Computational Electromagnetics and Applications
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Lecture No. 20
Finite Element Method-I

We have looked into the Finite element formulation itself. We saw that it has various elements various matrices but I told the devil is in the detail. So at this point it is good to start with the detail analysis on how the matrices are filled what are there inside the matrices which will also lead us to looking into the basis functions and the mapping from a local to global basis functions and things of that sort will come into play. So at this point it is good to go into the discussion on the formulation itself.

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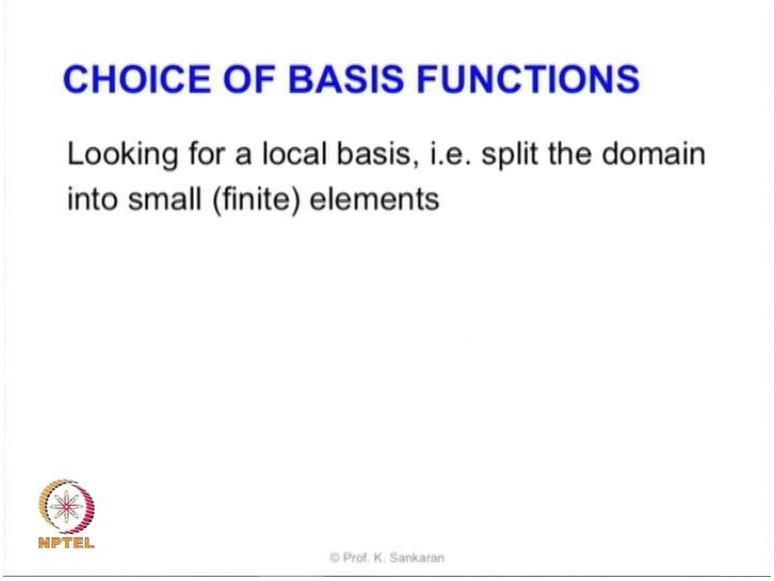

So the first question comes is what is going to be the choice of the basis functions? Because we told we are going to expand the unknown inside each of the finite elements using certain basis functions whose properties we know clearly. So the question comes what should be the choice of that basis function. So let us look at the entire domain, for our analysis let us keep the domain to be a one dimensional base. We are going to define the starting point and ending point which we call it as the boundaries and the intermediate points are going to be discretized.

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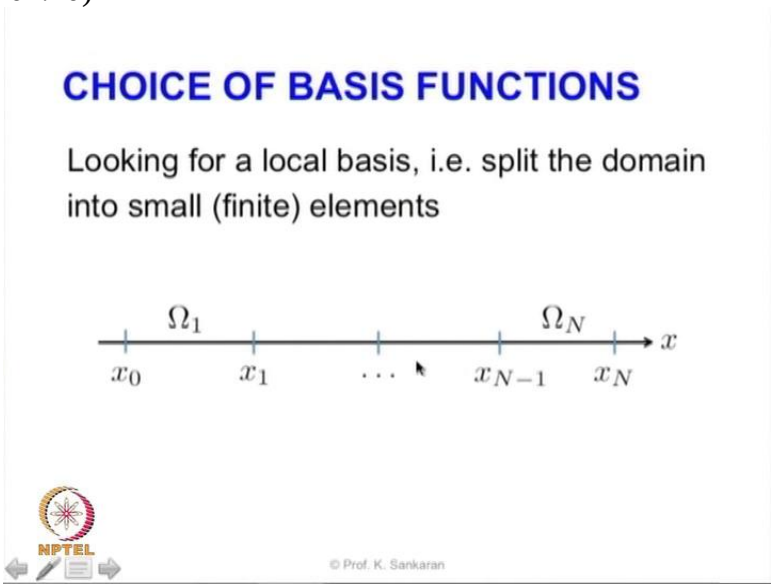

In other words split into finite elements and that is what you see here in this slide. So the starting point and ending point of the domain is fixed s we have x 0 x N and then the intermediate forms are arbitraly broken. It need not be equally split it can also have smaller parts and bigger parts regardless of what is the size of the individual elements. We call them as finite elements and they are defined as sigma 1, sigma 2 until sigma N. As you can see there are totally N plus 1 points. So N plus 1 points in (())(02:15) space gives to N finite elements.

