Computational Electromagnetics and Applications
Professor Krish Sankaran
Indian Institute of Technology Bombay
Lecture No. 23
Method of Moment

So in todays module we are going to look into one of the most important method and for the good reason the method is also called as mother of all methods. We have discussed some of the theoretical background related to method of moments.

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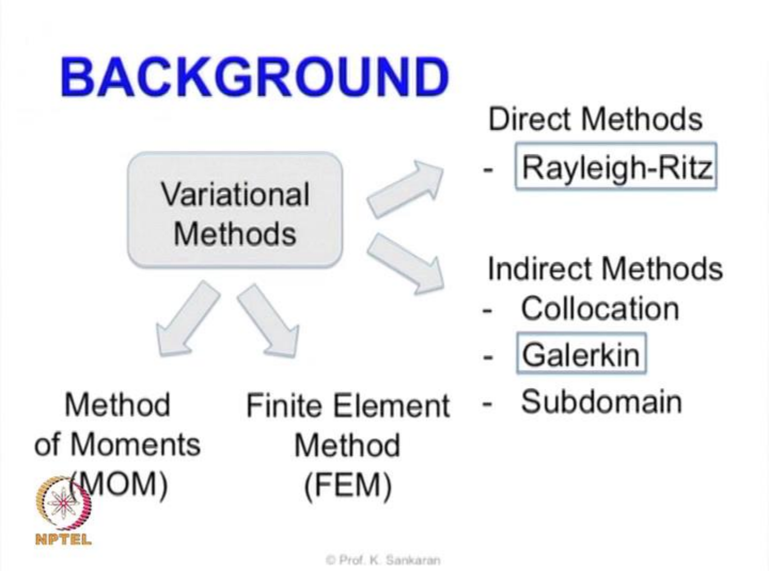

While we discussed the variational method and also Finite element methods. With that being said they are quite prepared to look into the basic idea behind method of moments. At large we will start looking into some of the motivation behind it historical aspect as well, and take you through some examples.

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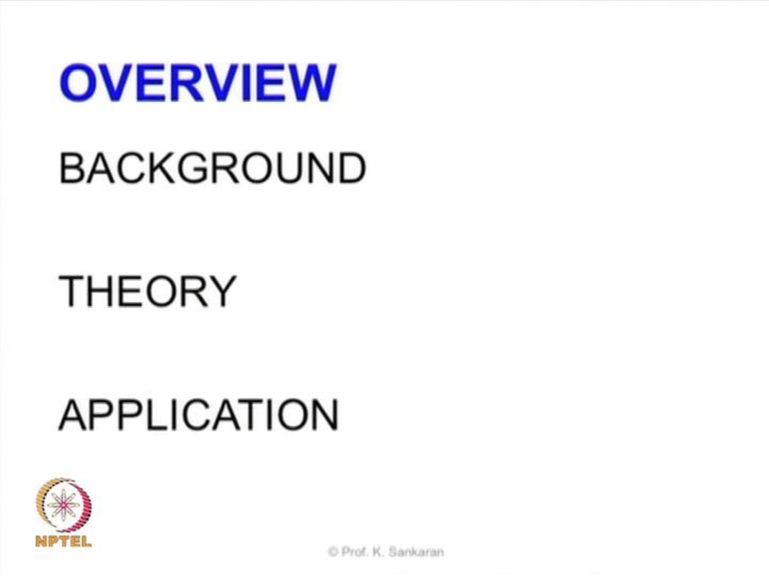

so in todays module we will start with the background as usual. We will look into the theoretical aspect of the method of moments and we will look into some of the applications.

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With that let us start with the background itself.

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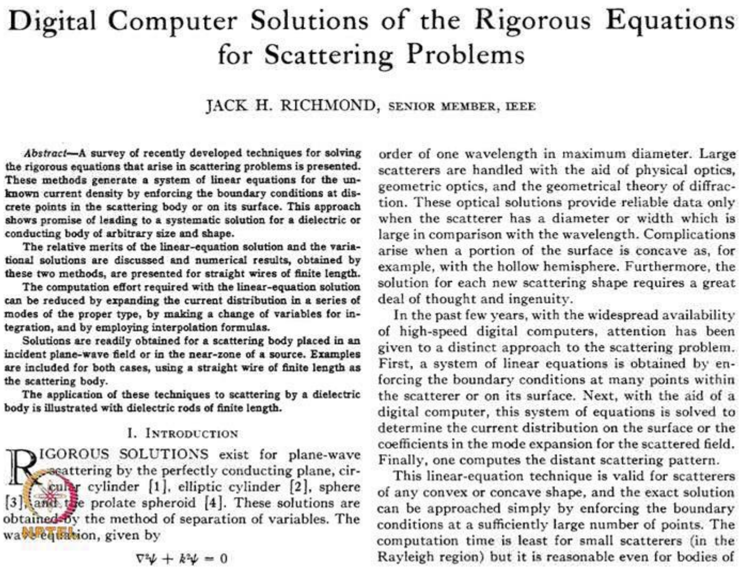

Digital Computer Solutions of the Rigorous Equations for Scattering Problems

JACK H. RICHMOND, SENIOR MEMBER, IEEE

Abstract—A survey of recently developed techniques for solving the rigorous equations that arise in scattering problems is presented. These methods generate a system of linear equations for the unknown current density by enforcing the boundary conditions at discrete points in the scattering body or on its surface. This approach shows promise of leading to a systematic solution for a dielectric or conducting body of arbitrary size and shape.

The relative merits of the linear-equation solution and the variational solutions are discussed and numerical results, obtained by these two methods, are presented for straight wires of finite length.

The computation effort required with the linear-equation solution can be reduced by expanding the current distribution in a series of modes of the proper type, by making a change of variables for integration, and by employing interpolation formulas.

Solutions are readily obtained for a scattering body placed in an incident plane-wave field or in the near-zone of a source. Examples are included for both cases, using a straight wire of finite length as the scattering body.

The application of these techniques to scattering by a dielectric body is illustrated with dielectric rods of finite length.

I. INTRODUCTION

RIGOROUS SOLUTIONS exist for plane-wave scattering by the perfectly conducting plane, circular cylinder [1], elliptic cylinder [2], sphere [3], and the prolate spheroid [4]. These solutions are obtained by the method of separation of variables. The wave equation, given by

$$\nabla^2\psi + k^2\psi = 0$$

order of one wavelength in maximum diameter. Large scatterers are handled with the aid of physical optics, geometric optics, and the geometrical theory of diffraction. These optical solutions provide reliable data only when the scatterer has a diameter or width which is large in comparison with the wavelength. Complications arise when a portion of the surface is concave as, for example, with the hollow hemisphere. Furthermore, the solution for each new scattering shape requires a great deal of thought and ingenuity.

In the past few years, with the widespread availability of high-speed digital computers, attention has been given to a distinct approach to the scattering problem. First, a system of linear equations is obtained by enforcing the boundary conditions at many points within the scatterer or on its surface. Next, with the aid of a digital computer, this system of equations is solved to determine the current distribution on the surface or the coefficients in the mode expansion for the scattered field. Finally, one computes the distant scattering pattern.

