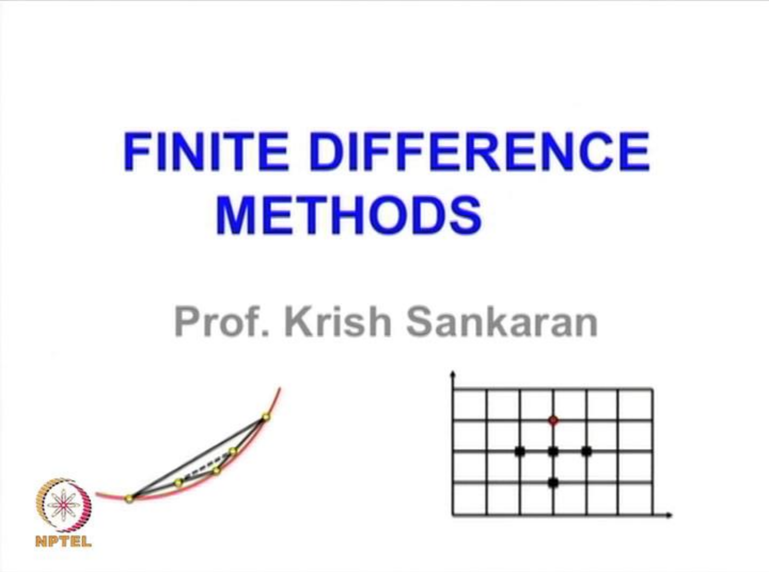

We have looked into very simple basic methods like Finite difference methods

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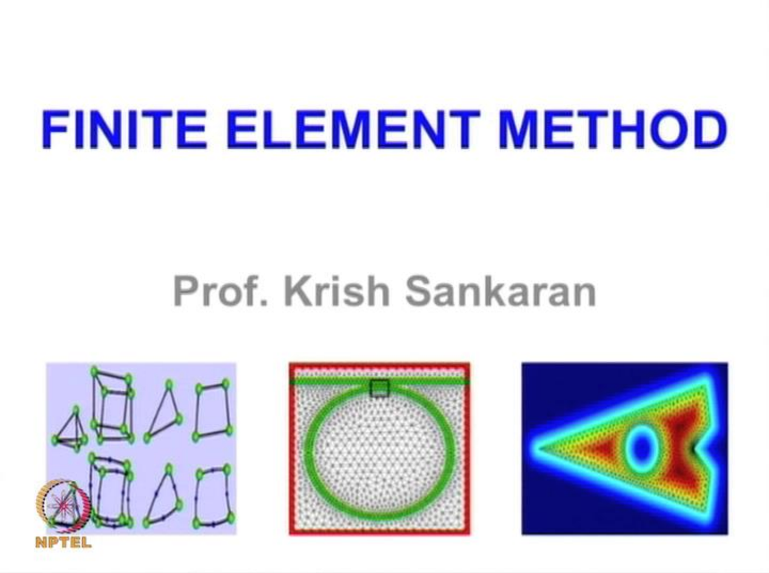

Finite Element method

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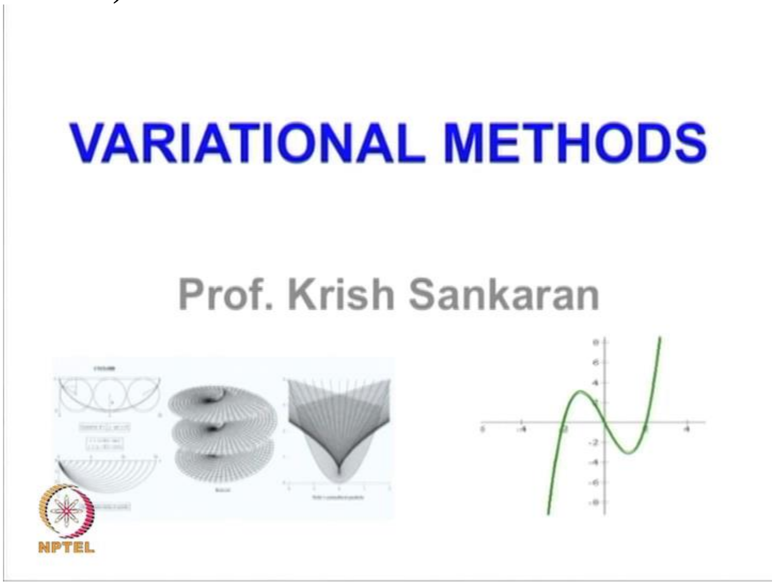