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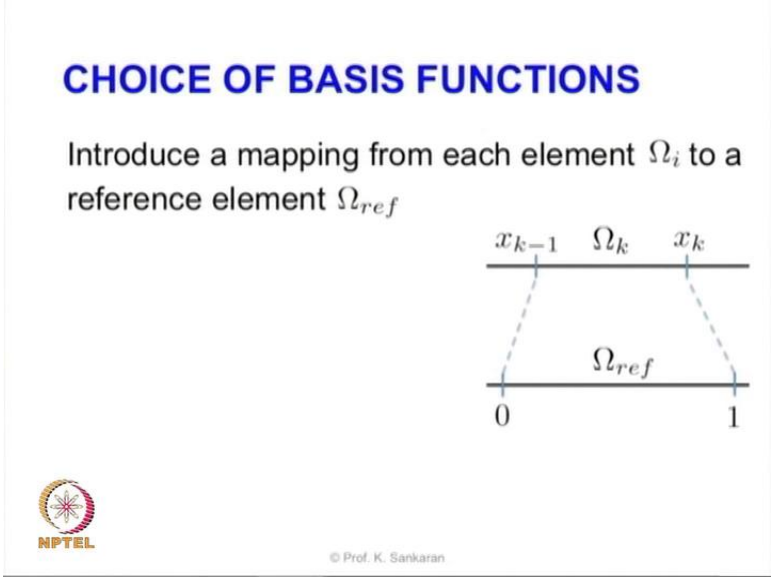

So now we are going to introduce a mapping from each of these elements sigma i to a reference element which we call as sigma ref. What we are essentially doing is a kind of a transformation which goes from local to global elements. So let us see little bit more in the slide here.

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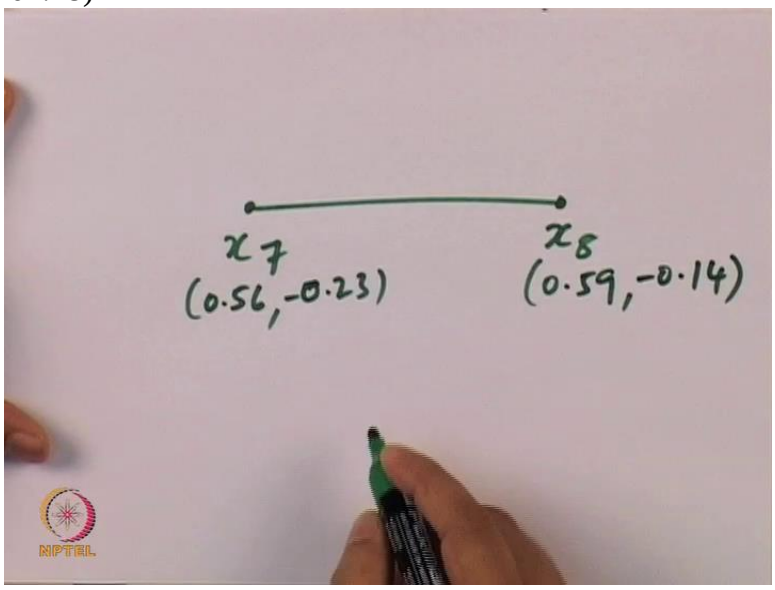

So what we have got is arbitrary number let us say x 7 and x 8. And let us say the coordinates of x 7 and x 8 are some arbitrary number (0.56, minus 0.23) and then let us say x side is (0.59, minus 0.14) so it is quite cumbersome to take each of these individual files and then identify what is going to be the points individual points that we are interested inside the domain.

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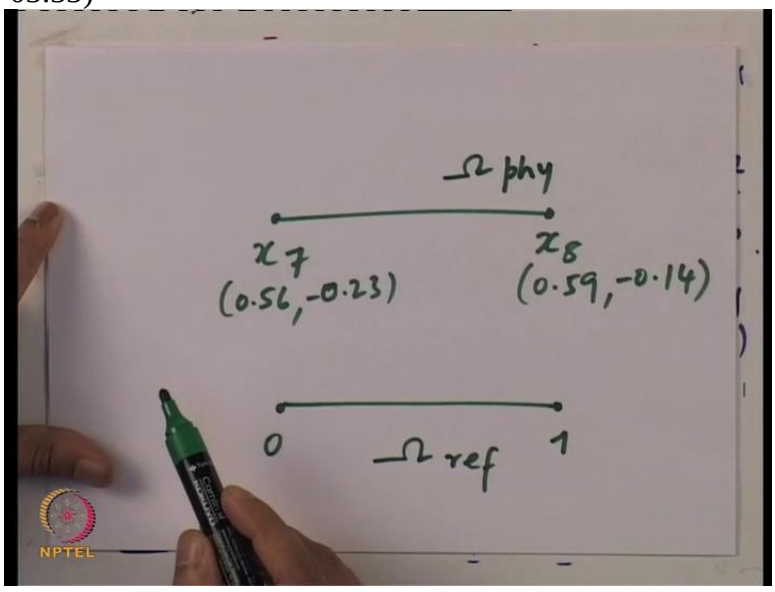