This linear-equation technique is valid for scatterers of any convex or concave shape, and the exact solution can be approached simply by enforcing the boundary conditions at a sufficiently large number of points. The computation time is least for small scatterers (in the Rayleigh region) but it is reasonable even for bodies of

So the method of moments actually started in 1965 by J.H. Richmond who actually introduced this method to Engineering Electromagnetics community at large mainly for catering problems. With that being said the problems what he approached is mainly looking at scattering applications. And later on couple of years later it was actually Harington who introduced the method to a wider applications like antennas and various applications in engineering. And he popularized the method further the introduced various mathematical aspect using simple example, so his publications both the books and also the transaction papers which he introduced method of moments are even today the most important references for method of moments.

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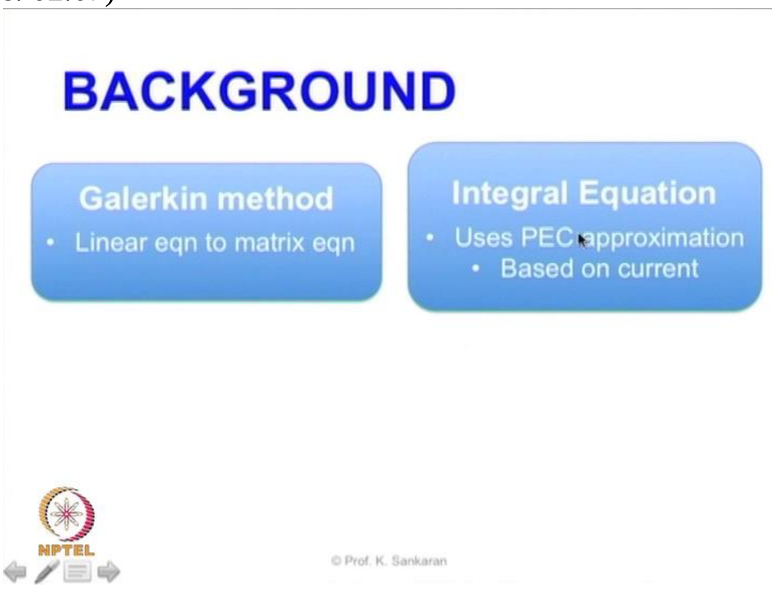

With that being said, there is similarity to Galerkin method or the Finite element approach which we discussed, for example we will see here in the Galerkin method we are basically going from a linear partial differential equation to a matrix equation. In the case of method of moments we are going to start with a integral equation not the partial differential equation like in the case of Galerkin method. So we are going to start with the integral equation, and we are going to use certain PEC approximation on the boundary conditions. Perfect Electric conductor approximation.

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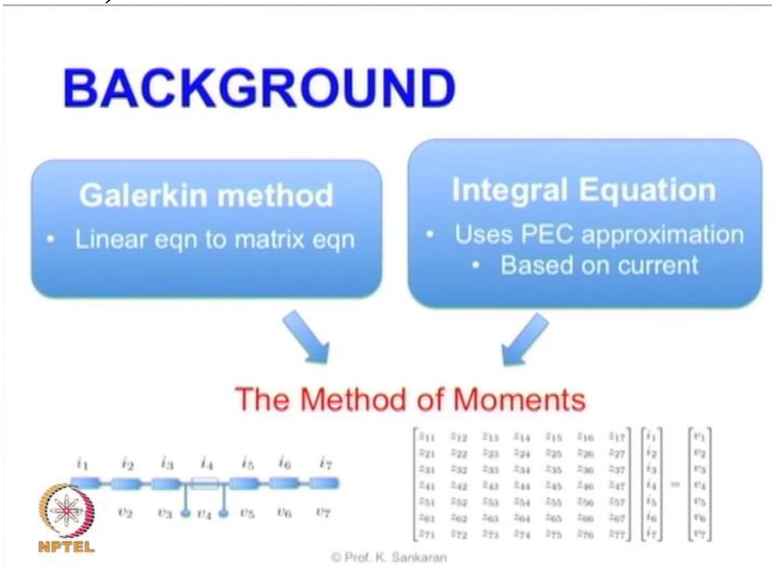

And also the entire approach is based on currents so what we in essence get is similar set of matrix equation but in the case of the method of moments what you get is basically the impedance matrix and the current vector and the voltage vector. This is basically kind of Ohms law, where you have the r represented by the matrix here the I represented by the vector here, and voltage represented by the vector on the right hand side. So in a sense it is a very similar approach to the Galerkin method.

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but the theory is a little bit different. So we will start with the theoretical concept itself.

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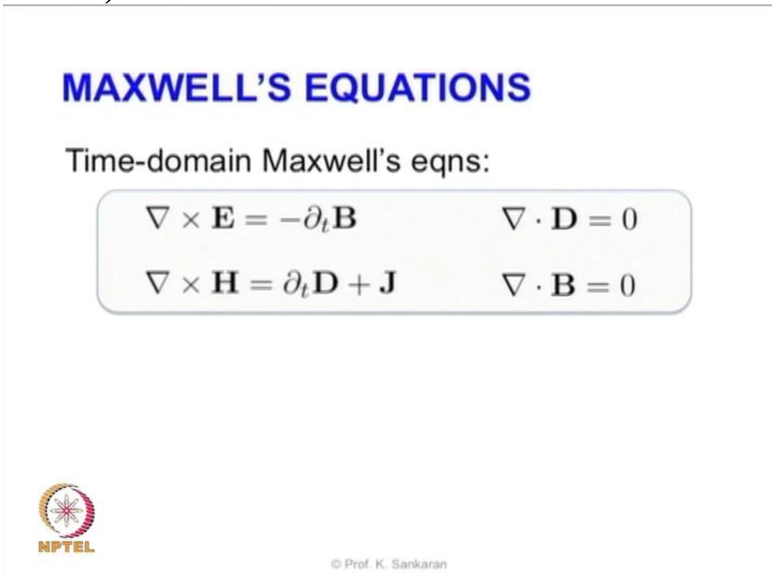

As usual we will start with the time domain Maxwell equation represented by curl of E and H field. And similarly we should also have the diversions condition. If we cannot define the diversions condition, it is difficult to uniquely compute the E and H, so both the curl formation and the diversions equations are important for us to uniquely compute in addition to the boundary conditions as well. So we are not discussing here the boundary conditions. So this is a time domain Maxwell equation.

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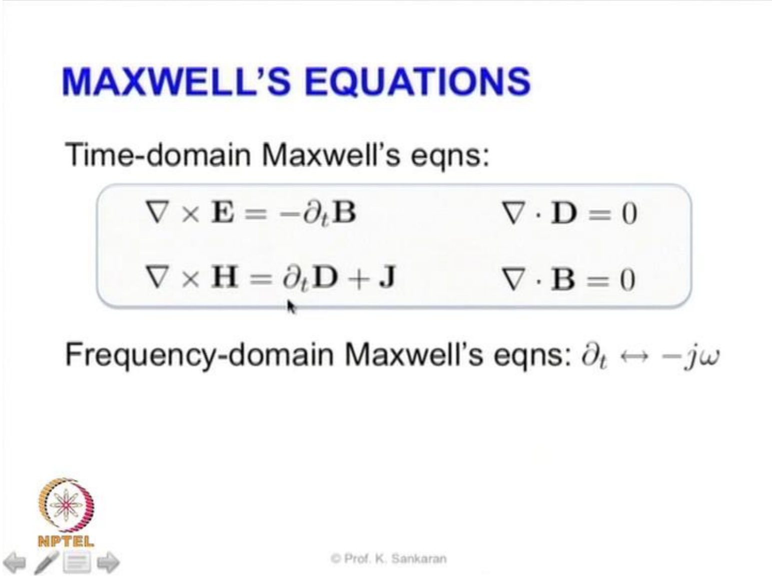

and when we go into the frequency domain we can substitute for the time derivative minus j omega. Please be aware that some literature use j omega. If you use the convention of j omega instead of minus j omega your signs here has to change accordingly. But the general formulation is pretty straight forward.

