Computational Electromagnetics and Applications
Professor Krish Sankaran
Indian Institute of Technology Bombay
Lecture No. 21
Finite Element Method -I

We are going to do some mathematics now. But I assure you that I will make it simple so that you can follow step by step for a problem that is very common in electromagnetics which is Poisson Equation.

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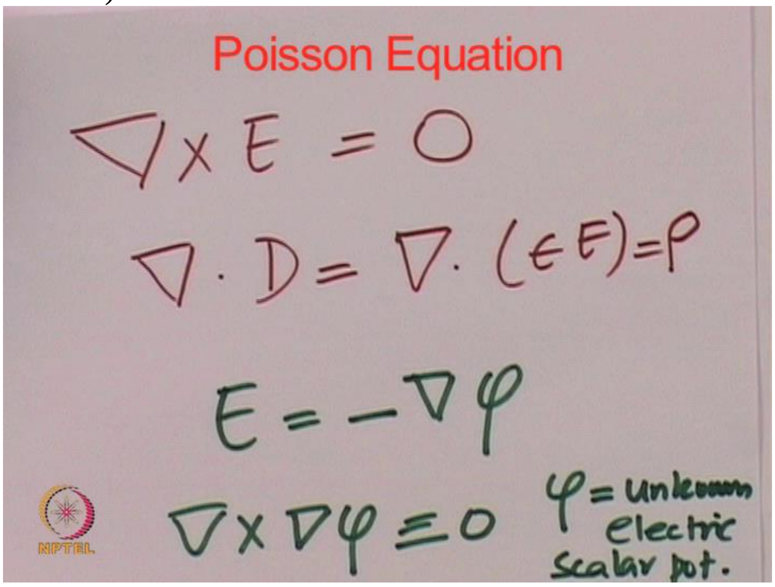

So when I say Poisson equation we are going to do this in 2D to make it little bit more interesting. We have been working only on one dimensional finite elements.

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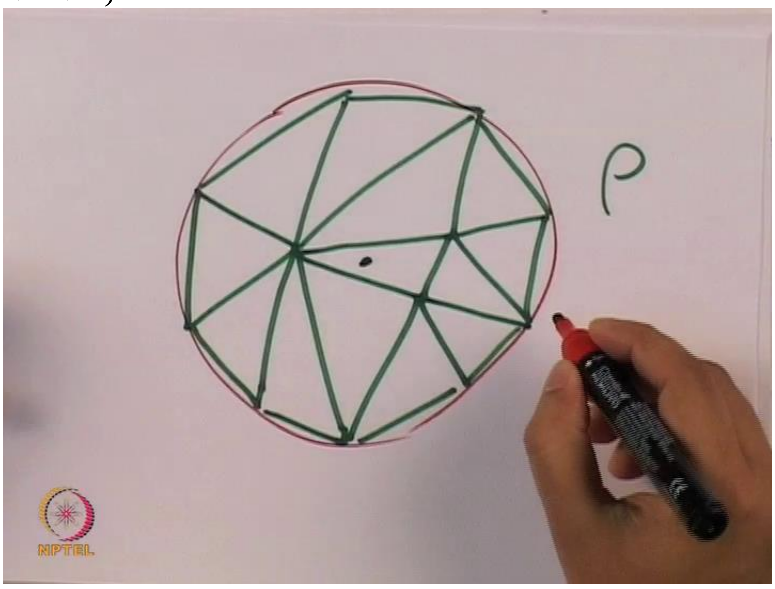

So let us look at a 2 Dimensional case. So when I say 2 Dimensional case assume that my domain is going to be a circulate domain. And it is going to be descritize with some triangles.

For the sake of simplicity I am making it really big. Something of this sort. So then the entire domain is going to be descriptive. So now for this domain I am going to solve the Poisson equation and assume that my Rho the volume charge density, here in this case will be the surface charge density in a 2 dimensional case is going to be at the centre of this domain.

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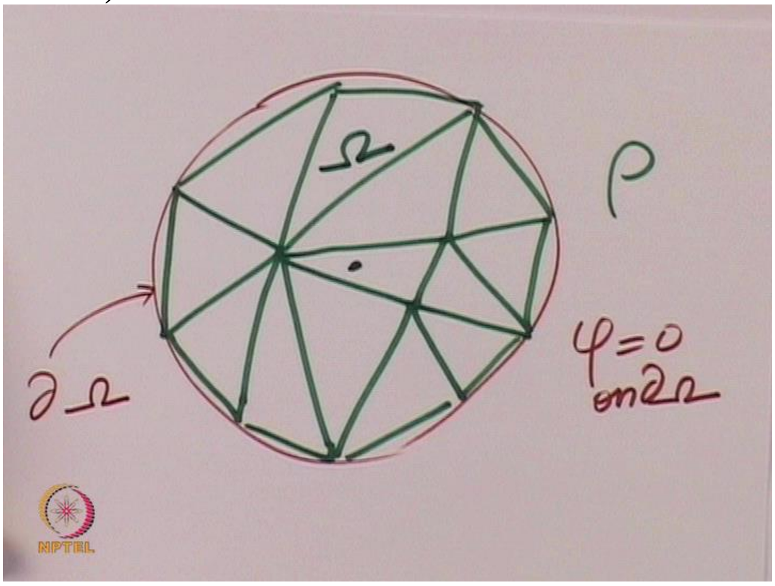

And what I am going to do is? I am going to force the value of the potential to be 0 on the boundary of the domain. So my domain is going to be the value omega and this is going to be my Doe of omega and the potential is going to be 0 in all these places so this is the starting point. So this starting point I am going to take you step by step how to solve this problem? I have given you the domain geometry but let us look into the governing equation itself. So what we will have is? A set of equations given by the Maxwell equation itself.

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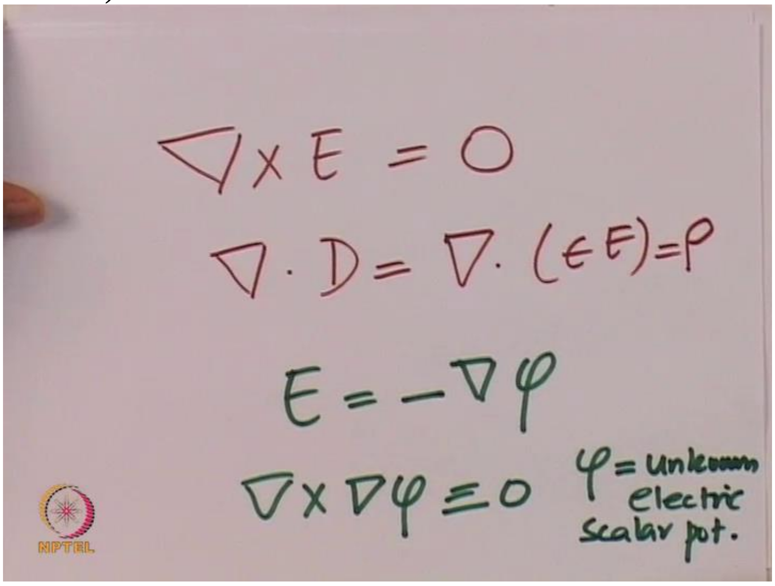

So what we will have is the curl of E is equal to 0. And the diversion of D which is equal to Diversions of (Epsilon E) is equal to Rho. So for us the first equation can be satisfied by representing the electric field as let us say E is equal to minus gradient of the potential Phi. Because we have the vector identity the curl of the gradient is equal to 0. And now Phi is the unknown that we have to calculate unknown electric scalar potential. Ok! So with this we are started getting to the Poisson equation and now we have to also additionally give you certain boundary conditions. So let us write the boundary conditions as follows:

