Computational Electromagentics and Applications
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Lecture No 40
Algebraic Topological Method (ATM II)

In the earlier module we looked into the operator which is called as a boundary operator we set the boundary operator operates on a chain and takes it from k plus 1 to k.

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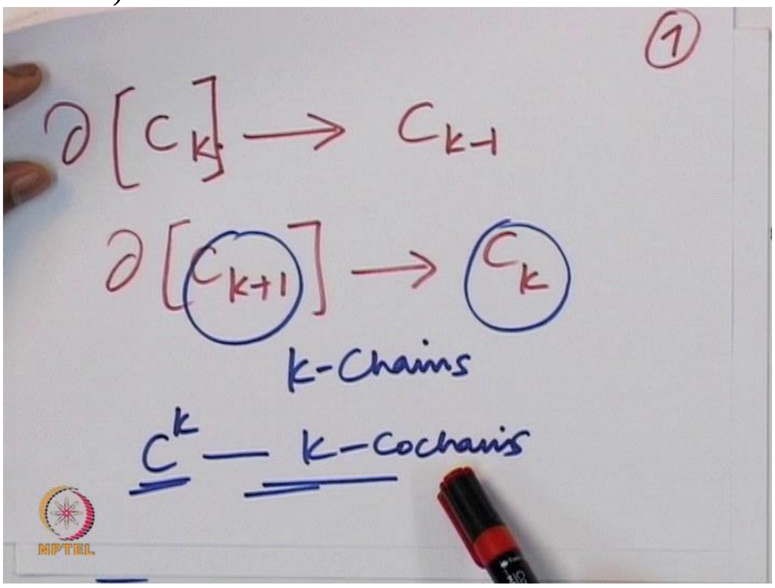

In other words what we have is the boundary operator operating on a chain k chain will take it to K minus 1 so we can say the mapping is from k to k minus 1 are we can even say if you want to keep K plus 1 and K as a basic thing we can say when you are operating it on K plus 1 it should give an output which is k. So we are using subscripts to represent k chains so this is something you should remember so these are chains k chains .

So we will use the word cochain and you use a superscripts so when we put CK superscript what we are talking about are K cochains so don't make a mistake with subscript and superscript make sure that you are following the thing subscripts they are called as chains and superscript they are called as cochains so I said the boundary operator is going to take a k plus 1 chain tour K Chain the co boundary operator does the exact opposite where is instead of all the k chains it acts on a k co chains.

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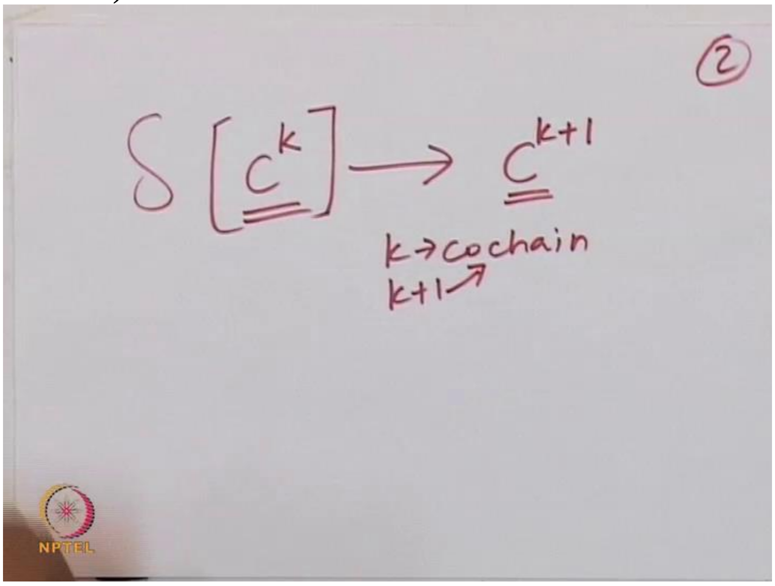

So let’s see what it does we say the co boundary operator represented by symbol tilde the small delta it takes a value that is in[C k and I said it operates on a k co chain and the co chain value is written on the superscript and it gives a value that is in ke plus 1 remember these are K co chains and K plus 1 cochains. So it takes a value from c k to c k plus 1.

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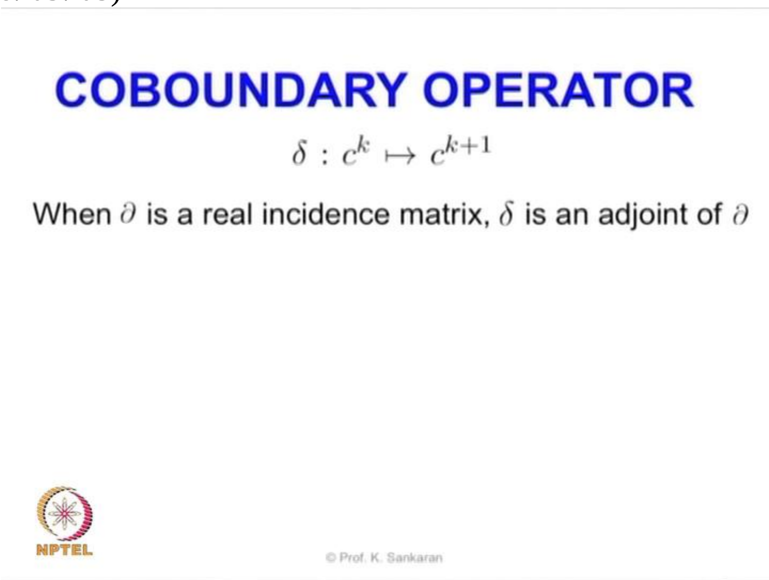

And that's what we have here in this slide so what we have is the co boundary operator takes a value of a k co chain and takes it took a plus 1 so when we have a matrix represented by a set of real number.