Variational Methods

And we also looked into some of the advanced methods like

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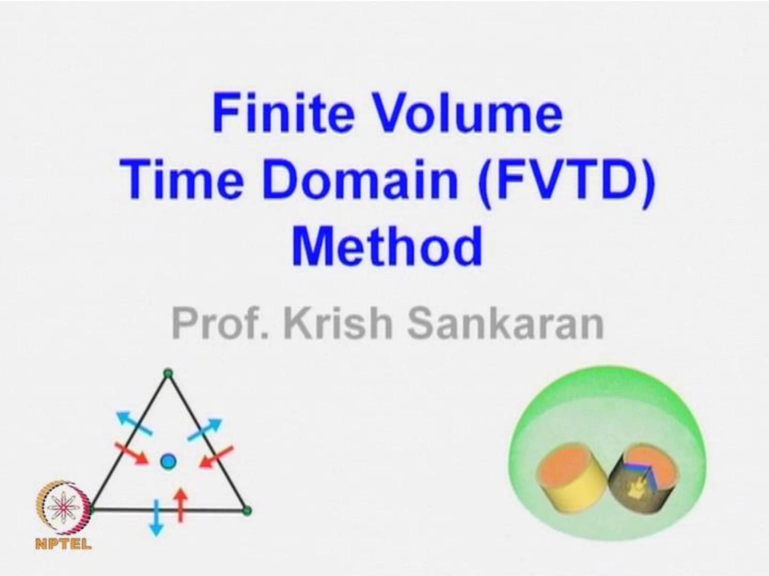

Finite Volume method.

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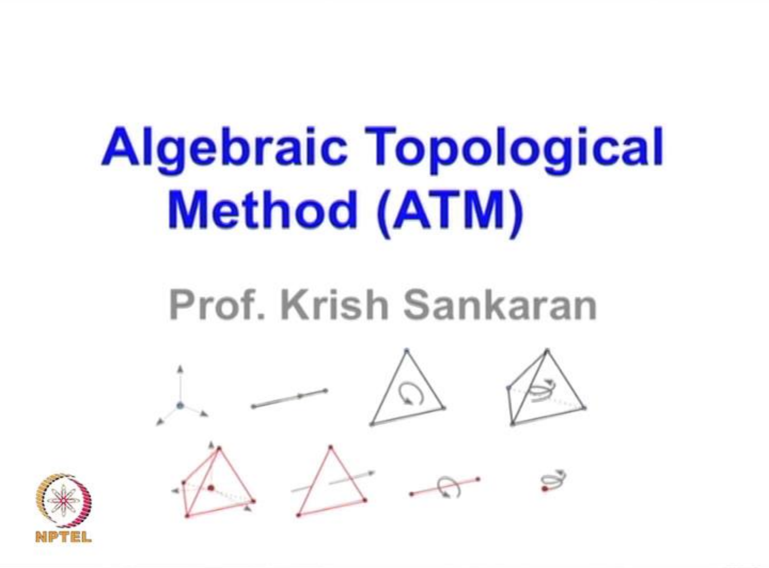

Algebraic Topological Method

We have very briefly gave you in this module the overview of

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Finite Difference based Mimetic methods

With that being said there has been a lot of learning both for me teaching this course on an online platform like this. We have given our best in combining the theory with numerical experiments.

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And not only that wherever possible we wanted you to come and experience the Lab tours so that you can understand what are the devices we are modelling.

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Because sometimes we have no clue about a wave guide.