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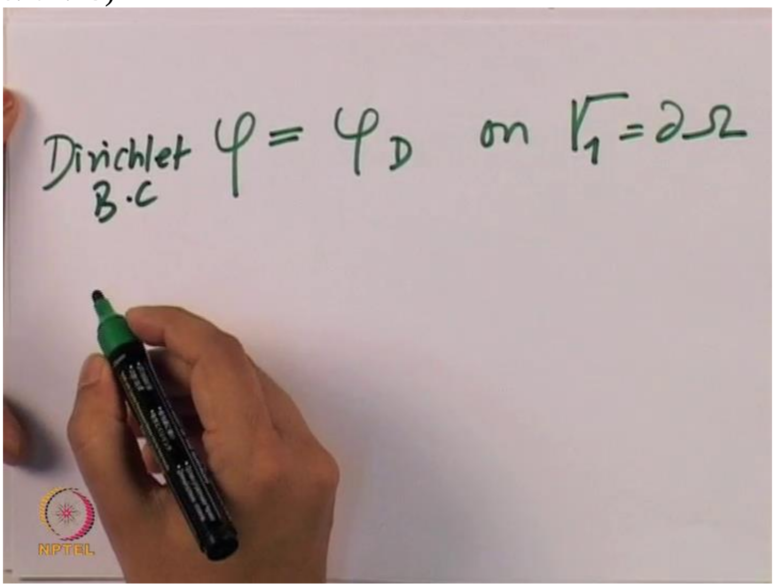

So we have two sets of boundary conditions. One is we say the potential is going to be equal to D on certain boundary elements which we have represented as square root equal to Doe omega for certain gamma 1 of this particular boundary element. Let us say for certain parts we say it is going to be equal to this value. We can also say in a different way this is a Dirichlet boundary condition.

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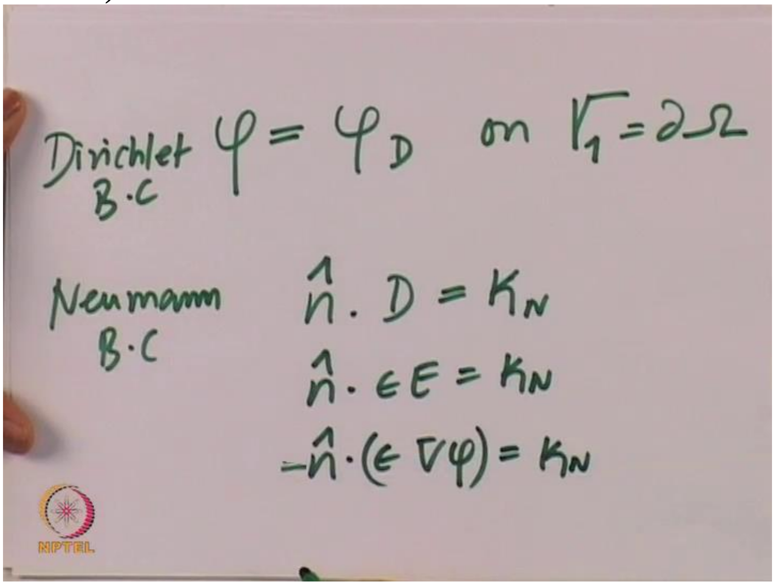

We can also have the Neumann boundary condition which is going to be given by the normal component of the electric displacement D is equal to certain value let us say Kappa N. So this is we are defining the normal component of D and we know D is given by Epsilon E. So this is equal to n Epsilon E equal to Kappa N and E can be written as minus of the gradient of the value that we have written. So basically what we have is n E the gradient should be equal to kappa N.

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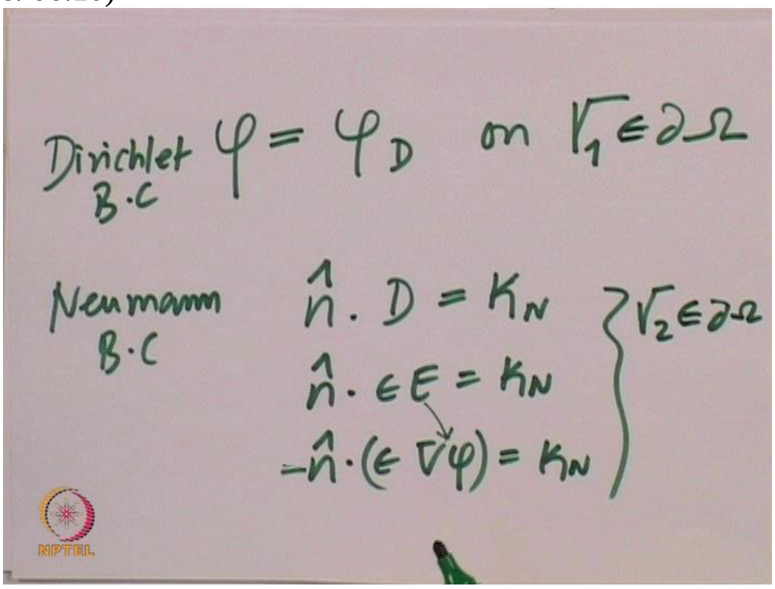

So I am substituting for this value here and epsilon is the value of the permittivity, so we are going to call this as for certain domain, 2 that belongs to this is a symbol for belonging to so let us say this is equal to certain part of the domain value. So what we have done so far is basically assigning the value to the boundaries based on certain boundary conditions. For simplicity in our case we can also say the value of Phi equal to 0 at the boundaries when we do the simulation we will use that condition. But this is a more general way of looking at the problem.

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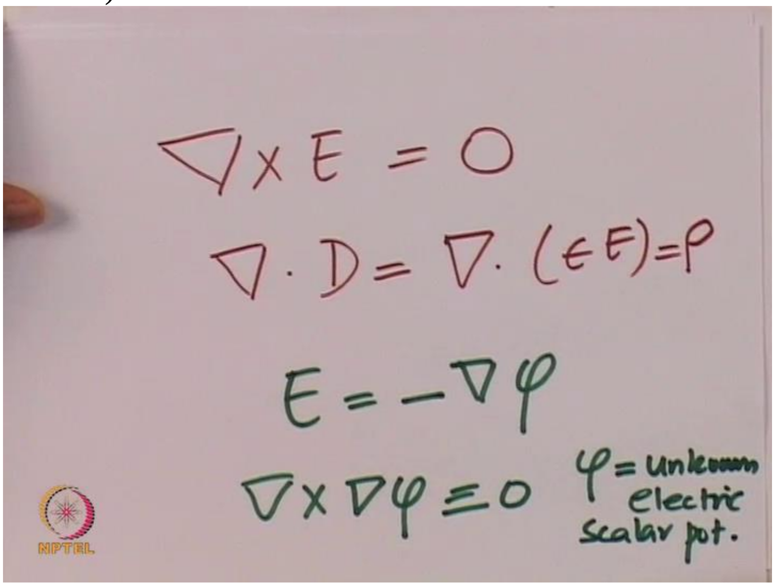

The problem is defined we have got the governing equations they are given by this equation here.

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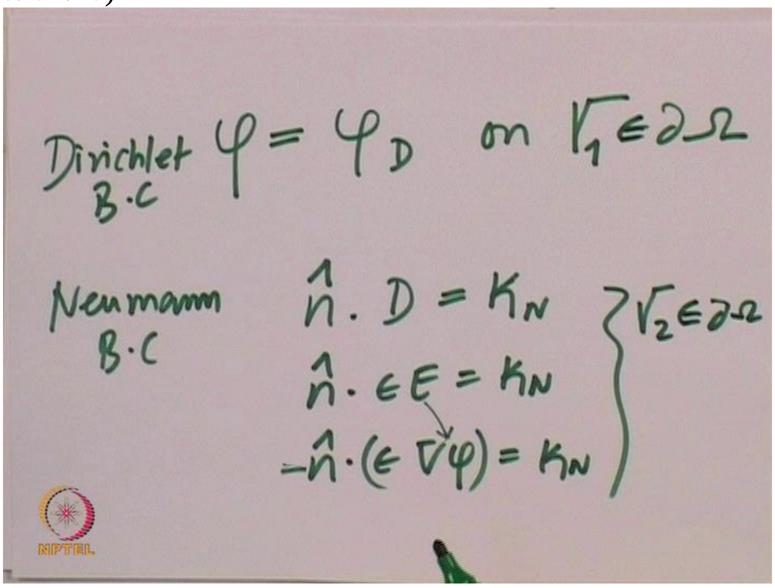

We have now got also the boundary conditions which are given by this equation. Now we are setting place to run into the finite element formulation based on the theory that we have learnt. Don’t be afraid we are going to go step by step.