So what we do is we kind of forget this and we try to map any physical element to an arbitrary reference element whose point is always going to go from 0 to 1. The reason for doing this is very clear where each of the points inside the physical element can be uniquely mapped to a reference element. And we know that this mapping is unique because each of the points will be uniquely referred here. So that is what we are going to do. So instead of worrying about the individual coordinates on these physical elements we can directly map that physical element to global element for a reference element. Which we call it as ref. So this is called as a reference element and this is the physical element. The mapping will give us the behavior of or the positioning of each of the points on the physical element.

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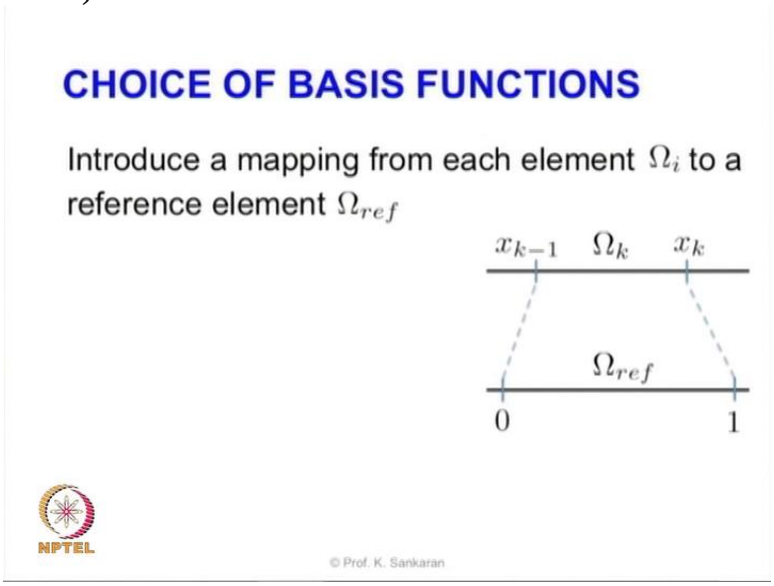

So that is what we are doing in this slide. So we say x k minus 1 which in this example which I have given will be x 7 and this will be x 8 and we are trying to map that point.

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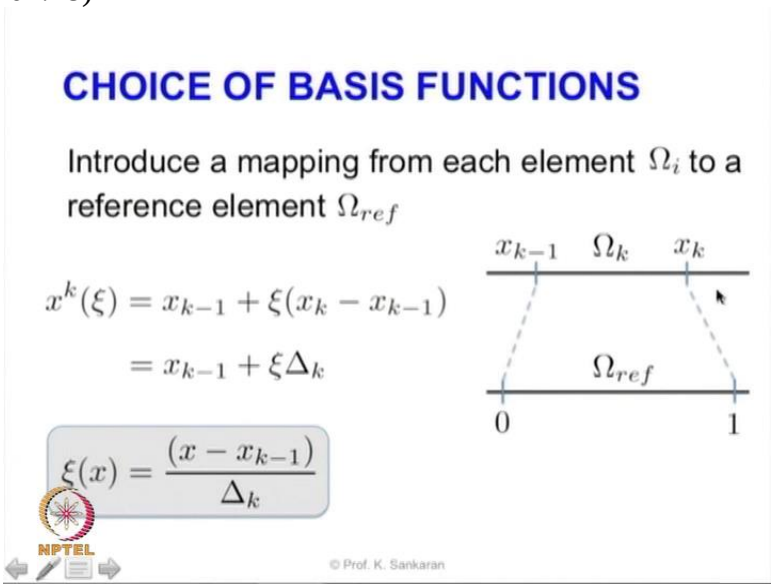

So what we are doing here is we can know that each of the points inside the physical element will have a unique point in the reference element. So for example what I have set here is when the value Eta is equal to 0 I will be on the point x k minus 1, exactly at the starting point. So if I put Eta equal to 0 k x will give me x k minus 1. Similarly if Eta is equal to 1, what I will get is? Directly this x k minus 1 and minus x k minus 1 will get cancelled. So we will go and get the value x k. And the superscript here gives the coordinate this is the k th element. So this is a k th element that is why there is a x k here.