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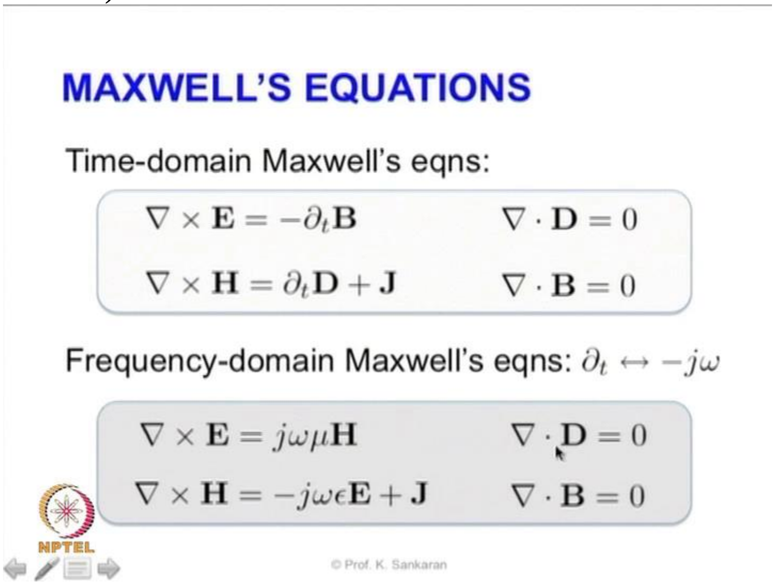

So when you go for the frequency domain formulation so the minus j omega is getting substituted for dt and the equations are getting transformed accordingly and the diversions equations remain the same.

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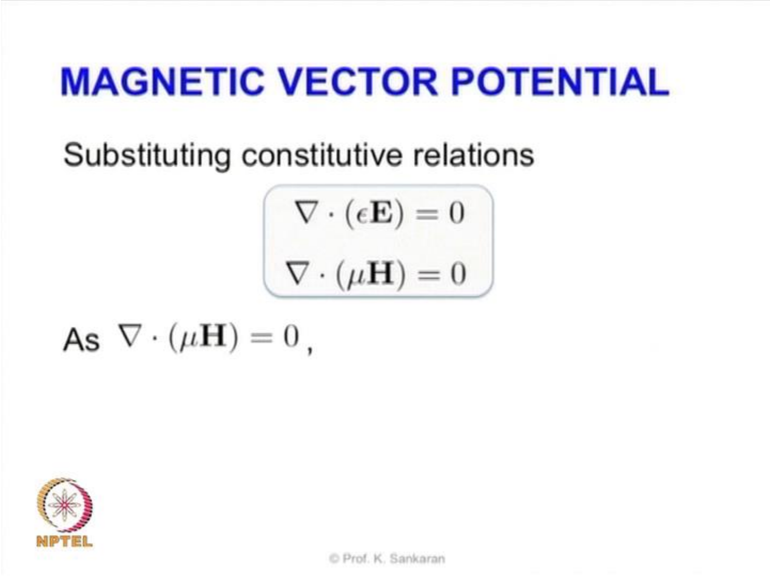

When we can substitute the constitutive relationships basically the constitutive relationships are the relationships between D and E, and D and H. When we substitute them the diversions conditions become like this. What we essentially have is diversions of Mu of H is equal to 0. So we take that one and we say when the diversions of something is equal to 0 that means that something is actually a curl of some other thing. So that is the basic mathematical identity.

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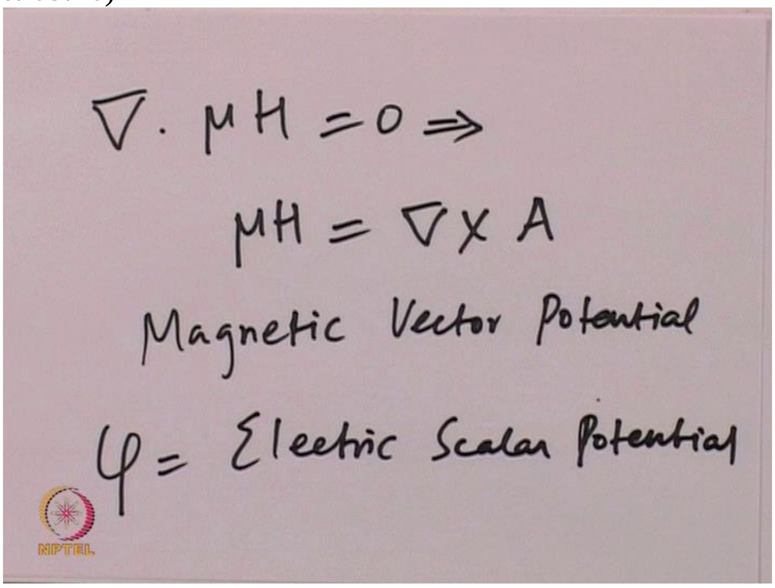

So when we say the diversions of (Mu H) is equal to 0.What we are going to say is we are going to say this MuH is actually curl of some other variable, and that other variable we call it as A which is a vector. And this is nothing but the magnetic vector potential. Remember the counter part of it in the case of electric will be the Electric Scalar Potential. As compare to the Electric Scalar Potential which is very much a physical quantity which you can basically mesher in laboratory experiments, magnetic vector potential is purely a mathematical construct. It is a purely mathematical construct in the sense there is no physical counterpart for it.

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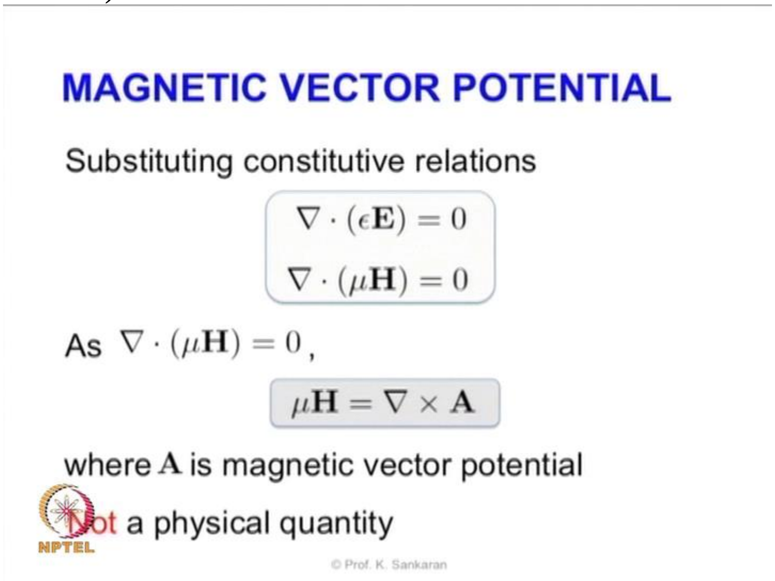

So we do that mainly to do the analysis simple. So the magnetic scalar potential there is no physical counterpart to it. So we will start with substituting this A into the first curl equation what we had.