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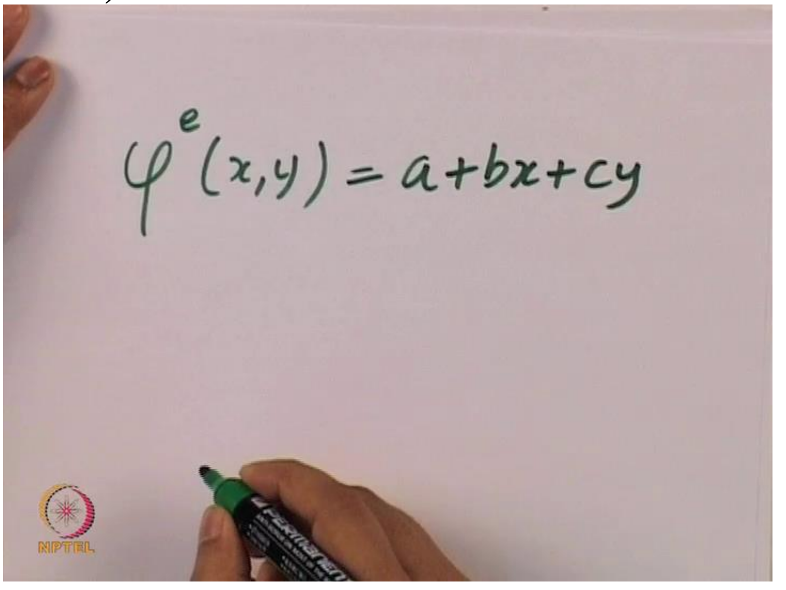

So what we will do now is we will say the value of Phi potential inside a domain. So we say now Phi inside an element e is going to be a function of x, y where x and y are the x and y coordinates we are in a 2 dimensional domain. So it is going to be a function of x, y and I am going to say it is going to be equal to some value constant a plus bx plus cy.

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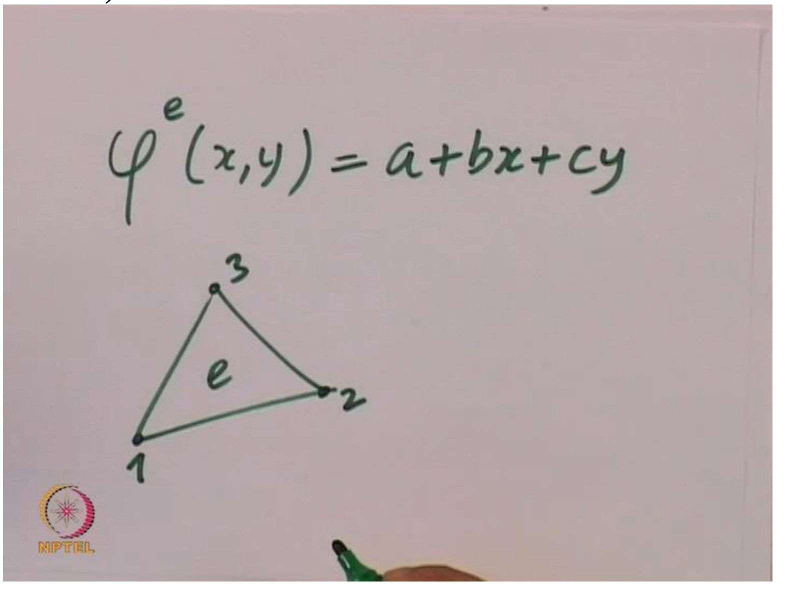

So what I have done is inside each of the triangular element the value of the potential is going to be based on certain number so this is e this element will have node 1, node 2 and node 3. And I am saying we are going to have linear triangular elements. So we will have at the point within a particular triangle we will have point 1 which is first degree of freedom, second degree of freedom, and third degree of freedom. So we can write the basis functions as the function of the three nodal values.

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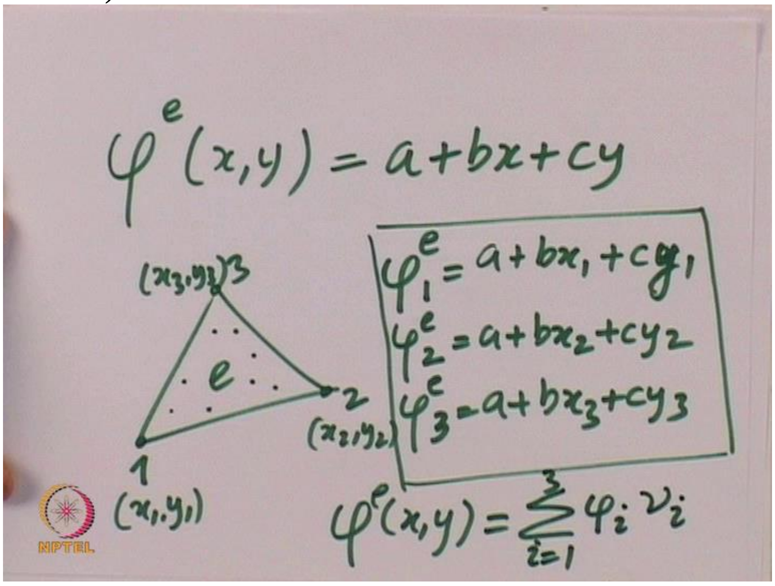

So what we are going to have is Phi e 1 is equal to a plus bx 1plus cy 1. And similarly we will have Phi e 2 equal to a plus bx 2 plus cy 2 and Phi e 3 is equal to a plus bx 3 plus cy 3, where the value (x 1 y 1), (x 2 y 2) and (x 3 y 3) are the coordinate points of these particular nodes for the particular element e. So now we can write the value of the entire thing what we have written here.

The potential value inside the triangle at each of the nodes as a function of certain constant a, b and c. What I am going to do now is I am going to write the entire thing as a function of a weight multiplied by a basis function. So inside the element I can find any point. The any point inside the element the value can be written as certain element x, y is equal to I am going to write it as some weighted sum of sigma i goes from 1 to 3 Phi i which is 1 to 3 multiplied by certain basis function v i. So this is what I am going to do I am going to expand this further.

So stay with me it is going to be interesting if you follow it one by one you are going to get the logic behind it. And when you look into the code later on or into any professional software it will be easy for you to understand the physics behind it. So this is the way we are going to do.

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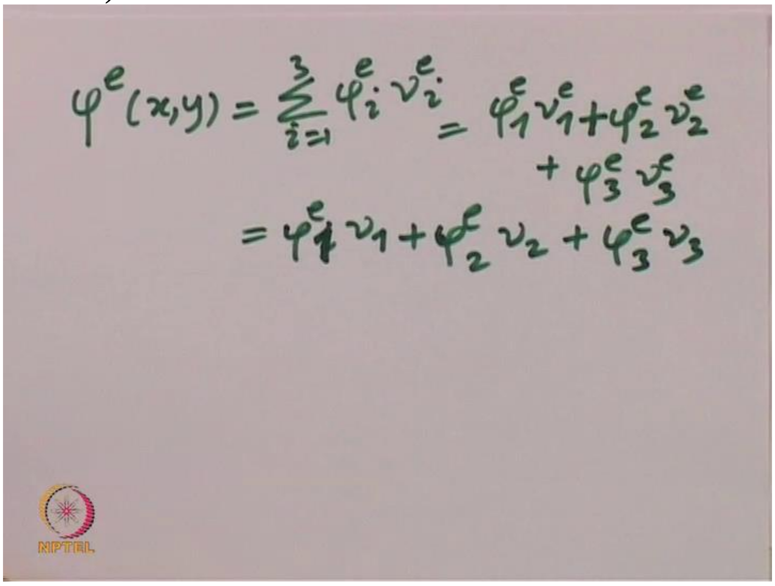

So what I did is I set Phi e (x, y)equal to inside a particular node any point can be written as sigma i equal to 1 to 3Phi e i v e i. So I can expand this pretty much as this is equal to Phi e 1 v e 1, so this is the first element so you can put as element as e plus Phi e 2 v e 2. So if it is going to be specific for the element we can always put the element number on the top plus Phi e 3 v e 3. So since the basis function is going to be same for all the elements I can probably write it simply as Phi e 1 v 1 plus Phi e 2 v 2 plus Phi e 3 plus v 3, I have removed the superscript on the basis functions because it is going to make things easier.