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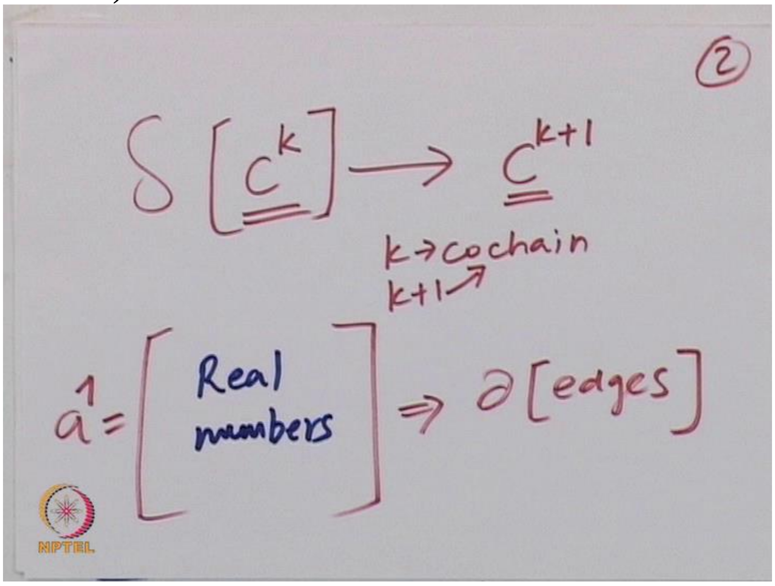

So what we have is a set of real number so let's say a one is given by a set of real numbers a one is the boundary operator operating on edges the co boundary operator of this when a one is real so the matrix contains only real numbers the co boundary operator and boundary operator are somehow related.

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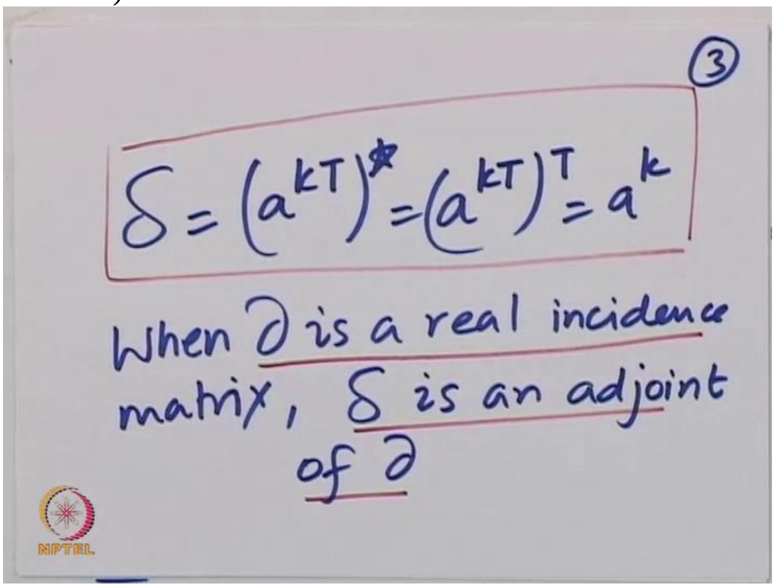

So what we have is the co boundary operator return as the Delta small delta is equal to (a kt) remember this for the transpose of the boundary operators(a kt) transpose which is nothing but a of a k. So which is nothing but so what you are saying that is when partial the boundary operator is a real incidence matrix the co boundary operator small Delta is an adjoint of delta the partial derivative so this is most important when this is real incidents Matrix this will be an adjoint of this so that is what we are saying here so if it is an adjoint this relationship is valid.

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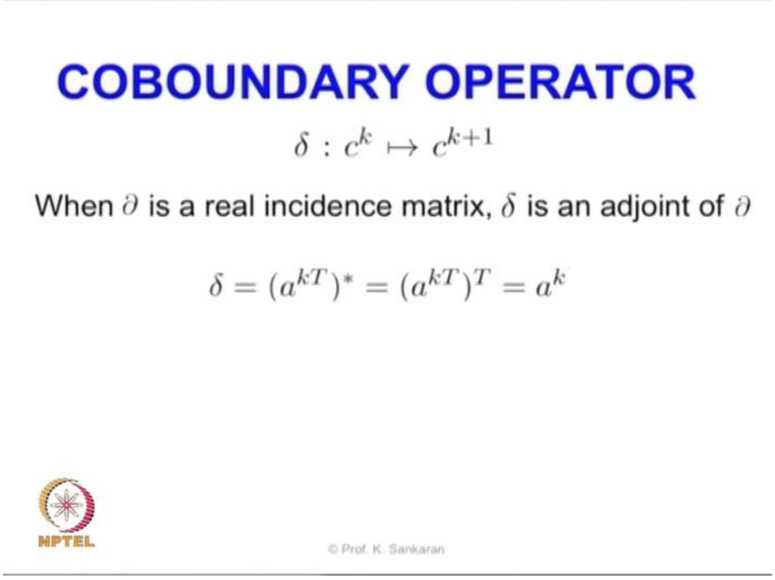

So that being said what we have is a very very unique property of the co boundary operator remember I promised while we started this discussion that we are going to avoid using vector calculus and partial differential equation so until now we have not done that we have been coming back and forth into Sam case of vector calculus or partial differential equation but now I am going to keep up my promise so I am going to define using the co boundary operator the operators that we use in vector calculus namely the divergence the curl and the gradient so let's see what is the physical connection or how the co boundary operator is related to these operators namely divergence curl and gradients using a simple example.

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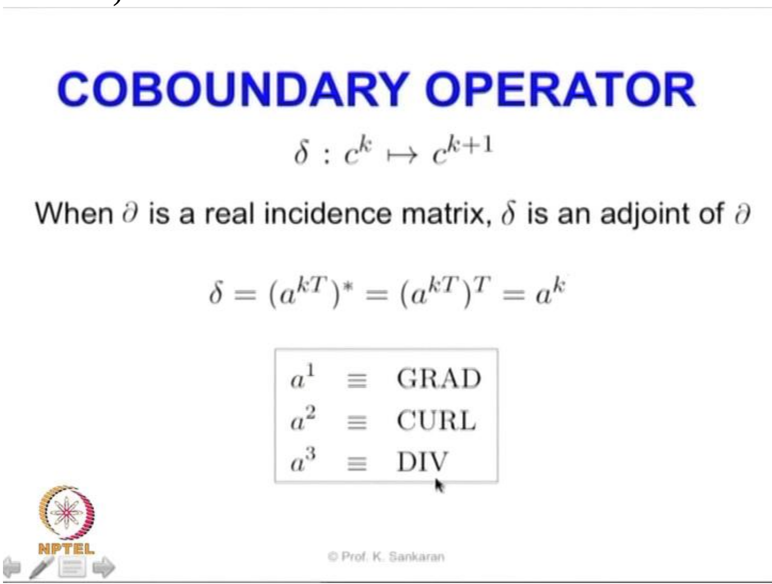

So before I tell how they are related let me already say what are the relationship so the relationships are the first co boundary operator is nothing but the gradient the second co boundary operator is a curl third co boundary operator is a divergence I am going to explain you how this relationship is using a simple example as follows.