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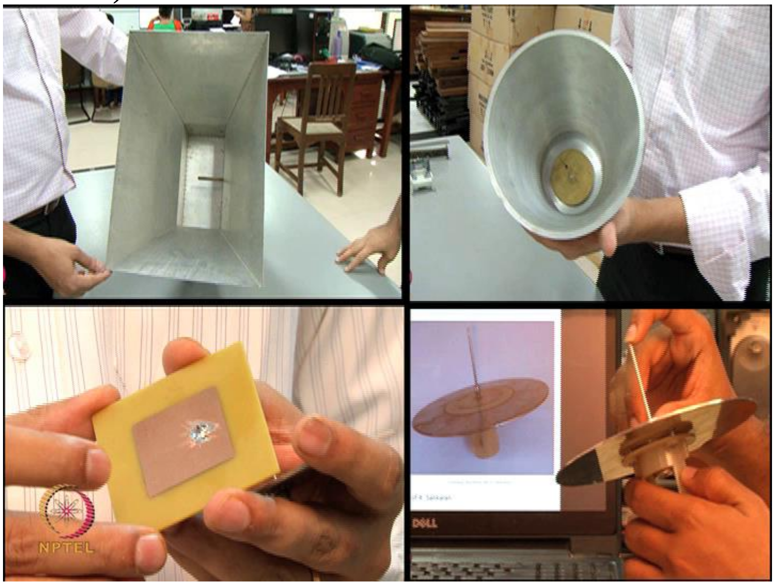

About an antenna

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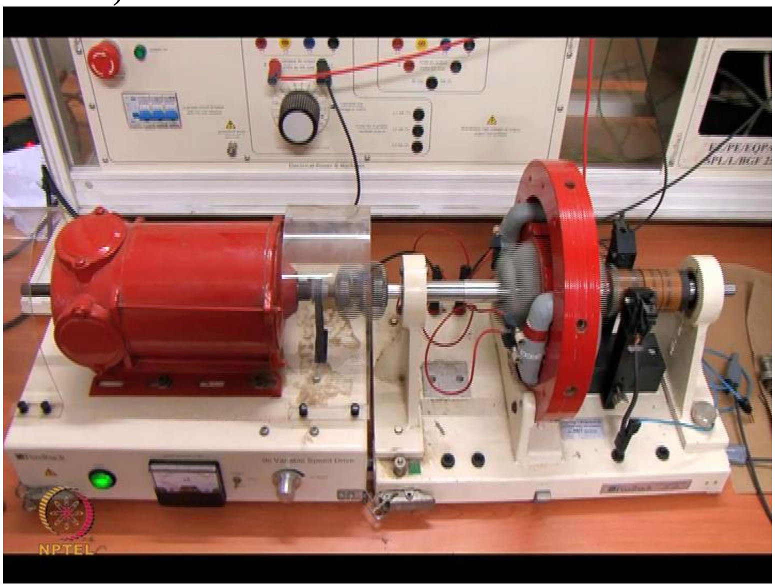

About machines transformers. So we wanted to break that barrier and we wanted to introduce you to physical aspects of devices that we are modelling . So in that sense we wanted to take that devices in our hands feel them physically and understand them there parts their aspects their functionalities their important dimensions so on and so forth. So that is what we did during the Lab tours.

And I feel Lab tours are more and more valuable for people who are coming from outside electrical engineering. If you are coming from departments like Physics Applied mathematics it is going to be very important element so that you can understand physically what are the devices. Because some people would have never seen antennas. That is something we like to break and that is the reason we are integrated where ever possible the lab tours into the course work and further more we are in a course called computational electromagnetics. So we cannot do away with simulations. We have to do some simulations and we have to make sure simulations are integral part of this journey and I am sure you have seen that we have given you a lot of exercises we have tested lot of exercises during the course and we also encouraged you to program your own codes and test it.

And I am sure that you have discussed various aspects of simulation during the last 12 weeks or so in the forums and our teaching assistants were able to answer those questions much more supportively. And I personally hope that you have learnt a lot about the basics of computational electromagnetics. Not only that your prospective and your understandings and your ability to model electromagnetic problems have changed in that sense I feel that we have given you a new foundation we have given you a new bases to begin with your journey in the electromagnetic field it does not end with the code it has to continue if you are an engineer working in a company its your daily routine if you are a scientist working in lab this is going to be your bread and butter. And also you are a PhD student this is going to support you in a very long way. So in that sense I hope I personally believe that this course has given you the prospective. I would like to hear about your opinions on the forum.

With that being said I wish you all the best for all your future projects and also I wish you best of luck in the forthcoming examinations. Thank You!