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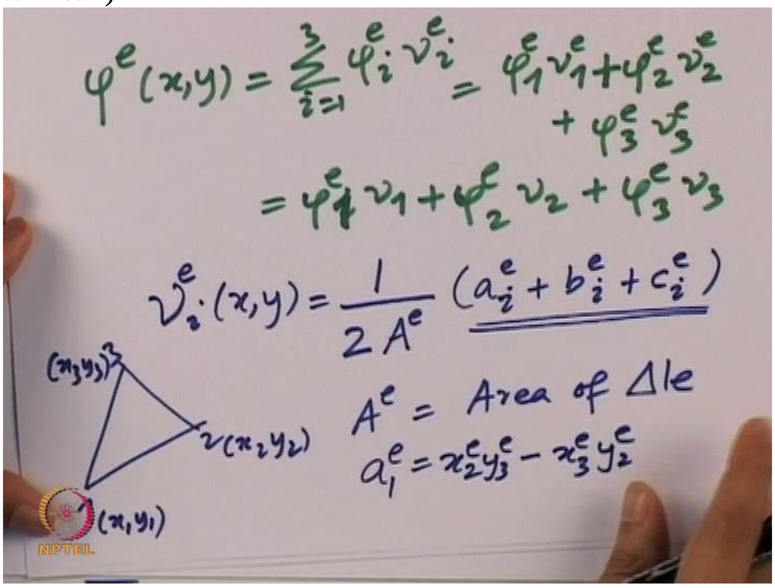

So now the value of those basis functions let us say V e i(x, y) is equal to is going to depend on the geometry itself. So I said 1,2,3 so inside the triangle the basis function is going to have a form 1 by 2 multiplied by A e multiplied by (a i e plus b i e plus c i e). So let us say I keep the e here. So this is the way it looks. So where A e is equal to the area of the triangle. So what we have done is we have assigned the value of the individual coordinates and this individual coordinates are going to give us uniquely the value of area and the various other parameters. So the basis functions value is fixed once the geometry is fixed. So that is what we have done here. So let us take this further.

So we have got the value of a e i, b e i, c e i. The question is what are these values? I said the moment the coordinates are fixed the basis functions are fixed but we do not know how this to compute this a e i, b e i, c e i. They are nothing but depends on x, so this is (x 1, y 1), (x 2, y 2) (x 3 y 3). So inside a triangle there are going to be dependent on the coordinates. So inside a triangle for example a 1 e is equal to x 2 e y 3 e minus x 3 e y 2 e. So I can take out all the e's out. So I can going to write down simply in terms of a 1 is equal to whatever coordinate system. So that will make it look easier. So we have run for a 1 so we will write it for a 2.

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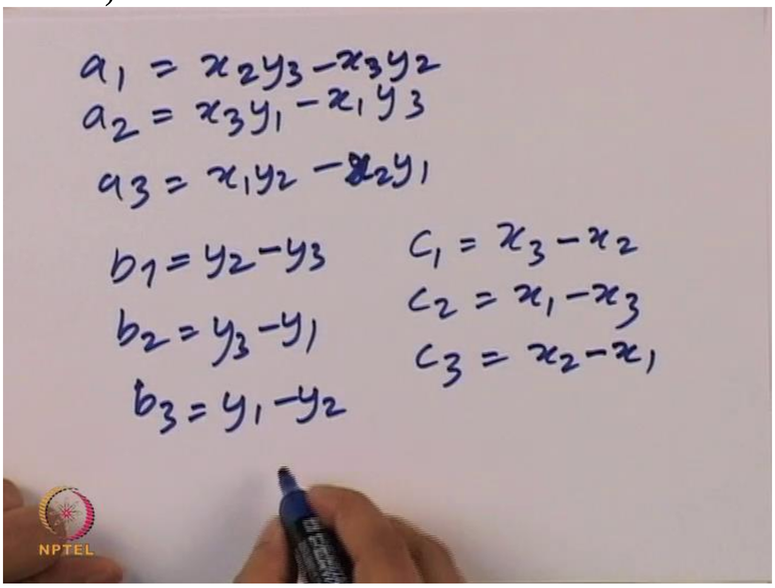

So a 2 is equal to x 3 y 1 minus x 1 y 3 , a 3 is equal to x 1 y 2 minus x 2 y 1. Similarly we will have 3 equations for b and c. So b is going to be b 1is equal to y 2 minus y 3, b 2 is equal to y 3 minus y 1 and b 3 is equal to y 1 minus y 2. So you see there is some sense of symmetry here. When there is 1 here 1 is missing at the right hand side, when there is 2 ; 2 is missing at the right hand side so on and so forth. So similarly c 1 is equal to x 3 minus x 2, c 2 is equal to x 1 minus x 3 and c 3 is equal to x 2 minus x 1.

So let me add it for a 1 in one sheet so it will be easy for you to get a grasp of it, a 1 equal to x 2 y3 minus x 3 y 2.

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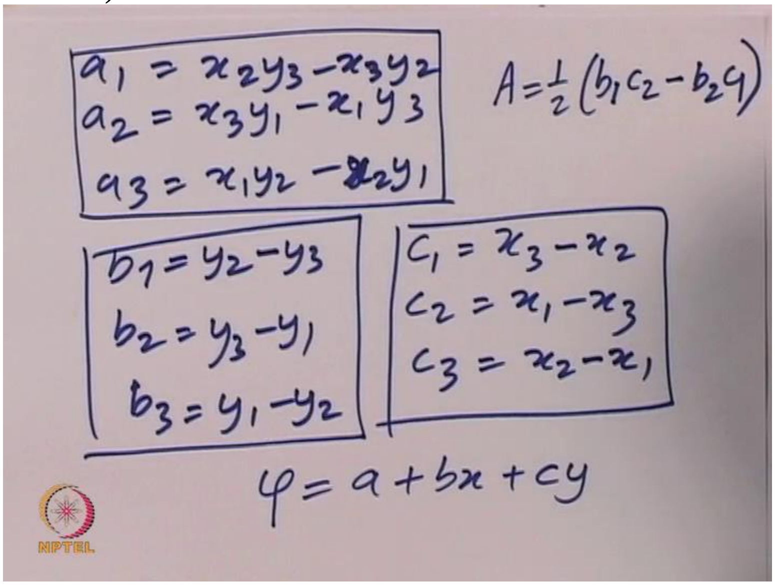

So this is based on our assumption that the function is going to have a value Phi equal to a plus bx plus cy. So each of the terms will have the dimension that is a multiplication of x and y, so a is a multiplication of x and y, b is a multiplication of x and ys and c is a multiplication of x and ys. So that way you can be sure that this is correct. And we can also write, so this is for a this is for the b this is for the c. We can also write the area is equal to 1 by 2 multiplied by (b 1 c 2minus b 2 c 1). So this is the way we compute the area of the triangle or area of the element.

So in a nut shell what we have done is the moment our coordinate systems are fixed our geometry is discretized we know the nodes we know the elements. The value of the basis function is also fixed. If we say we are going for a linear interpolation then we can accordingly find out what will be the value of the basis functions for each of the elements. So that way we are on a safer ground. So we are going to go now one step further.