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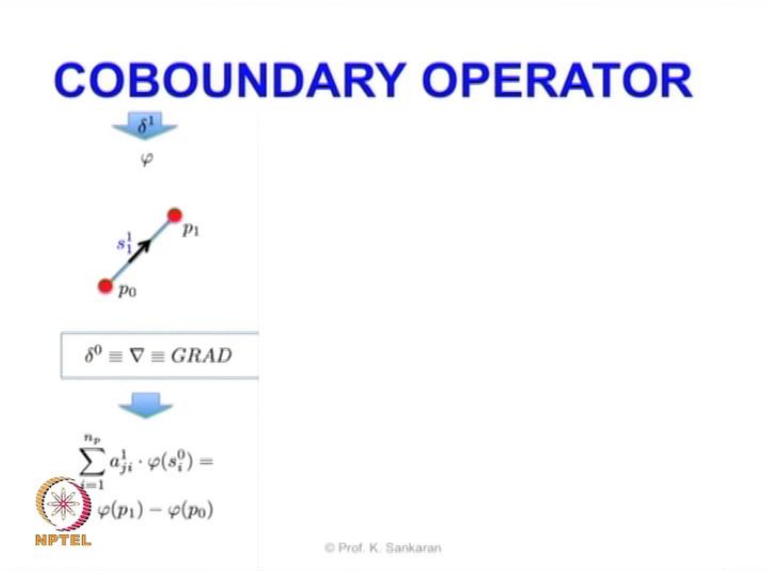

So let's take this case the case where we have a line with two points so I do the co boundary operator the first co boundary operator on this line on which certain cochain are there so the coach in which I am going to operate is defined on the points so the cochain is nothing but the potential electric scalar potential which are defined on those points remember the co boundary operator transforms a K cochain took k plus 1 co chain.

So let's see how this cochain the 0 cochain is going to be transformed to one cochain remember this is call electric scalar potential are defined on points and they are called as 0 questions and you will see how this 0 questions gets transferred one order higher so what you are saying is when we see hear the value of the one co chain is given by the expression so it is operating on the individual points where the potentials are located we have Phi of P1 minus p h i of P0 so what we have is nothing but the potential difference define on this line the potential difference is nothing but the electromotance which is a 1 co chain.

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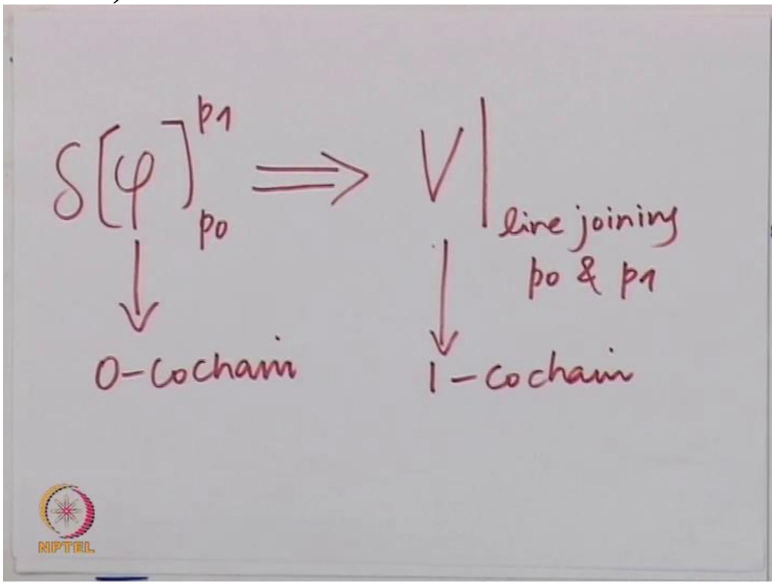

So let’s see so while we do this one these are defined on the two points which are basically p0 and P1 what we are getting his we are getting we define on the line joining P0 and P1. This is the 0 cochain this is a 1 cochain so as we expected the co boundary operator is taking a 0 co chain into one co chain.

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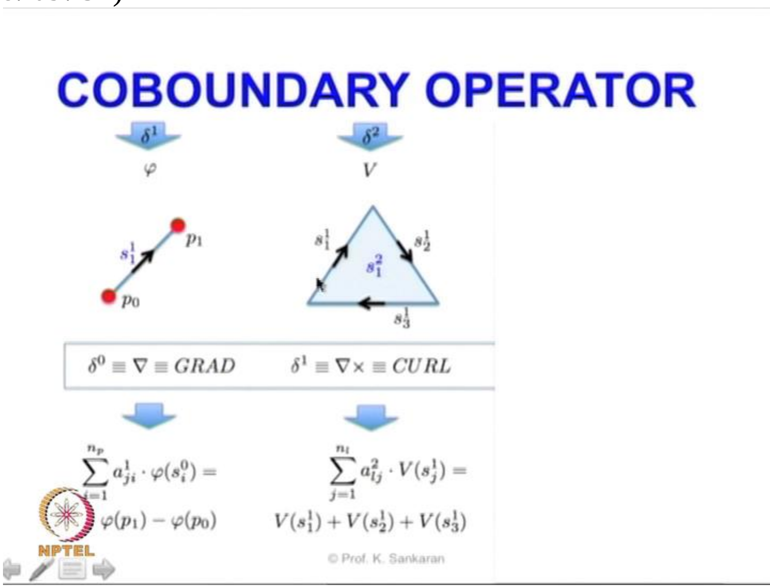

Let’s see what it does to two co chains and in this case the second question is operating on the surface values so the values what it operates on our basically the boundaries of the surface so the boundaries of the surface what we have got are nothing but the potential differences are the electric motors so the potential difference is here is between this point, and this point, and this point, this point , and this point, and this point. So what you are computing here is the coach and operating on the potential differences the potential difference are 11 cochain and what you are getting is the definition of the one cochain getting transform to order higher so what we have got here is the values represented s plus V (s 11) plus v (S 1 1) plus v (s 1 2) plus v (s 13) so is plus because the direction in which you are computing is they are all in the same direction if one of them is in the opposite direction we will have a minus.