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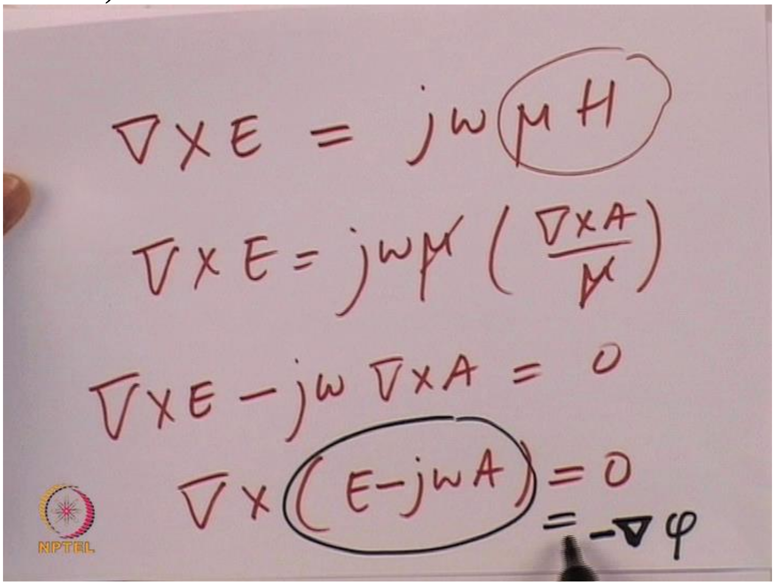

So what we had is we had a curl equation that says the curl of E is going to be given by j omega MuH. So what I am going to do here is I am going to take j omega Mu outside and for H I am going to substitute curl of A divided by Mu). So once I do this what I am going to do is I am going to bring them to the left hand side So curl of Eminus j omega curl of A equal to 0. So once I have this as you can see in the slide I am going to write this equation as Curl of

(E minus j omega A) equal to 0. So now we are going to look into another vector identity when the curl of something is equal to 0 that means this something in going to be a gradient of something. so in this case we take this gradient of Phi, which is the electric scalar potential and by convention we use the minus sign. So this particular thing will be equal to this.

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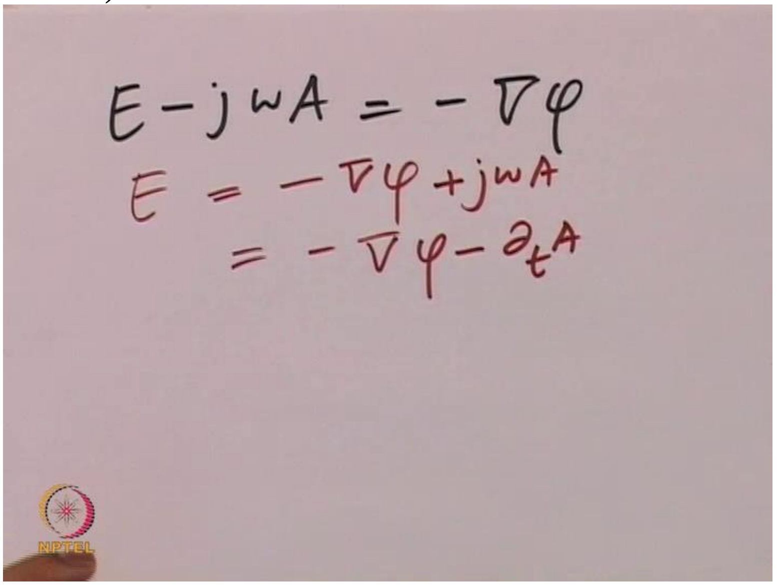

So what we have got now is E minus j omega A equal to minus the gradient of electric scalar potential. So what we can write down we can write it in the time domain form which will be useful for us. So now what we will do is we need to find the way to solve the problem of addressing more variables than the degrees of freedom. S somehow the mathematician use this term called gauging. In this particular gauging is referred to Lorentz gauging named after the great Dutch Physicist called Ludvik Lorentz. And the reason for doing that is to cope up with a degrees of freedom of the variable itself. We are not going to go into the mathematics of it but we will see how did we come to that Lorentz Gauge here?

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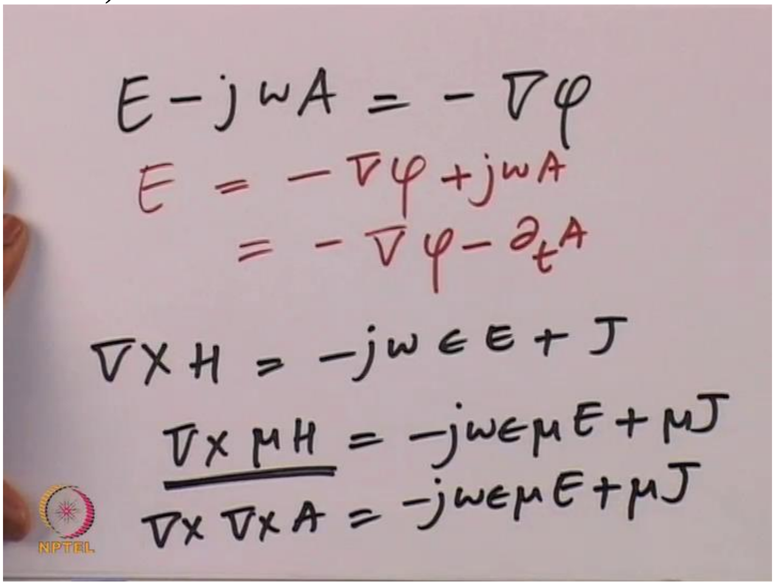

So what we have here is? We start with the basic Maxwell equation the curl of H is equal to minus j omega Epsilon E plus J. So when we multiply Mu towards this entire equation, so what we will get is, but we know for this one MuH is the curl of A. So curl of curl of A is equal to minus j omega epsilon Mu E plus Mu J.