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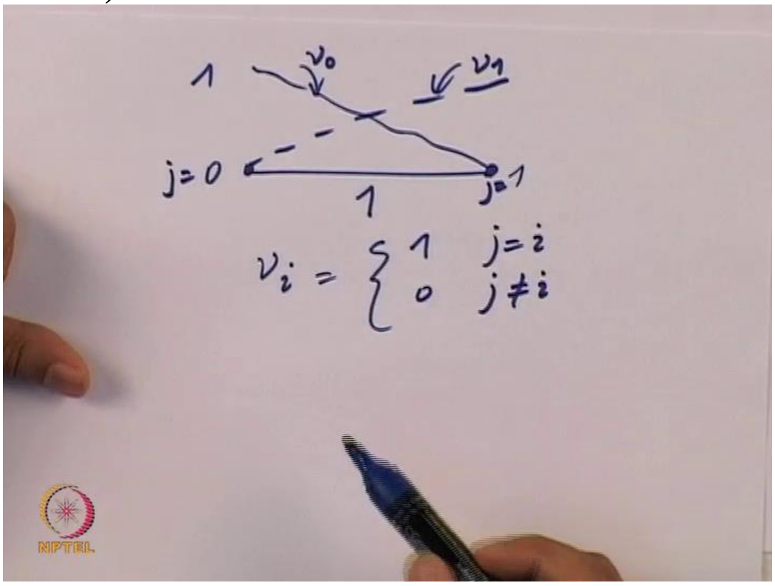

Remember I told you in the case of one dimensional problem that you have a point you have two degrees of freedom and the basis functions are going to go from in the linear manner 0 to 1. So let us say this is element so 0 to 1 so this is element 1. For the element 1 the basis function that will have the main contribution will be this one and this will be the second basis function for the particular node you can decide if this is 0 and you have to assign the value of the basis function particularly for the node for this particular node this will be the main basis function because this has the maximum contribution for this particular node this is the basis function that has the maximum contribution.

So in other words this will be the v 0 and this will be v 1 for the node 0 v 0 will have the maximum value. so you can write as v i is equal to 1 if let us say these are j so these are j equal to 0 j equal to 1 the nodal numbers so equal to 1, when j is equal to i. Otherwise it will be 0 for all when j is not equal to i. So this we can say within a element which of the basis function will have a maximum contribution. In this case what we have is this basis function is v 0 will be equal to 1 for the node when j is the same as the i. Likewise we are going to do the same thing for the two dimensional case.

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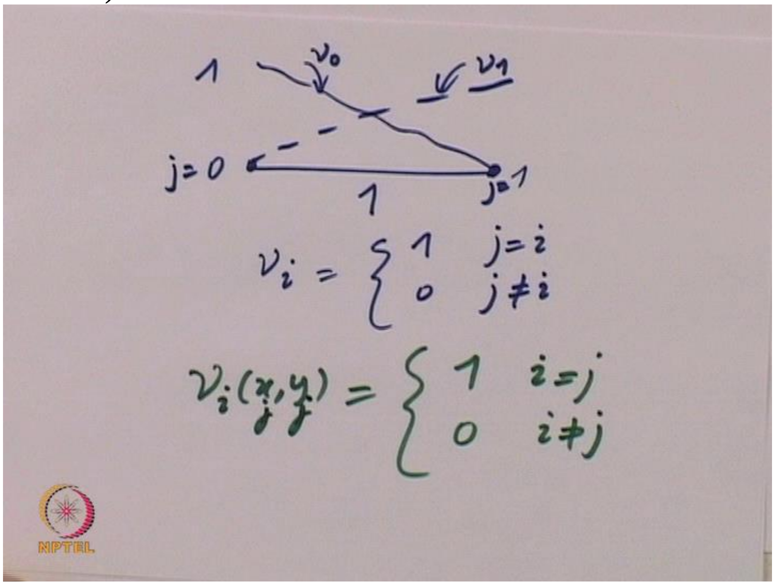

In the two dimensional case our basis function v i x, y is equal to 1 when the value is equal to let us say the coordinate here are j. When the i and j are matching so this is going to be 1 if it is 0 it is going to be when i is not equal to j. So I am at a point when x j, y j these are j and this is i. So which of the basis function will have the maximum contribution can be assigned from this particular property on the basis function.

So with that being said I am going to go forward and we are going to say certain properties about the boundary condition itself.

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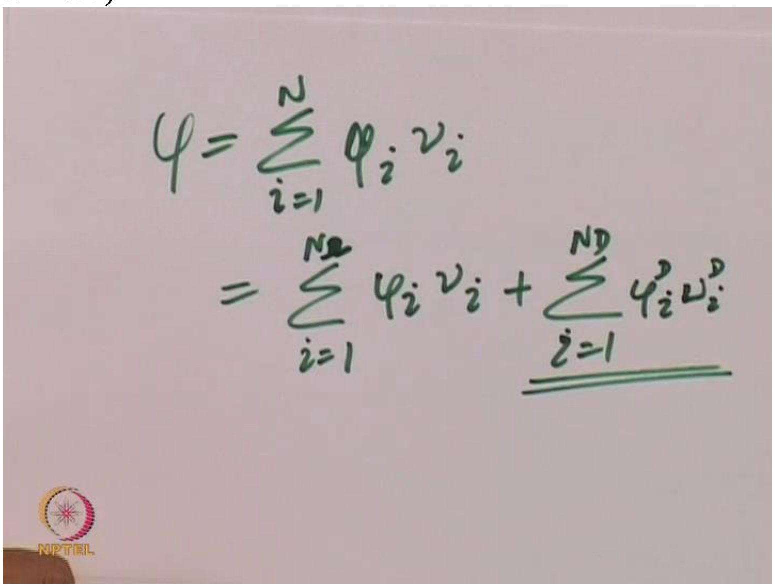

I said Phi is equal to sigma i equal to 1 to N for the entire domain this is going to be Phi i multiplied by v i and v are the basis functions. So this I can split them into the one which is inside the boundary and the one which are on the boundary. So this can be written as sigma so i goes from 1 to let us say N x Phi i v i plus sigma i goes from 1 to N D. So let me use a subscript here N this is the inside the domain and this is on the boundary of the domain which I have written as N D is equal to Phi i on the boundaries v i on the boundaries. So these are the boundary contributions we can separate them in order for us to do some manipulations.

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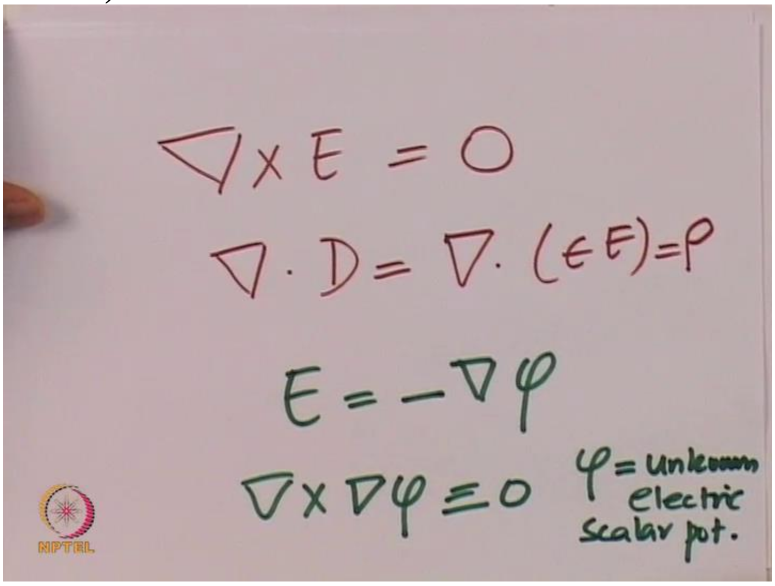

So now what we are going to do is we are going to use this equation into our initial formulation. The initial formulation gave us the value for partial differential equation for which we have to do some calculation. So I am going to substitute some value for Phi using this here and I am going to integrate it over the entire domain. So this is what I am going to do. Let us take it one by one. So what we have got is?