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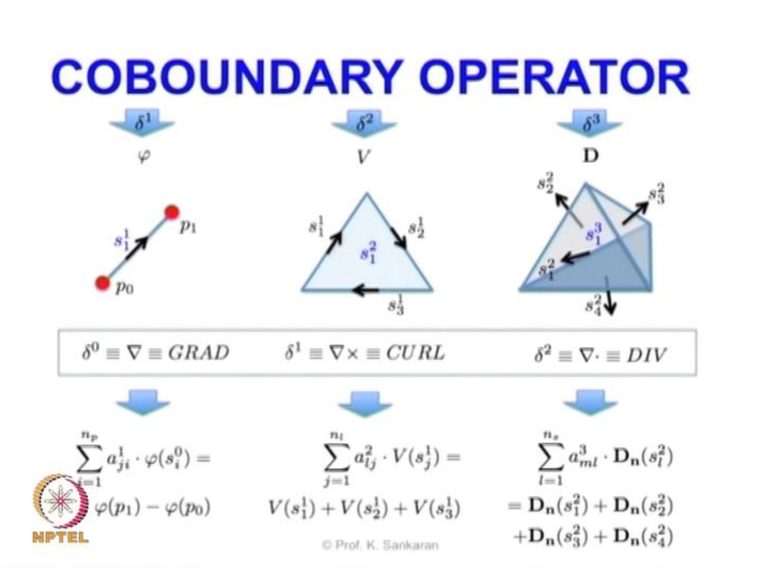

Similarly the third coboundary operator is operating on the the volume and it takes the value that is define on the surfaces one or higher so what we have got is a normal component of the director the electric displacement vector define on the normal components and we are doing the editions whether it is plus or minus depends on the direction in which the arrow is Pointing if the arrow is pointing in word it is described as minus. If the arrow is pointing out what we have a plus accordingly what we see here is the co boundary operator is able to give us illogical way in which we can define divergence curl and gradient using one single operator and this operator is purely determined using the geometrical.

Or in fact the topological aspect of your mesh for your domain so this is a big relief because you don't need to worry about how to compute the gradient how to compute the curl or to compute the divergence because once we have set the way to compute co boundary operator regardless of what the object it is operating on we will be able to find a way to compute the corresponding quantities in fact the best way to go ahead is once we know what is going to be the geometry we start to define the boundary operator and remember in our case they are going to be real mattresses once the boundary operator is known the co boundary operator is normally known so I am going to write down what I mean by that in logical way.

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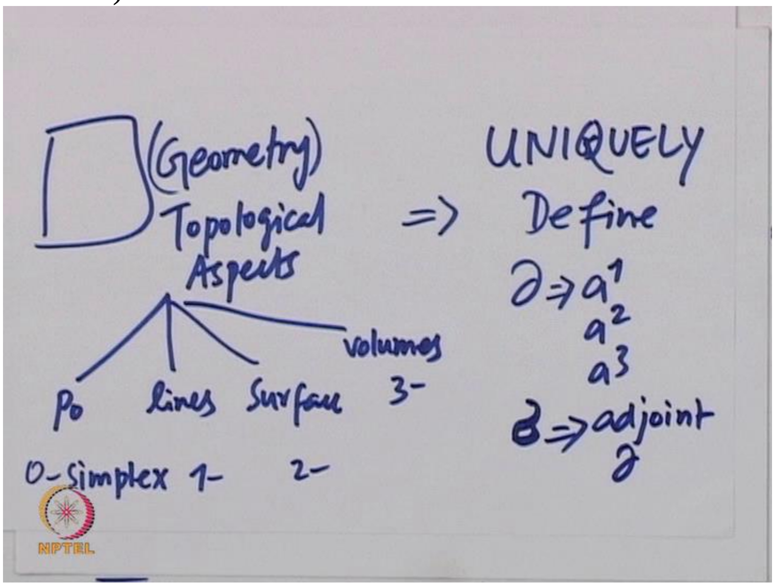

So what we see here is once we have the geometry so what I mean by geometry is a topological aspects so I know the points I know the lines I know the surfaces and I know the volumes so what I am talking about his Hero simplex 1 simplex to simplex and 3 simplex so once I know all of them the only thing I need to worry about is how I am going to define the direction for the simplex us once I define that I can uniquely define the boundary operators the boundary operators are here A1 A2 A3 for example so once this is known my coboundary operator is also defined because it’s going to be the ad joint of the boundary operator and once I know the boundary operator I have already found a topological way to represent divergence curl and gradient that’s it so the entire machinery of differential equation is represented by one particular operator and then you have everything so having discussed that now we are going to look into some of the properties of this boundary operator more in detail.

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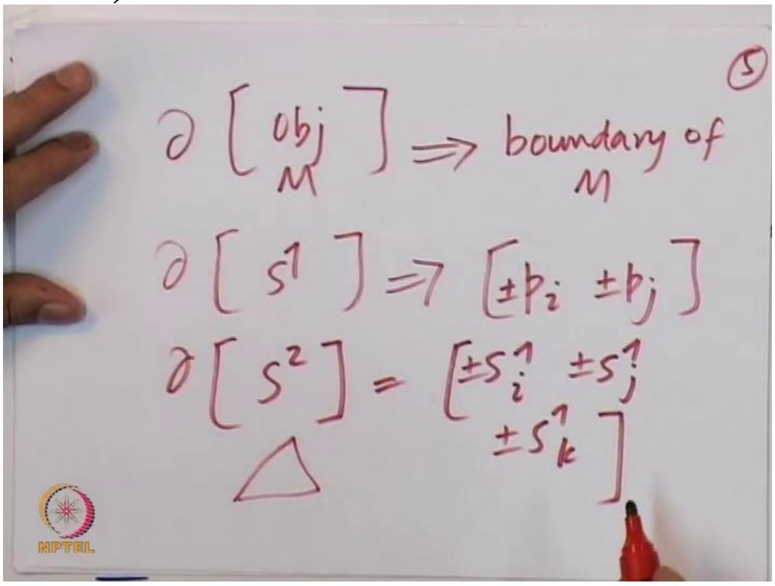

So I said the boundary operator is going to operate on object and use the boundary of that so let's say m, a boundary of m. So what you are saying here is it takes an object M and give the boundary of m so if I put here S1 it's going to give me two points [p i and P J] which are the boundaries of S1 similarly if I put S2 its going to give me the values of S1 of I S1 of J s 1 of k so i j k are the 1 simplex that are found in the case of the 2 simplex so assuming that this is a triangle the three sides are given by here and the signs of it is going to be weather its plus or minus is going to be defined by the orientation that we set so what happening now is we are able to find out that the boundary operator is operating on certain simplexes or objects and it gives what happens when we do the boundary operator on a point.