So similarly if we do the arithmetic and if we assume that the length of each of this element is given as Delta k. We can write the expression and get the unique value for the Eta x. So Eta x is nothing but the value of x that we are interested inside each of these elements and these are mapped to the reference elements. So the idea of working with reference elements is easier convention because we know each of these points will always vary between o and 1. So we do not need to worry about what is the starting point and ending point on the physical element.

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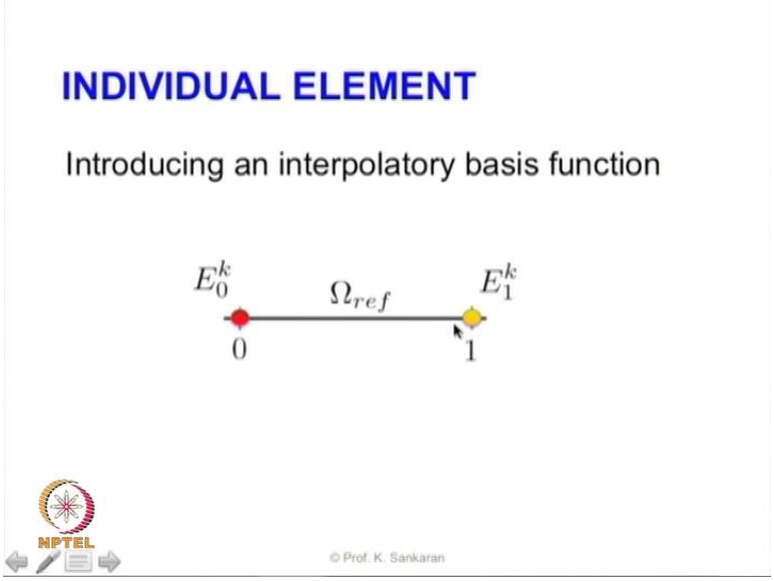

So while we do that we can also introduce the basis function itself. So I said the reference element will always have the point starting 0 to 1. So let us introduce certain nodes on the interface so what we have got here is we have got the inside the reference element there is a left point and the right point.

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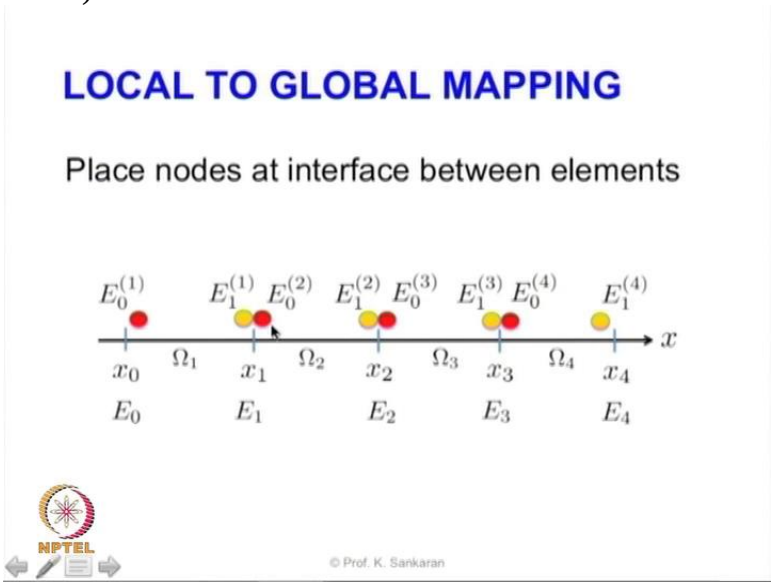

And that is what we see inside the each of the individual elements, inside each of the individual element there is a left point and there is a right point there is a left and the right point. So the left points are represented as red point and the right points are represented as the yellow points. And that is exactly coming from the mapping to the reference elements.

So what we essentially do here is inside each of this physical element the mapping is done to the reference element. So this element is individually mapped to the reference element, this element is mapped to the reference element; this is mapped to the reference element so on and so forth. While we do that what we do is basically, inside each of the physical elements we have got local degrees of freedom. So the degrees of freedom is nothing but the points where we measure or assign the field quantities. So inside the physical element sigma 1 there are two points where we assign the field values. So those are exactly the end points of those elements.