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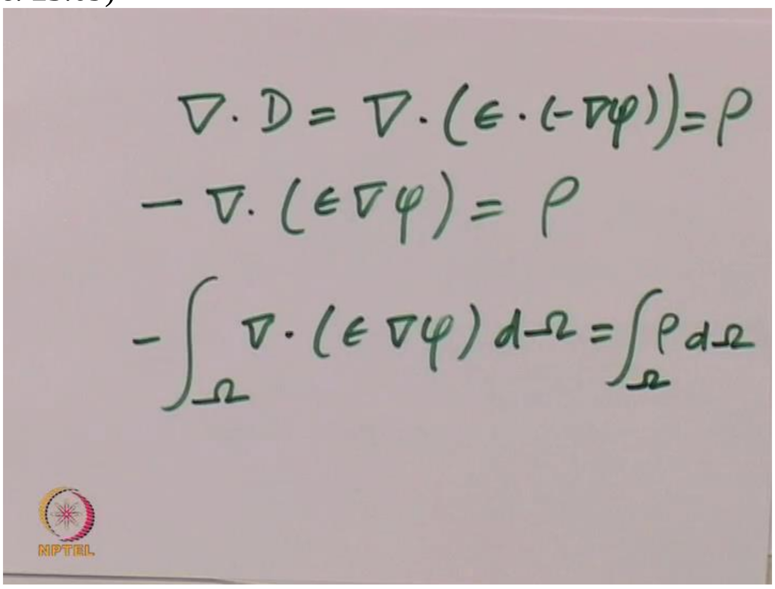

We have got a value which is the diversions of D that is equal to Diversions of Epsilon multiplied by minus gradient of Phi equal to Rho, this we have. So I can take the minus sign out what I will have is minus diversions of (Epsilon gradient of Phi) equal to Rho.

So I am going to integrate it along the volume. So I am going to integrate it inside the domain and I am going to have the equation with a minus sign out (Epsilon gradient of Phi) d omega equal to integral of Rho d omega, so this step is clear. So I said we can do finite element in two ways we are going to use the weighted residual method and then we can also use the Galerkin approach. The Galerkin approach and the weighted residual method only varies by the changing of the weighing function making them equal to the basis function.

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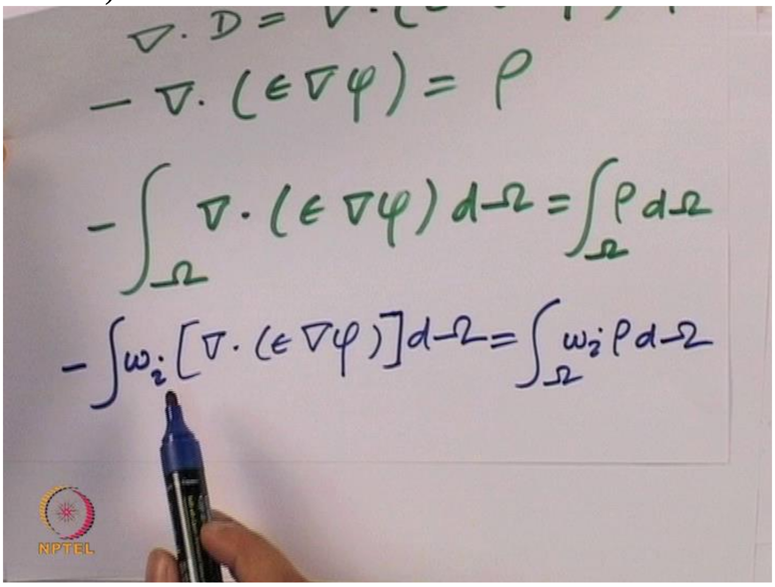

So let us take a simple starting point where we will start with the weighted residual and then we will decide how to go forward. So I am going to multiply this with the weighting function, so what I am going to do is I am going to multiply this integral minus w i it is going to be [diversions of (Epsilon gradient of Phi)] d omega is equal to. So now this is starting point, I am going to use the vector identity because here I have got diversions and the value of the Del is on the potential itself. I really want to move these diversions on the Phi to the weighted function. So that is why I am doing some manipulation.

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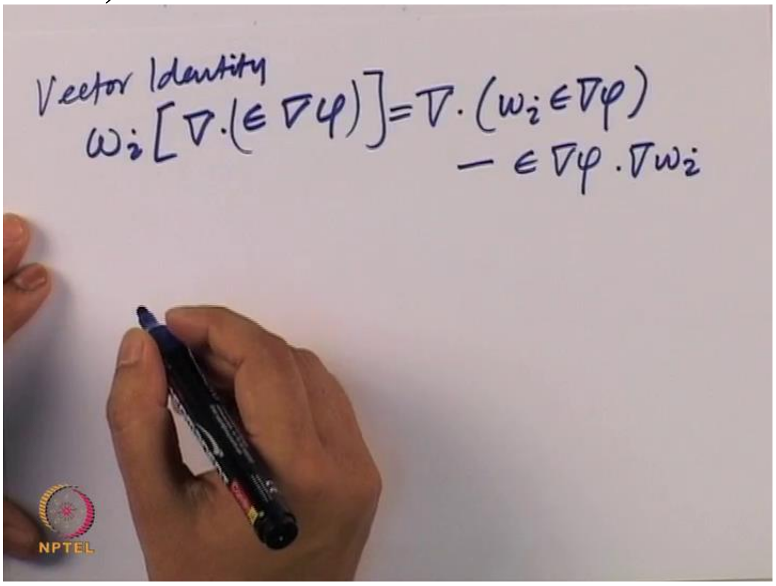

What I am going to do is I am going to use the identity w i [curl of (Epsilon gradient of Phi)] equal to curl of (w i Epsilon gradient of Phi) minus (Epsilon gradient of Phi) multiplied by curl of w i. So this is the vector identity that I a going to use in this particular equation.

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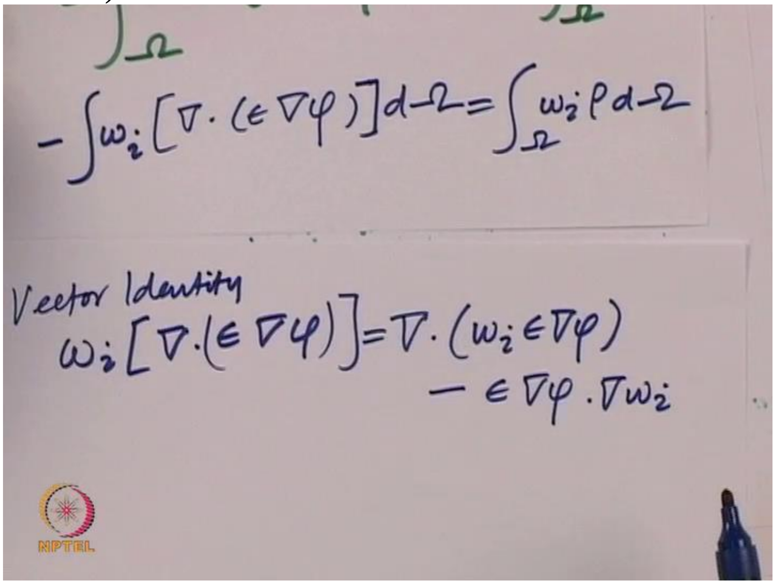

So I am going to substitute for this value here using the one on the right hand side.

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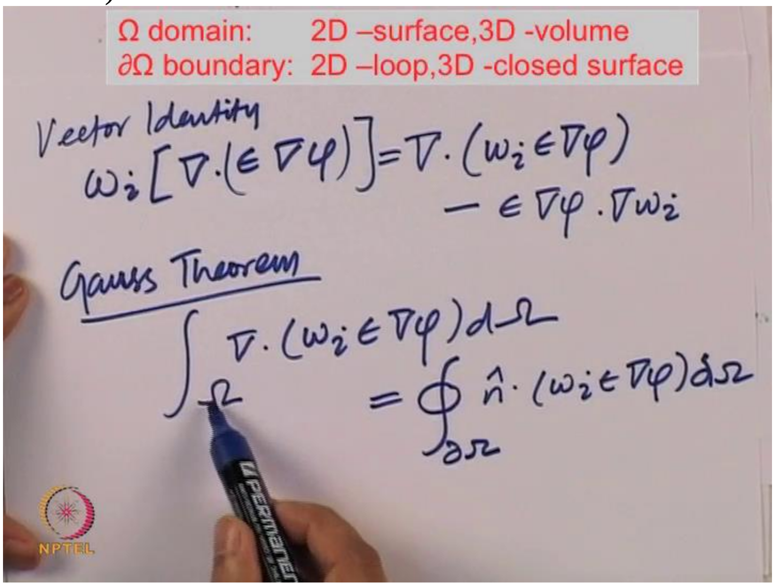

And I can apply the Gauss theorem also. The Gauss theorem is basically saying the surface integral diversions of omega i epsilon phi equal to so when I have a surface integral over that particular surface it is going to be the boundary of the surface which I am going to write it as the normal component of the value of the field that we are going to compute. So this is the Gauss theorem that takes the diversions of a field over a domain is equal to the normal component of that field over the closed boundary of that surface. This particular thing is the volume this will be a closed surface, if this a surface this will be a closed boundary.