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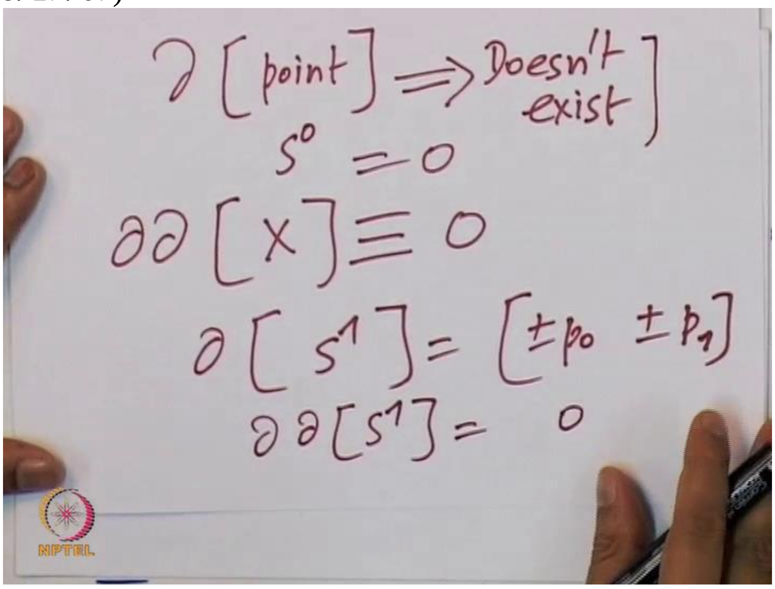

So when here a boundary operator operating on a 0 simplex doesn't exist in other words when the boundary operator is operating on s0 we get a value 0 so that is one thing that we should understand second thing is when we do the boundaries operator twice the value regardless of what we put will always be equal to 0 let me explain this so the boundary operator of S1 will be 2 points which are plus or minus p0 plus or minus p 1 when I do it again what I am doing as I am doing the boundary operator on those points because the boundary operator on the points are 0 this will be against 0.

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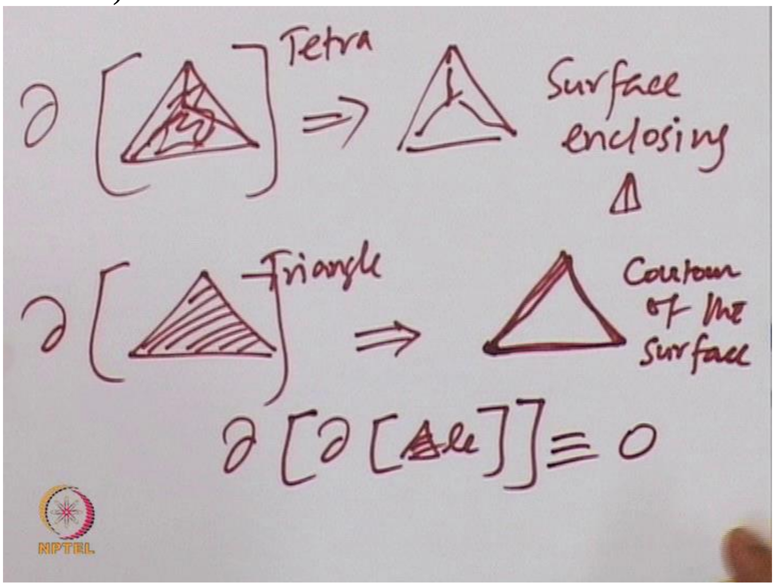

Second thing is if you take let's say a tetrahedron and you are doing a boundary operator on the tetrahedron what you will get is a closed volume let's say this is the face so I am doing the boundary on this one I am just getting the surface of this so this is the surface and closing this tetrahedron what happens now is when I do again the boundary operator on this one I will get a 0 this will be much more clear when we take a simple example of a triangle let's say we have surface represented by so this is Tetra so this is Triangle so when you are doing the boundary operator on the triangle what we are getting is we are getting a closed contour that is contour of the surface so when I do the boundary operator again on this particular thing what I see is the boundary of that doesn't exist because it starts from a point it goes in One direction it goes to other direction and it comes to the same. So the point is circular so when it circles the boundaries again equal to 0 so in other words when we take a circle remember apologist is a person who cannot differentiate between a coffee mug and a doughnut for him any closed loop is a circle same like a circle and for a circle the starting point and ending point are on the same point.

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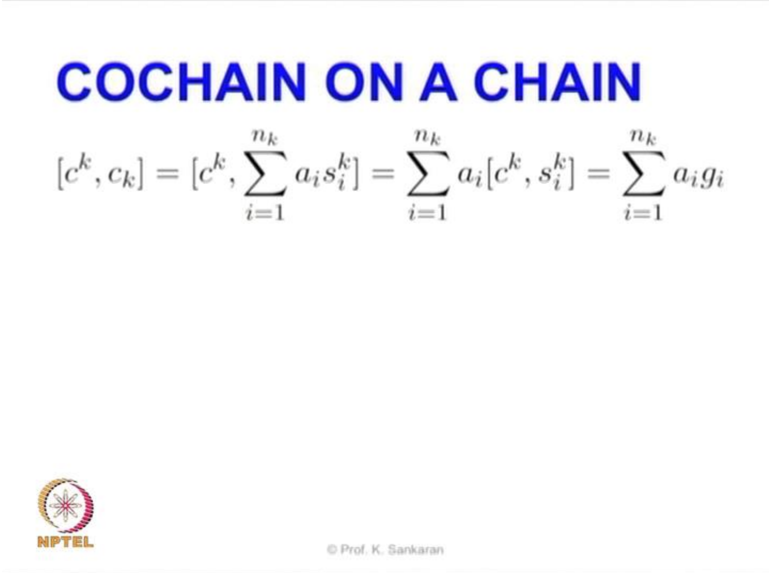

In other words boundary of a circle is 0 so that is a logic so first property what we have got is we have got any time you do twice the boundary operator regardless of what is their EX this will be always equal to 0 so this is number one so the second one is when you are computing the value of a cochain on a chain it has to be explained a bit in a detailed so let me explain this more in a detail so what I am doing here is I am computing the value of a cochain or chain remember the cochain are nothing but physical variables that are located on a particular chain.

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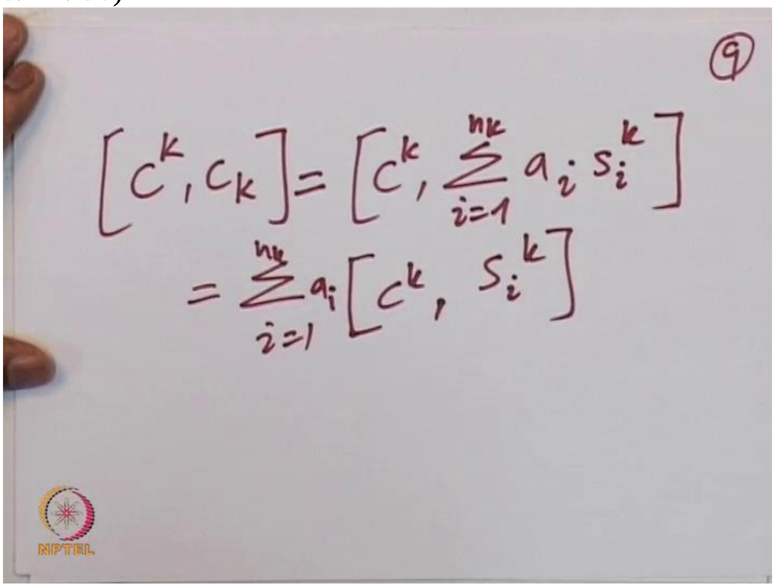

So if I want to compute the coach in which is written as a superscript or chain what is happening here is I am basically using the value of the chain which is written as Sigma i equal to 1 to 10 of k where the dimension k is the dimension of the simplex that are involved in the chain so we have a i s i of k so this one we know of keychain can be represented as

Coefficient incident coefficient X the value of the care simplex so what we are doing now is we are moving the summation out of the bracket so this is equal to C of k multiplied by s i of k. Remember I told you the value of cochain that are defined on individual chain they are represented as g remember that we used the word g to represent the value .