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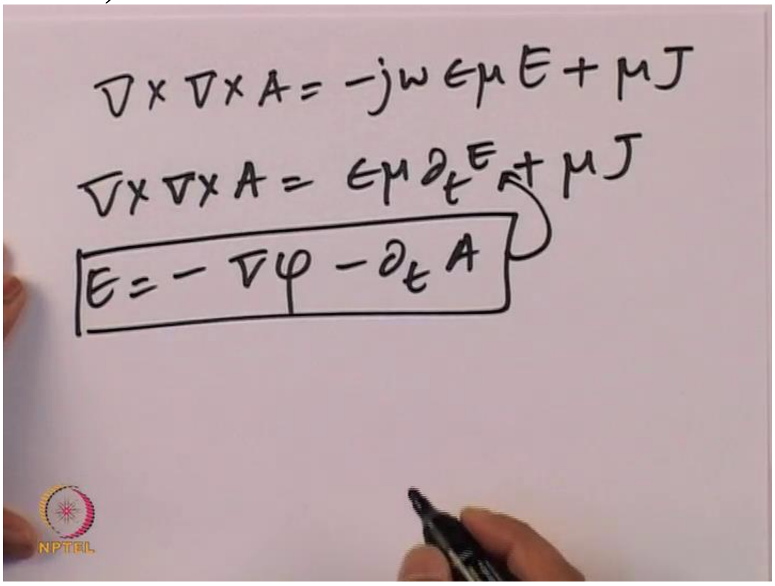

So with this we are going to manipulate the equations further. So we will what we have got now is Curl of A is equal to minus j omega Epsilon Mu e plus Mu J. What we will do is we will substitute the value for E in this equation. We can transform this entire thing in the time domain. So what we will get is curl of curl of A is equal to minus j omega will be dt so we will get Mu Epsilon dt E plus Mu j and we know E equal to minus and we will have the value of minus dt A. This we know we can substitute this value for this E here.

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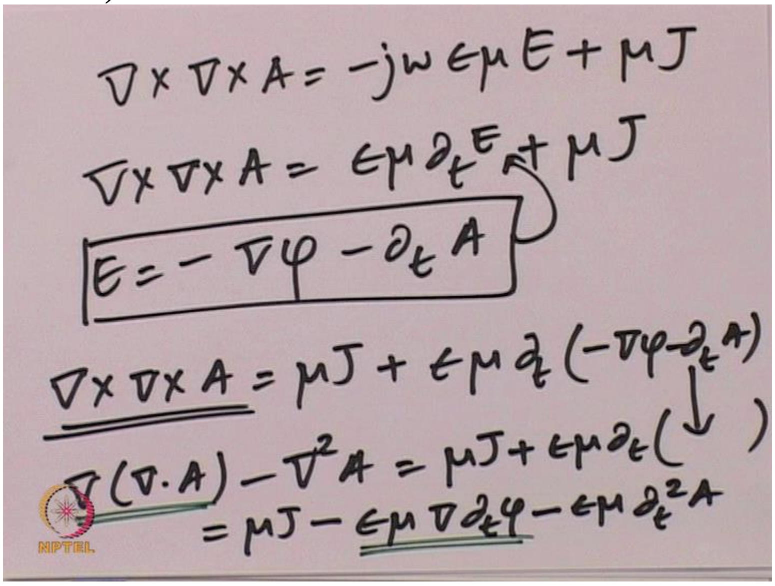

So what we will get is curl of curl of A is equal to Mu J plus Epsilon Mu d t (minus gradient of Phi minus d t A). This left hand side has a vector identity which is nothing but gradient of the diversions of A minus the laplatian of A is equal to Mu J plus Epsilon Mu d t of this particular variable here. So I can expand this inside term as follows. What I will get is Mu J minus Epsilon Mu the gradient of time derivative of Phi minus Epsilon Mu d t square A. So you can notice we have two terms we have 1 term the gradient of the diversions of A on the left hand side and we have the gradient of the time derivative on the right hand side. So we are going to bring them on one side and then rearrange the equation as follows.

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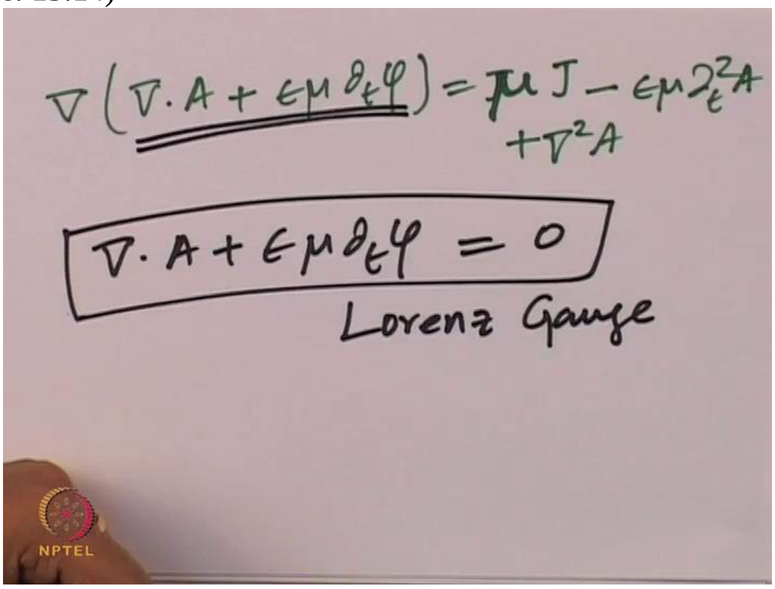

So what we have now is the gradient term diversions of A on one side plus Epsilon Mu d t by the time derivative on the electric scalar potential. And the right hand term which is Mu J minus Epsilon Mu d square t A. And then the Laplatian term from the right hand side which is plus laplatian of A. So what you see here is the condition that will lead us to the Lorenz Gauge. The Lorentz cage is nothing but this term we are setting it to 0, so the diversions of A plus Epsilon Mu d t Phi is set to 0. So this is the Lorenz Gauge. And that is what you will see in the equations to come. So when this is 0, so gradient of something 0 essentially you can equate this to 0 and then you can compute the values accordingly and you are coping up with certain degrees of freedom issues.

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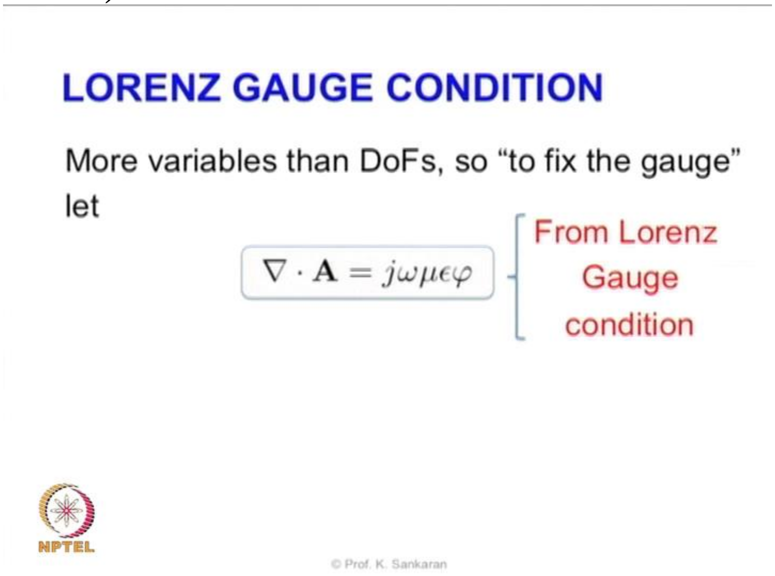

So the idea behind the Lorenz Gauge is exactly that so what we have done now is the derivation for Lorenz Gauge.