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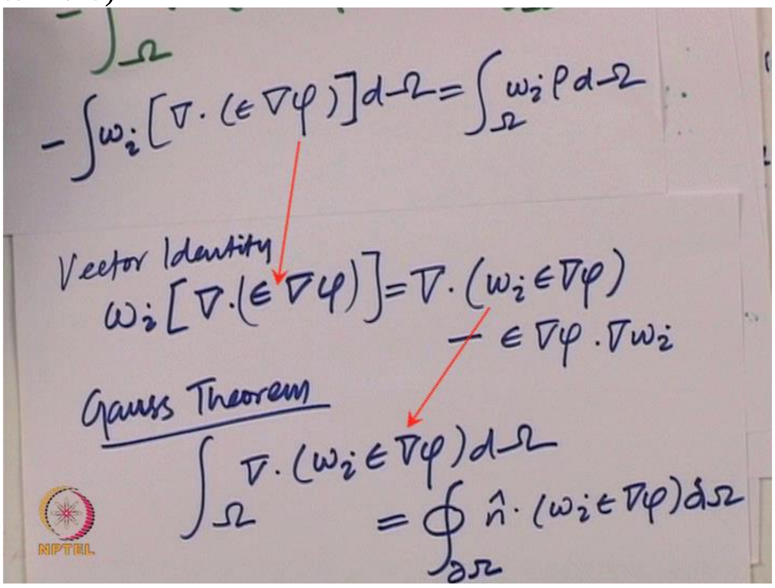

So what we are going to do now is we are going to use these two formulas onto our governing equations to simplify this because we wanted to take out the Del on the Phi outside and then we wanted to put it on the weighting function itself. So this is the manipulation we are going to do let us see how the equation itself is getting transformed.

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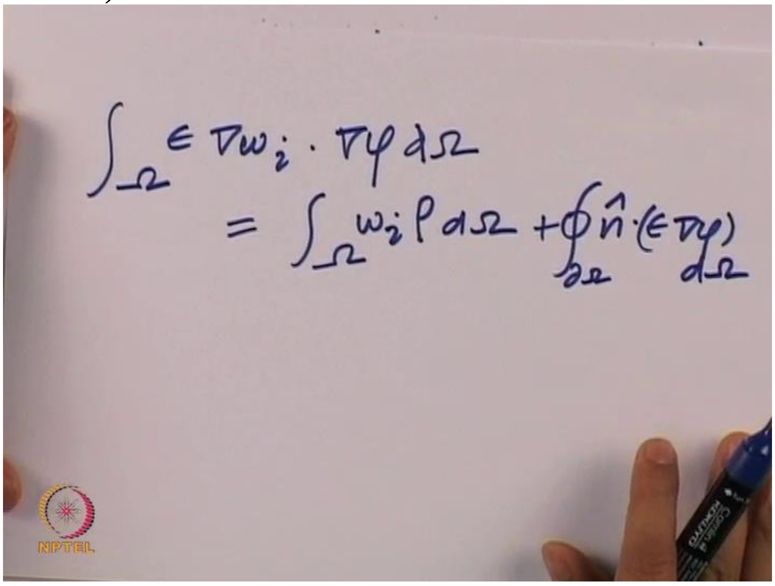

So we have epsilon del omega i dot del of phi is equal to the integration over the surface Rho times thing plus the closed boundary what we have talked about closed boundary the normal component of the same flux field that we are talking about times so what we have done is we have transformed the equation that we have got here we have applied the vector identity to transform this and we have used the Gauss theorem to get to the point where we have the right hand side integral will be represented as the surface integral or the closed boundary integral for that particular equation.

So with that what we can split this boundary into both domain boundary whether it s going to be a Neumann condition or Dirichlet condition. I am not going to go into the detail of that but right now we know that we have given the value of Newmann condition and also the value of the boundary conditions which are given in the initial slides. So they are basically the values that we have taken.

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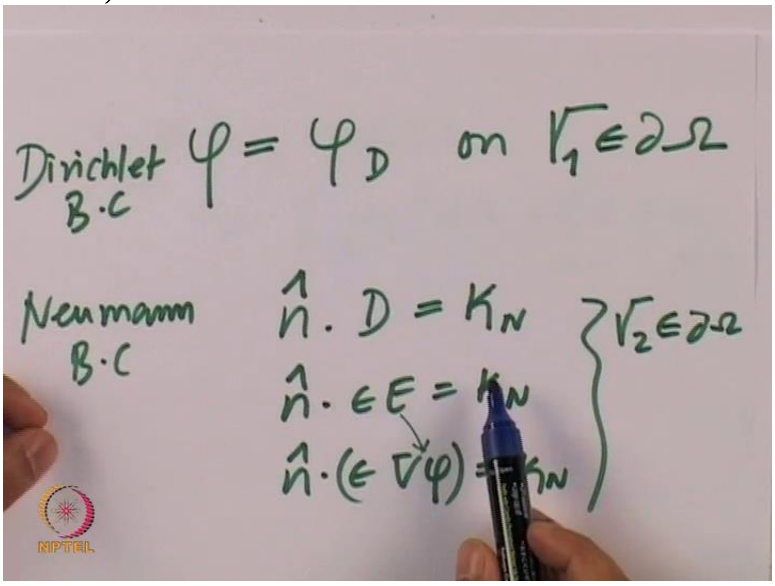

And they are given by the values that are here. The value of Phi on the Dirichlet boundary condition is like this and the Neumann boundary condition is like this. So the Kappa N value is here. So I am going to use the Neumann boundary condition and the Dirichlet boundary condition to reduce this particular formula even further.

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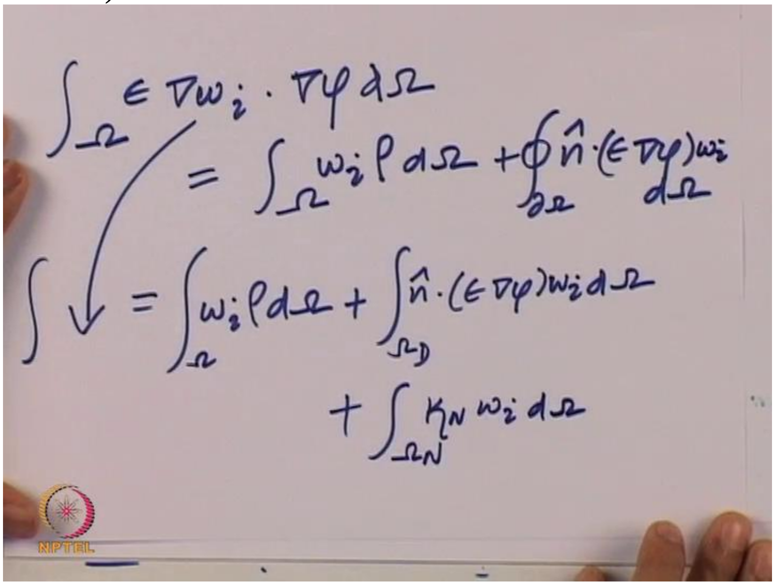

So what I am going to do is I am going to split this into the right hand side will stay the same. So this particular thing will be here is equal to the first part will also stay the same so it will have plus I have one part of the boundary will be let us say the Dirichlet boundary which will be n component the same thing multiplied by (Epsilon Phi) w i doe plus there is an another boundary that is the Neumann boundary which is going to be given by the same equation but the same equation this particular term is going to be equal to kappa N. So this value is something that we got from here we set this value is the value that we are having here and we are going to substitute for this Kappa. So that is what I am going to do here. Kappa N omega i so I forgot a omega here. So with this we have come to a point where basically you write the entire equation in a matrix form.

Right now what we have done is this particular side is still having the values for which the Dirichlet boundary condition has to be added so we have the right hand side which is here so we can write the entire equation in kind of a matrix form by substituting the values for Phi which are written as the expansion coefficients and also for the values of the expansion coefficients which we have already seen. So what I am going to do is I am going to transform this equation further into the matrix form but for that I need to write down in a more manageable form.