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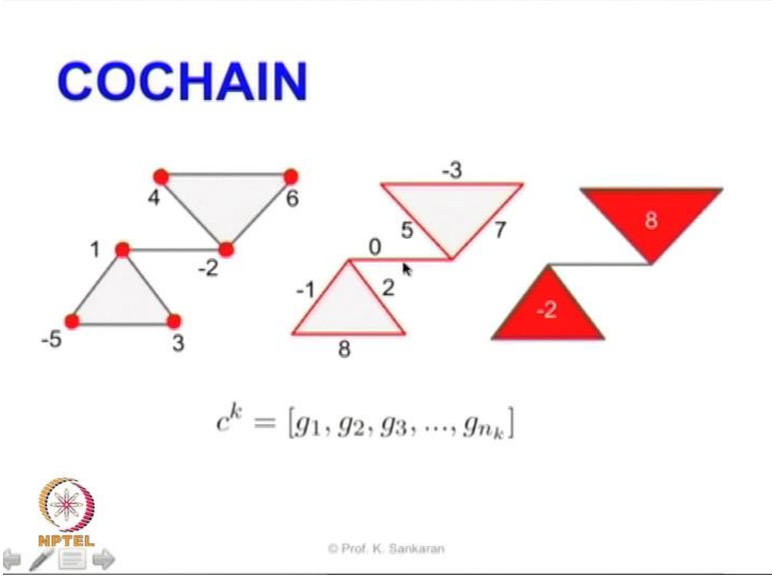

So let's go back in the slides to see what I mean by that when we started discussing the value of the cochain we said these are the values of the cochain that are defined on the individual points so when we are talking about 0 cochain we are talking about points of those notes and their value So g1 is the value that are stored in the nodes if c k is C 1 is equal to 1 we are talking about the values that are stored in those ages likewise these are the values so j i nothing but the values that are stored in simplex so we are going to substitute for that thing here .

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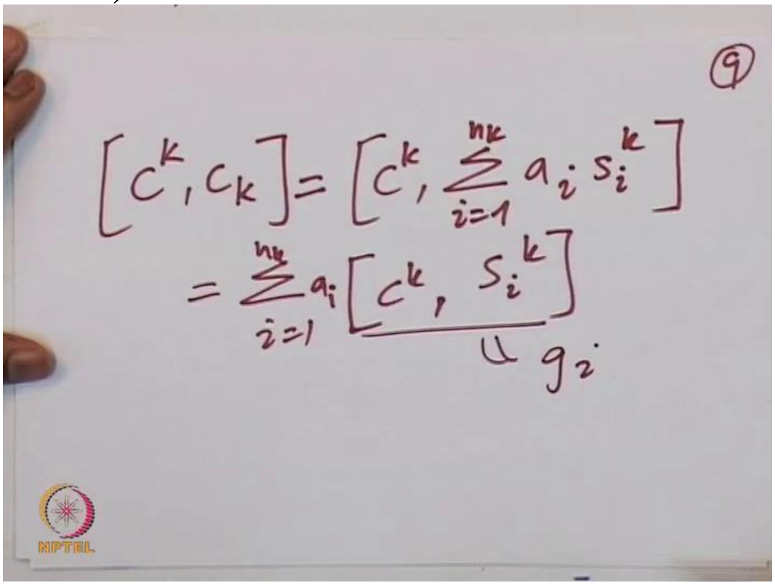

So the C of thing is equal to g of I and once you do that what we will see here is.

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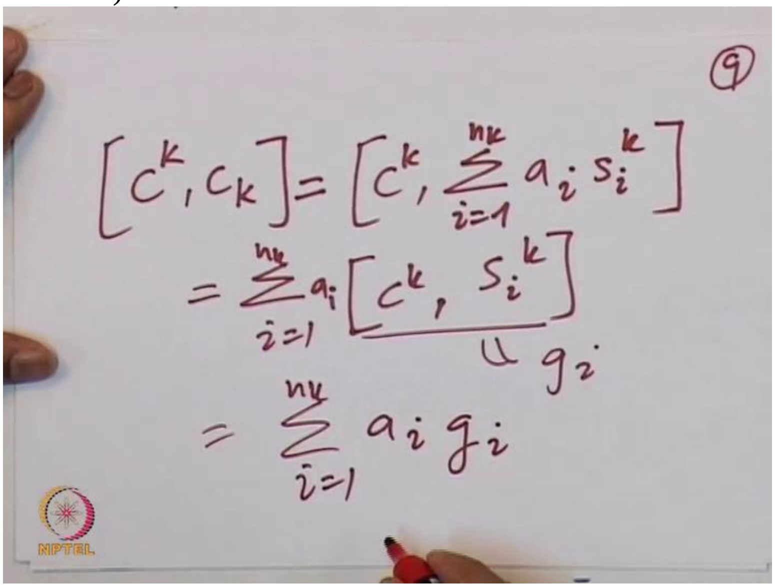

We see that the value of thing will be written has Sigma I equal to 1 to nk a i g i and that's what we have in this slide.

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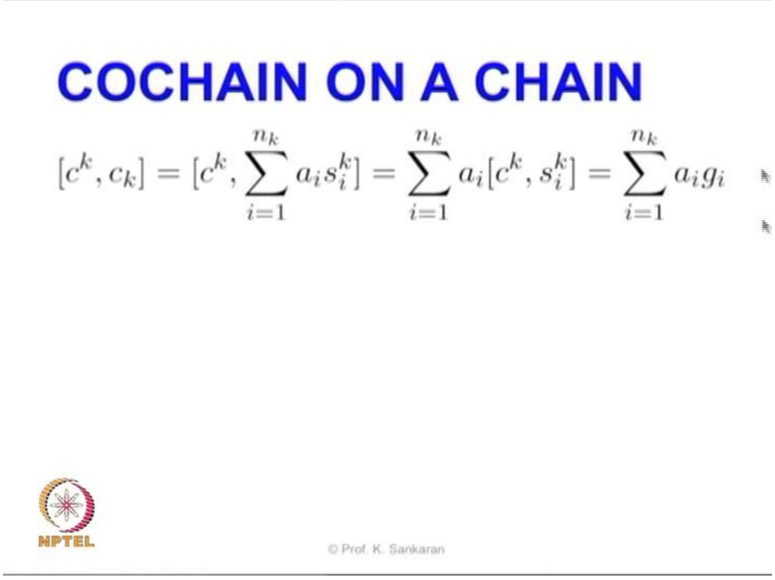

This expression can be written as a i g i so now we know that the value of the cochain can be directly obtained from the value of the boundary operator.