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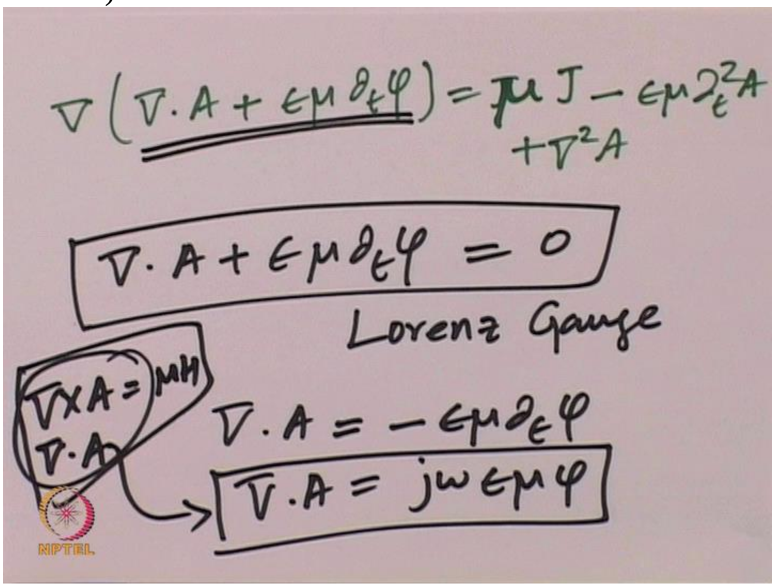

So you can rearrange this term accordingly, What you will get is diversions of A is equal to minus Epsilon Mu d t Phi and then you can substitute for d t as minus j omega and then minus minus get cancelled you will get j omega Epsilon Mu Phi is the diversions of A. 15:17

The reason for doing this is also due to Lorenz Helmond’s theorem. Helmond’s theorem says that its not enough to talk about only the cur l of defining the curl of A we need to also define the diversions of A to uniquely find the value of A. So we have defined curl of A as Mu H and now we have defined the diversions of A as j omega Epsilon Mu. So this is in fact the necessity for us to mathematically compute the value of Mu of A so the background behind Lorenz Gauge is something important. And that is what we have tried to explain you now.

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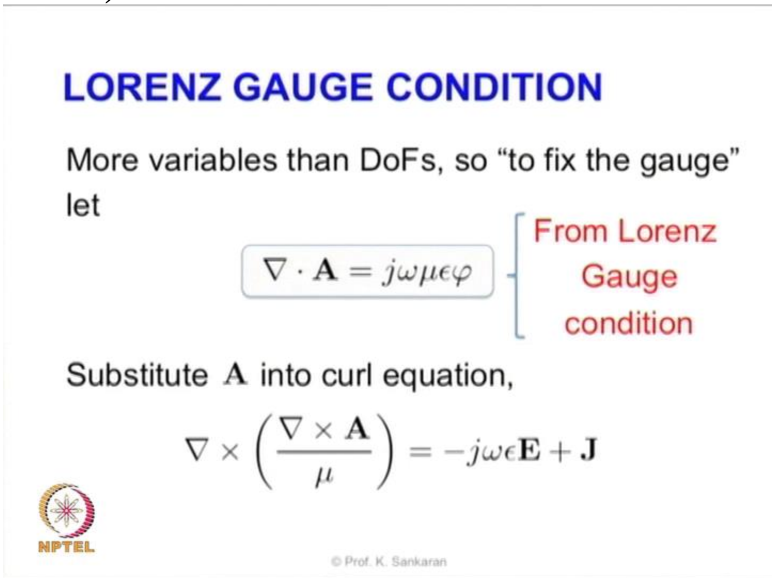

So let us go into the method itself with the Lorentz Gauge. We are going to substitute the value of A into the curl equation. So what you have got is the curl of A divided by Mu is equal to minus j omega Epsilon E plus j.

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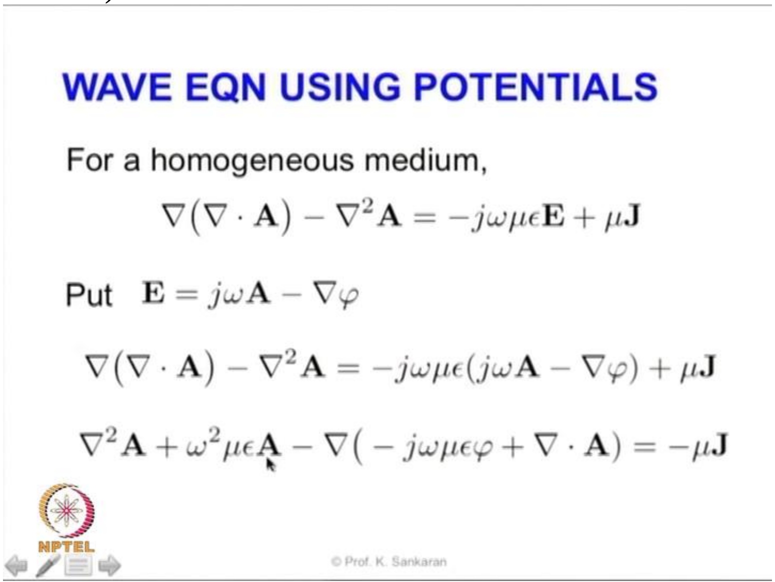

And for homogeneous equation what we have got is putting e equal to j omega A minus the gradient of Phi. You will get the value accordingly, so what I am going to do is I am going to substitute the value for E here and get an expression for curl of curl of A. This is curl of curl of A, I am expanding curl of curl of A using the vector identity which is the gradient of diversions of A minus Laplatian of A is equal to minus j omega Mu Epsilon multiplied by (j omega A minus gradient of Phi) which is the value of e plus Mu J . So rearranging the terms what I have got is an expression for the laplatian of A plus omega square Mu epsilon A minus I am taking out the gradient term outside like I did before. That is equal to minus MuJ.

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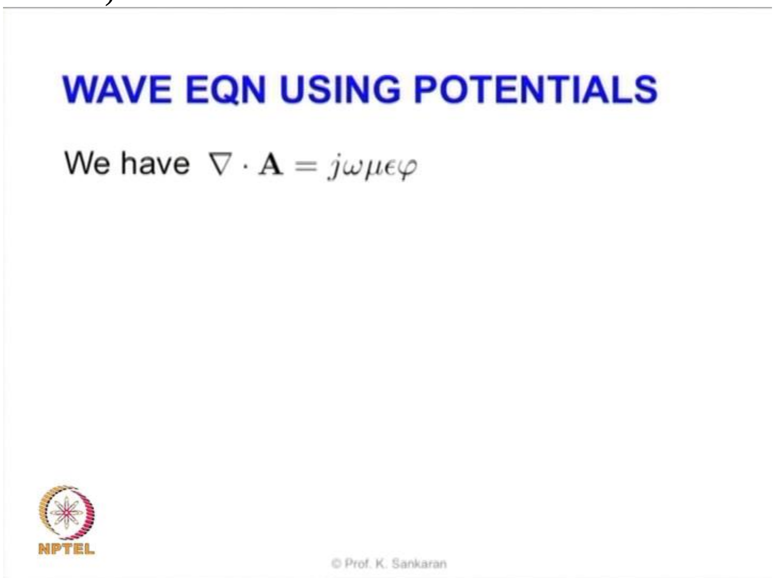

So we have the Lorenz Gauge here.