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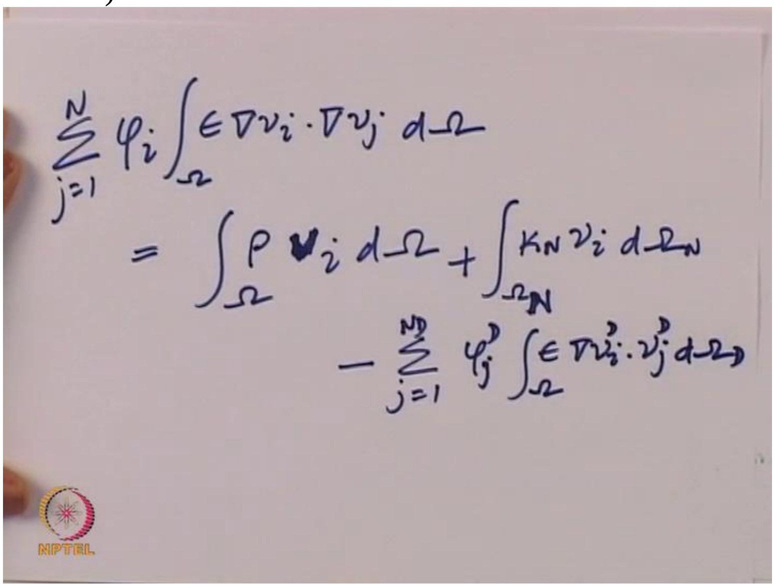

So what I am going to do is this is going to be equal to so j equal to 1 to N Phi of i integral over the domain Epsilon of v i and dot of v j over the domain is equal to integral over the domain the Rho of v i over this plus the Neumann boundary condition which is given by sigma d kappa sorry Neumann is N kappa N v i over the domain n and minus sigma j equal to 1 to Nd the Dirichlet boundary condition which is Phi of j of d integral over the domain Epsilon del of Phi of i Phi of j these are all the Dirichlet weighting functions multiplied by the domain which is the Dirichlet domain.

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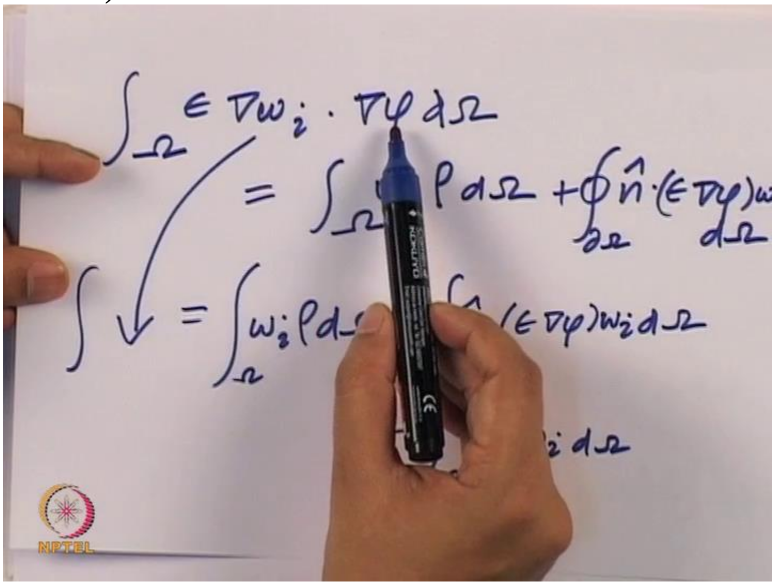

So what we have done now is we have written the earlier equation the equation which is here so it has certain functions I am going to take the values in such a way that I have applied the weighting functions and using the weighting functions. So I have used the value v i instead of w i because I have used the Galerkin approach and by doing that what I have done is I have written the form in a much more easy to manipulate form.

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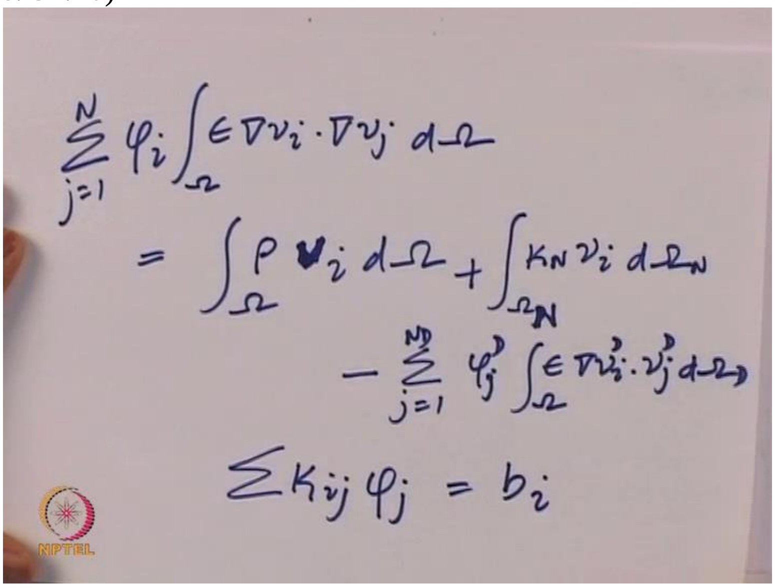

This is nothing but sigma of K of i, j multiplied by Phi of j is equal to b i. So the entire right hand side can be clubbed into one b. So the Phi will be here and the right hand side can be written as k i j and where k i j has to be expanded further that is what we will do now.

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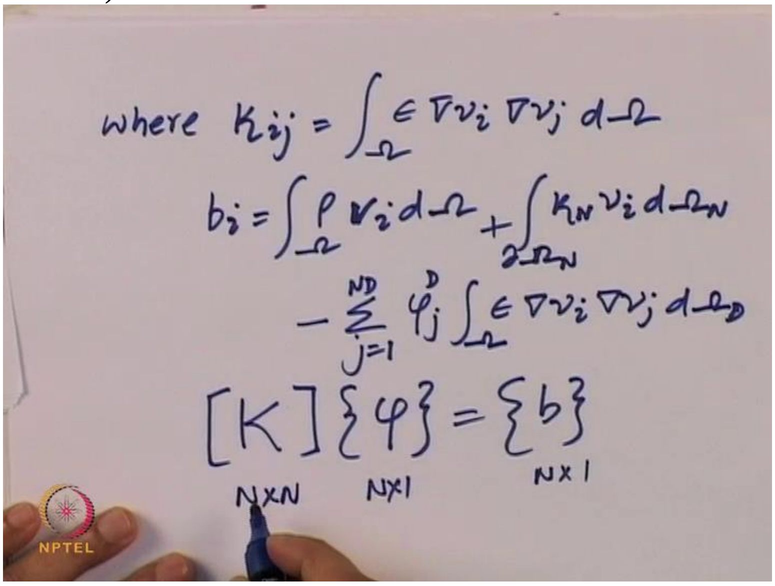

So where k i j is equal to inside the domain will be Epsilon Del v i Del V j over the domain. So this is the way we compute the case and b i is equal to integral over the domain Rho v i into integrated over the domain plus the Neumann condition which is kappa N multiplied by v i integrated over the Neumann boundary minus sigma j equal to 1 to number of points on the Dirichlet boundary multiplied by the potential on the Dirichlet boundary integral multiplied by weighting function which is again the basis function here v j over the domain of Diichlet boundary.

So what we have got is a kind of a expression form so this is nothing but [k] multiplied by the potential vector that we have to compute equal to the b vector, the potential will be a vector number of unknowns will be the number of nodes multiplied by 1 . So it will be let us say the number of unknowns are N so it will be a n,1 matrix and then this will be also N, 1 matrix and you will have the value for k which will be a N,N matrix. Where N is a number of unknowns or number of points in the domain.

So it has been quite a bit of mathematics here but at least you understand how the entire process is done step by step in order for you to get a sense of if you understand mathematics or not we will take it to the next level by showing a small simulation it is giving you a intuitive of understanding how the potentials are distributed around the domain what we have considered so we will stop here we will see you in the next module. Thank you!