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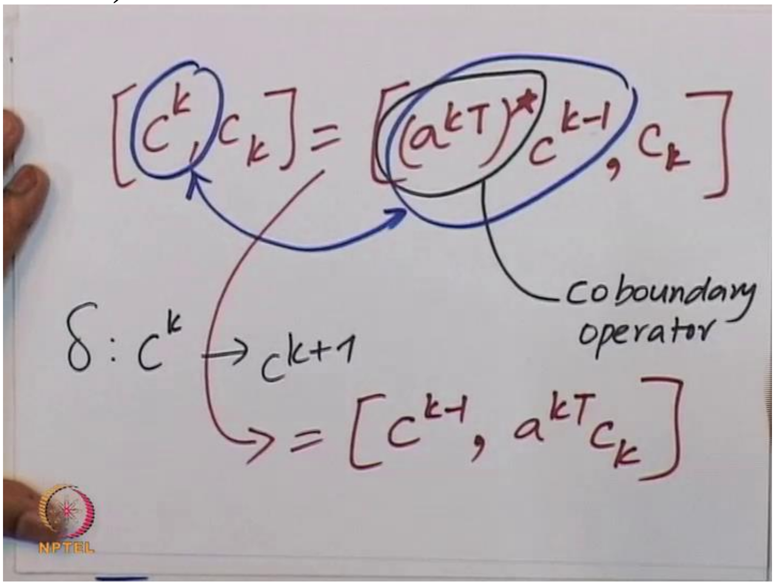

So in other words the value of c k define on k can be written as a k t conjugate a joint thing of that so what you are doing now here is instead of C k we are writing a term like this. You might wonder how this is true. We said the coboundary operator is the adjoint of the boundary operator and this is what we got initially this is the coboundary operator and the co boundary operator is operating on a k minus 1 cochain it takes the K minus cochain to a k cochain so that's what we said upper boundary operator text a value that is k to c k plus 1 we said it adds one value so in this case k minus one so it adds to improve the value buy one so it takes k to k plus 1 so it has to take k minus 1 to k so that’s what we see here the second thing is we know that we can change the order of this thing such that so this is again equal to we can take this with the co boundary operator to the other side so what will happen is we have c of K minus 1, so the conjugate complex goes away and what we will get is only the value that is returned like this so this is important.

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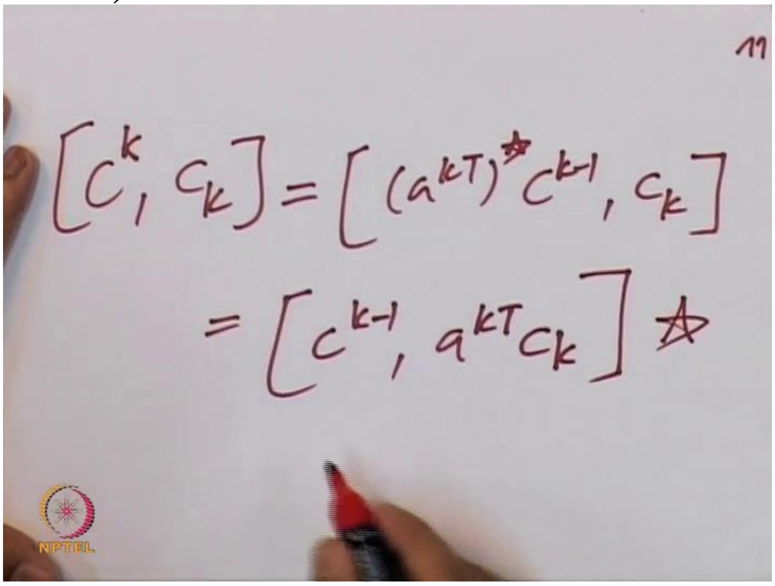

Let me write down this one more time we said c of K the cochain operating on a chain is equal to and we said this is equal to we take this one here and then we change the order on which the boundary operator it goes into the chain that ad joint goes away so what you will get is so this is a very very important thing but let me write it down in a more familiar form that you are used to the most familiar form you are used to is.

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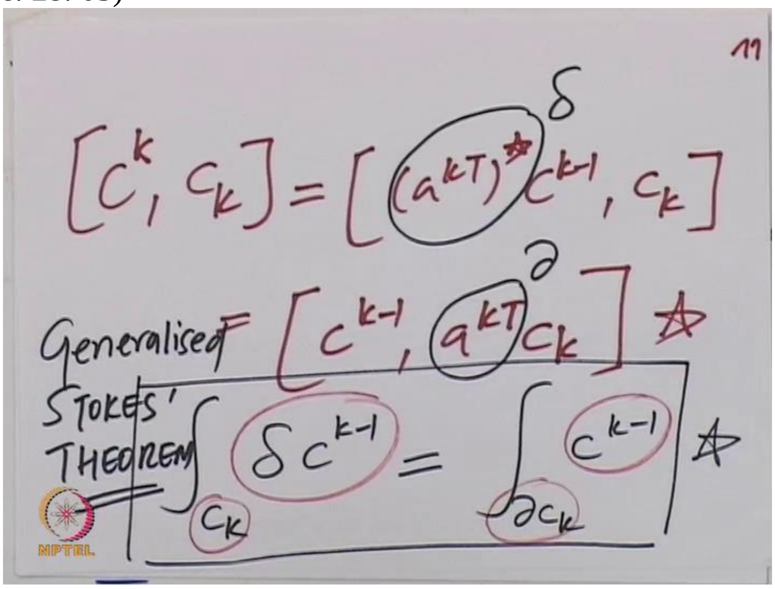

So I have a co chain so remember this one is the small delta it's a co boundary operator so show the co boundary operator operating on a c k minus1 and it is integrated over the sea of K is equal to value what we are getting here so always a second value all the boundary terms

I told you this one is the boundary operator which is written as partial derivative operating on C of k and this one is k minus 1.