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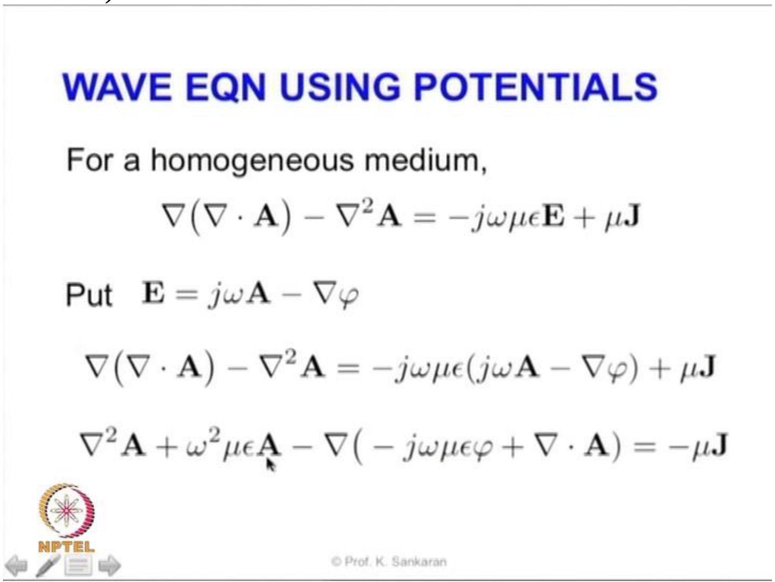

So essentially what I am going to do is I am going to substitute the value for this Lorenz Gauge and when I substitute this, this particular term will disappear because we will get a value minus j omega Mu Epsilon Phi plus j omega Mu epsilon Phi, so this term will disappear, what I will get is a Laplatian of A plus omega square Mu Epsilon A is equal to minus Mu J.

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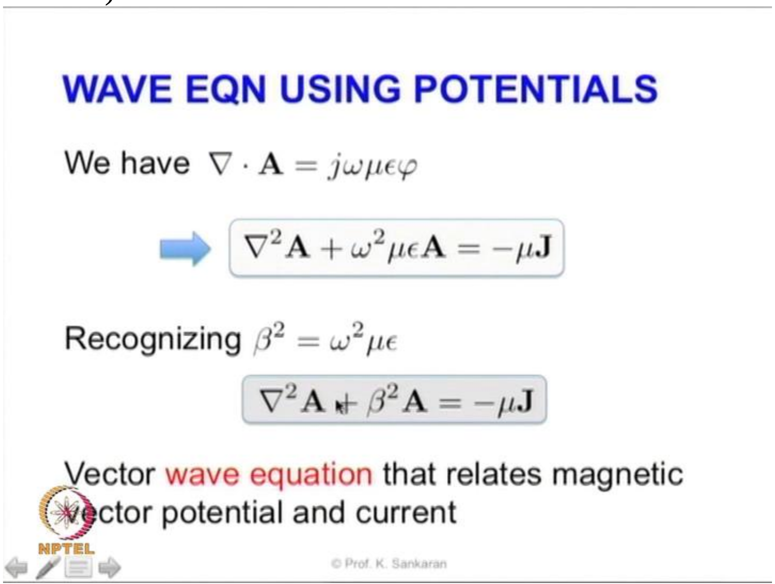

And that is what I have written here. And this is a wave equation using potentials so what I have used here is directly the magnetic vector potential and you can substitute the value omega square Mu Epsilon is equal to beta square. Once you do that what you get is the same equation written in terms of Beta and this is the vector wave equation that relates magnetic vector potential to the current. You have seen a similar equation related to the electric scalar potential but this is the magnetic scalar potential counterpart.

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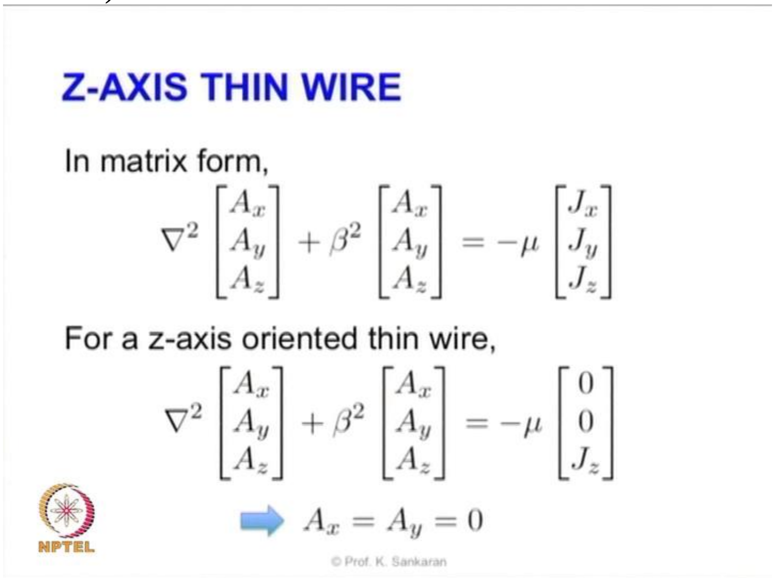

And in matrix form when A is a vector with Ax Ay and Az as its component similarly J is a vector using Jx Jy and Jz as its component we can write this in the matrix form. For a simple case where we have a z oriented antenna where the antenna is approximated as a thin wire you can assume that the current density is only Jz and Jx and Jy are 0. What you will get is an expression for the thin wire antenna as follows.

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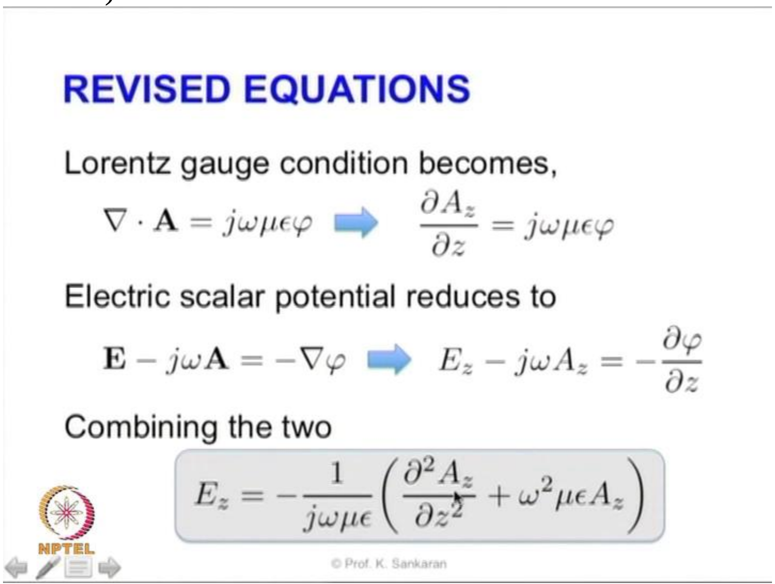

This expression essentially demands our Ax and Ay to become 0. And using the Lorenz Gauge what we get is basically the formulation that can be written of this form and the Electric Scalar Potential reduces to Ez minus jomega Az equal to minus dPhi by dz. There is only variation in the z direction and combining both this equation and this equation you essentially get the expression for Ez using only the magnetic vector potential. And that is the reason why we started with the magnetic vector potential you wanted to compute the value of the electric field as a function of the magnetic vector potential.

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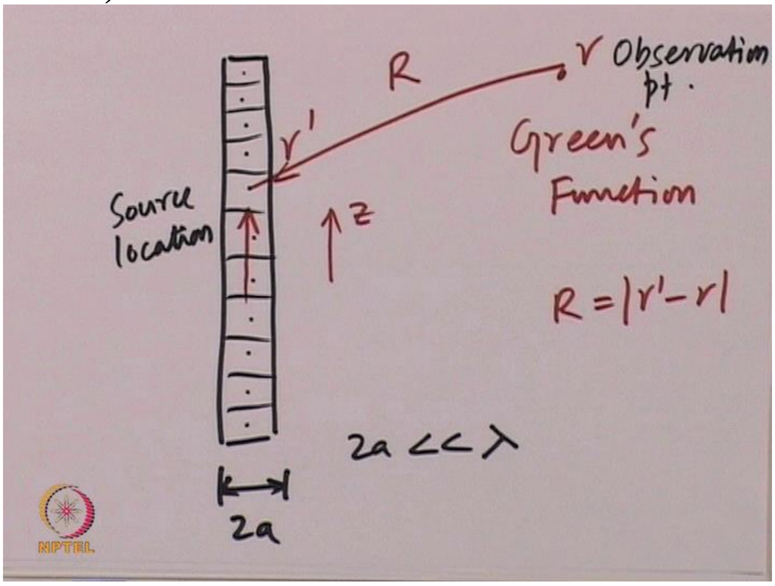

And the idea now is suppose that you have a simple antenna which is a very very thin wire antenna. And I am going to zoom it to show its cross section. The cross section here is let us say 2A and this 2A is very very very small compared to the wavelength. And now suppose that this antenna is oriented in the z axis so this is the z direction we can assume this antenna is decomposed into many small identical elements so let us say we are splitting it into small small elements. And inside these elements there are individual sources that we are going to talk about.

So what we are interested is we are going to find the value of a field at a point let us say r due to the location of some charges at r prime. So in other words what we are looking at is the response of certain field due to the source at certain location is represented using a function called as the Green's function. So Green's function is nothing but a response function which we are going to compute due to certain source at certain location. And assuming that the distance the vector r its magnitude is given by r prime minus r and we are going to compute the value of the field at this observation point. So this is the observation point and this is the source location. So what we are going to do now is basically compute the value of the field at this point due to this one.

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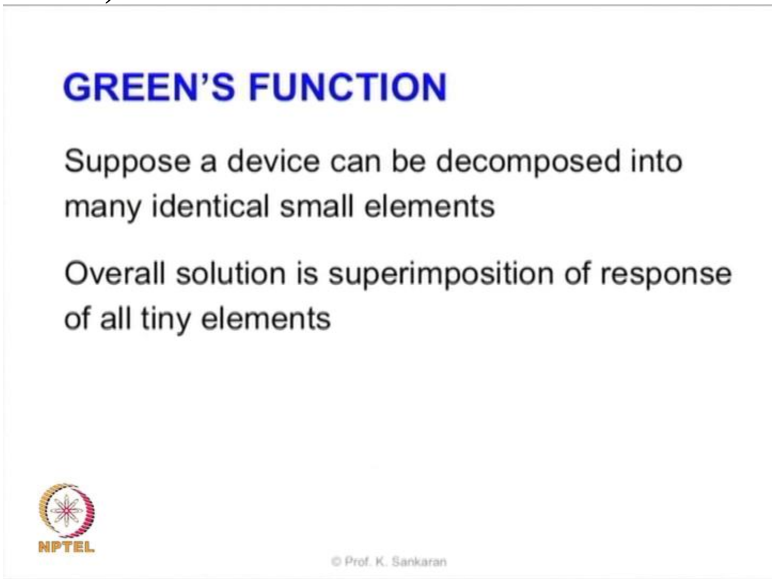

Essentially this is the idea behind the Green's function. And what we will do is the overall solution due to this entire antenna is going to be

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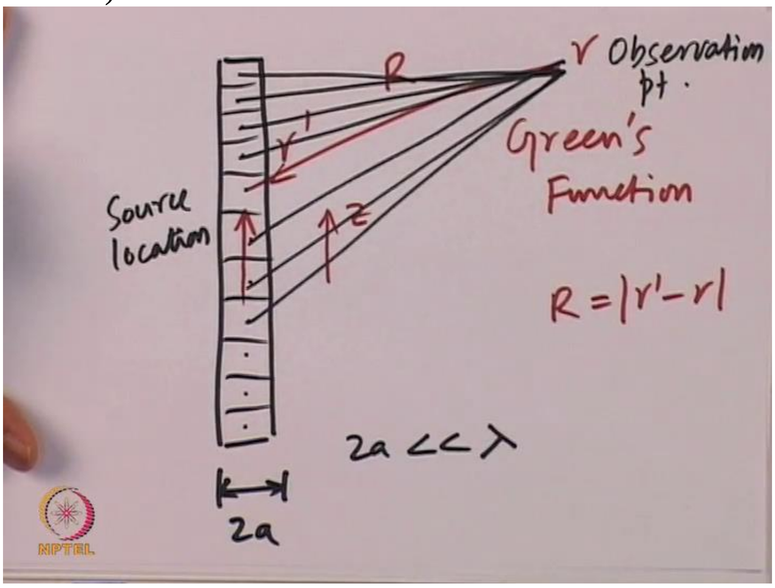

The superposition of this individual elements. So what I will do is basically compute at this point due to the super position of all these points. So you get the idea so we will use this principle to compute the value of the response the overall response.

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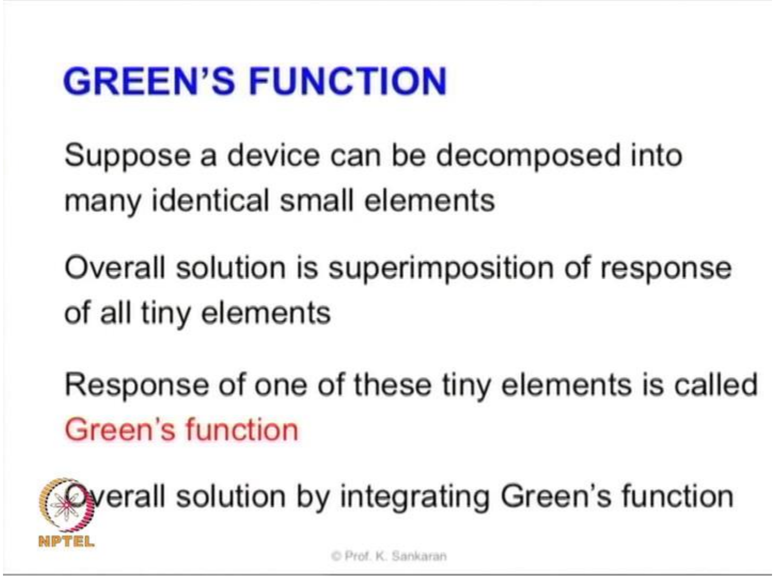

And the individual response of the tiny elements are the Green's function and the overall solution is going to be the integration of the Green's function itself.

So at this point we will stop and in the next module we will come and compute the value of the Green's function for this test case and take it from there on until then Good Bye!