And this is nothing but the generalized Stokes theorem and this is the most important relationship so what is happening here is the co boundary operator are operating on a particular K cochain and then we move the co boundary operator to the domain itself it becomes the boundary operator and it operates on C k minus 1 so in other words the co boundary operators operating on a cake or chain nothing but the integrals so these are the integrals and the boundary operator and the chain are the domain itself so this is the most important generalized Stokes theorem so this relationship is very very fundamental and crucial for you to understand the relationship between the coboundary operator and the boundary operator and that k cochain and Co chains so ok cochain is defined on the chain and then we are defining that way what you are talking about it the co boundary operator operating on a k cochain K minus 1 cochain and defined on the boundary so we can basically move the operators in such a way that we can get it.

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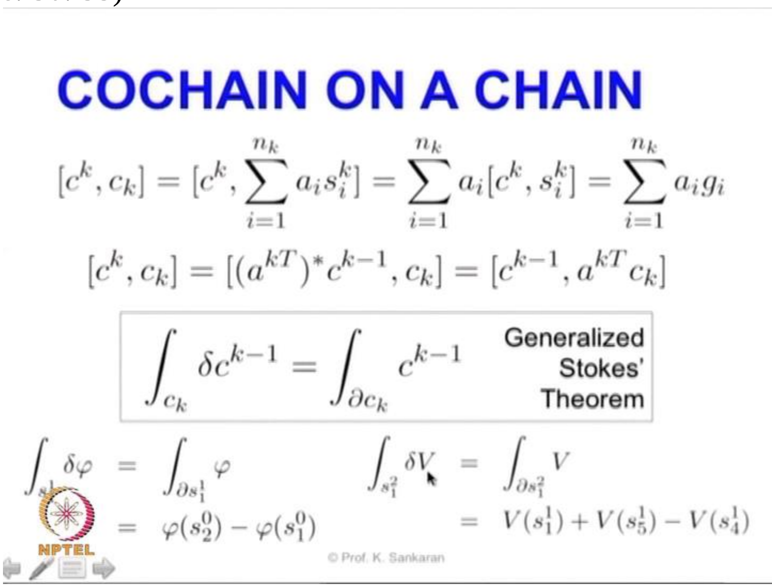

So this is the most important aspect of the algebraic topology so we get finalized form which is a generalized Stokes theorem so using that we can compute the co boundary operator operating on the 0 cochain as discussed here and that’s what you are getting here remember this is a same value that we define Alia and similarly this is the way we define the boundary operator operating on one cochain.

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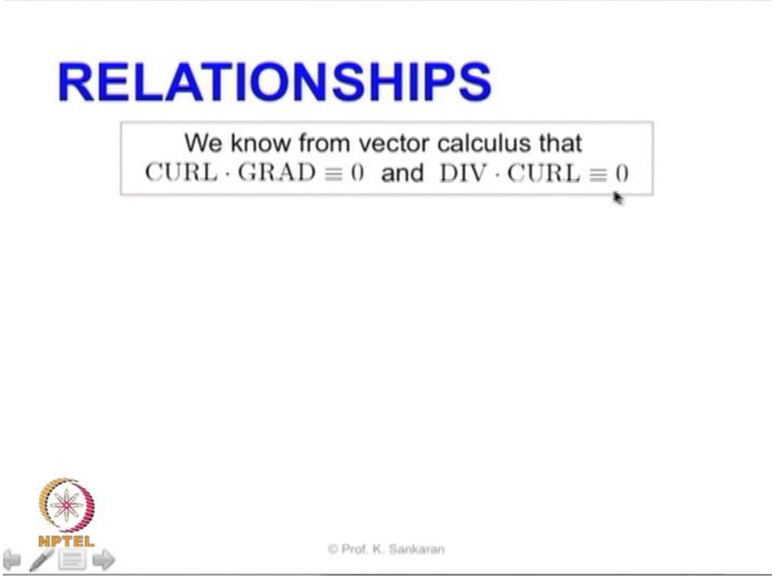

And the last relationship that we are interested is the relationship between the operators themselves so as you know from the vector calculus call of the gradient is equal to 0 and we know that the diversion of the curl is also equal to 0.

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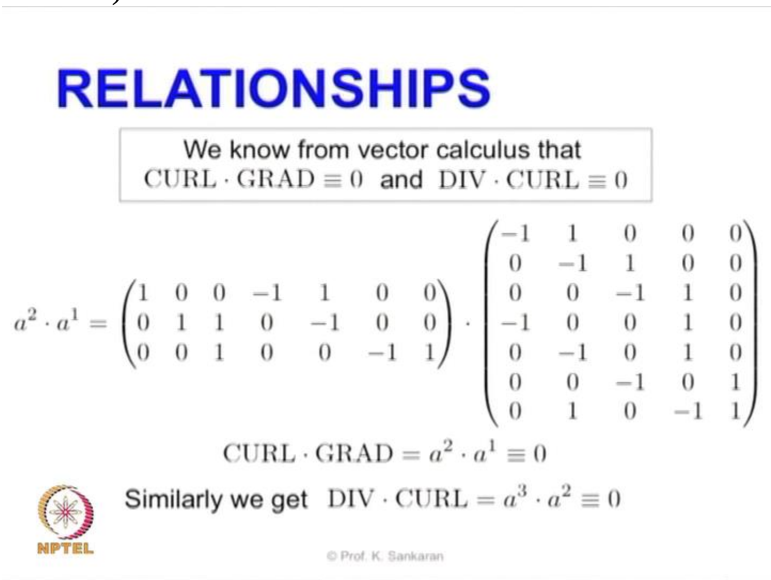

So this you can verify we know the co boundary operator for curl we know the co boundary operator for the gradient the dot product of them should be equal to 0 similarly we can also find the divergence of the curl which is a 3.2 should also be equal to 0 with this we come to the stop where we have discuss the various aspects of algebraic topology we have discussed what is the physical meaning behind two classes of variables namely the source variable and configuration variable reset the initial theory of algebraic topology from a very simple network what we considered where we discussed about K simplex we discussed about co chains we discussed about what is the boundary operator we discussed about what is the cake cochain and we discussed about what is a co boundary operator having discussed all these things we also found out the co boundary operator that is in fact the most important operator which gives us various ways of representing divergence curl and gradient once we have that we can also see they follow certain fundamental geometrical properties given by the vector calculus namely the curl dot gradient and also the divergence that curl should be equal to 0 so that way we see that they are related to vector calculus themselves and in other words they can also help us to get physical connection between algebraic topology and vector calculus so with this we will come to the conclusion of this module in the next module we will look how to model Maxwell equation using algebraic topology thank you!