So we have the left point and the right point. So similarly inside each of this physical element we have a left point and a right point. So these individual points are called as a local degrees of freedom. We call them local degrees of freedom because they are locally assigned to the physical element. Similarly there is global degrees of freedom which are represented by E 0, E 1, E 2, E 3 and E 4. They are assigned directly to the number of nodes inside a domain. So the degrees of freedom has two different connotations. The first one as I said has to do with the basic domain itself. So that is global degrees of freedom. So the global degrees of freedom is going to be equal to the number of nodes if you are talking about a nodal element method is going to be the number of element edges. If it is going to be the edge element method. So let us focus now on the nodal element. So the number of nodes inside the domain is going to give me the global degrees of freedom.

Similarly the number of edges of a element. So if you are in a 1D we have got only one choice where we have single line segments. If you are in 2D we have got multiple choice we can have a triangle where we have inside the each of the element three points which are represented by 3 areas.

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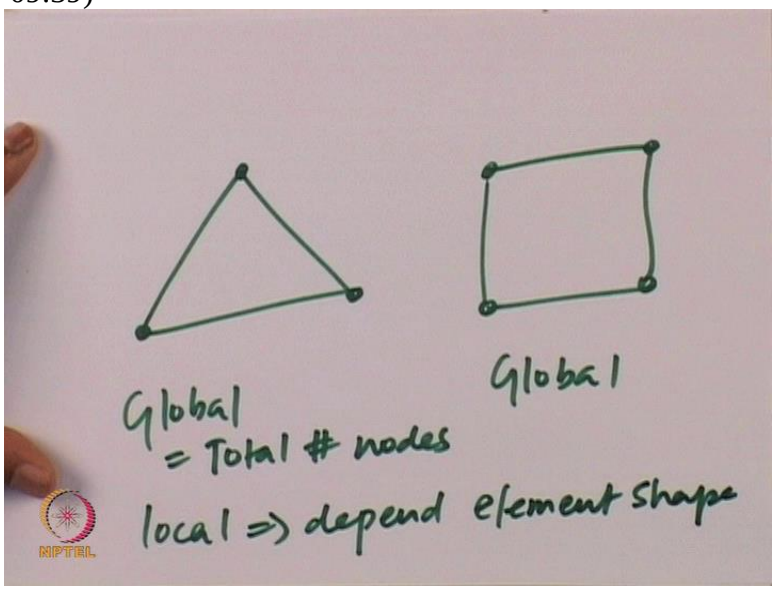

Let me explain this through the slide. So if you are in 2D you can have inside each of the elements three nodes or you can have a square where for each of the elements you will have 4 points so on and so forth. So here the global, so the global will be dependent on total number of nodes. Whereas the local will depend on your element shape. So the element is a triangle you have three degrees of freedom. If the element is square you have 4 degrees of freedom locally inside each of the elements.

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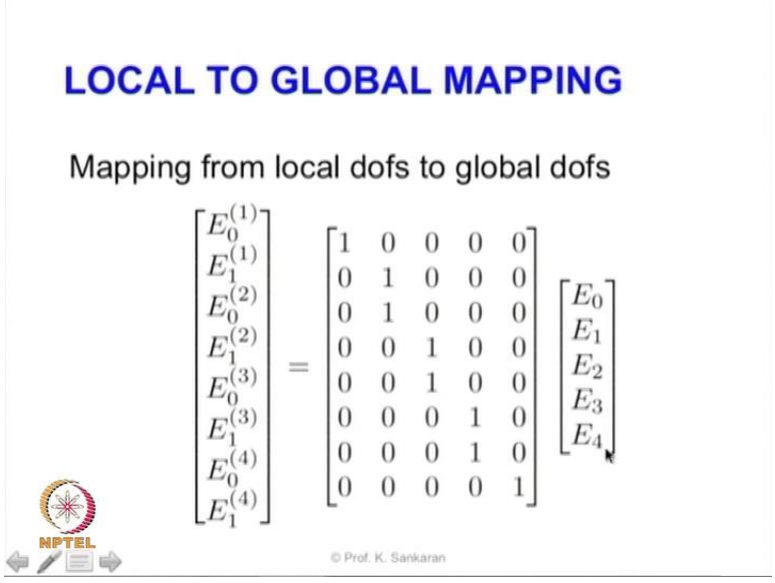

So what we are going to do now is we are going to map the values that we have locally assigned to the global values. And this is the local degrees of freedom to global degrees of freedom

mapping. So the first one has the left point and the right point, the second element which is written as superscript inside the bracket here 2, so these are the second elements left and right, third elements left and right so on and so forth. And then the global values here E 0, E 1, E 2, E 3 and E 4. They are assigned to the node 0,1,2,3 and 4.

So let us go forward.

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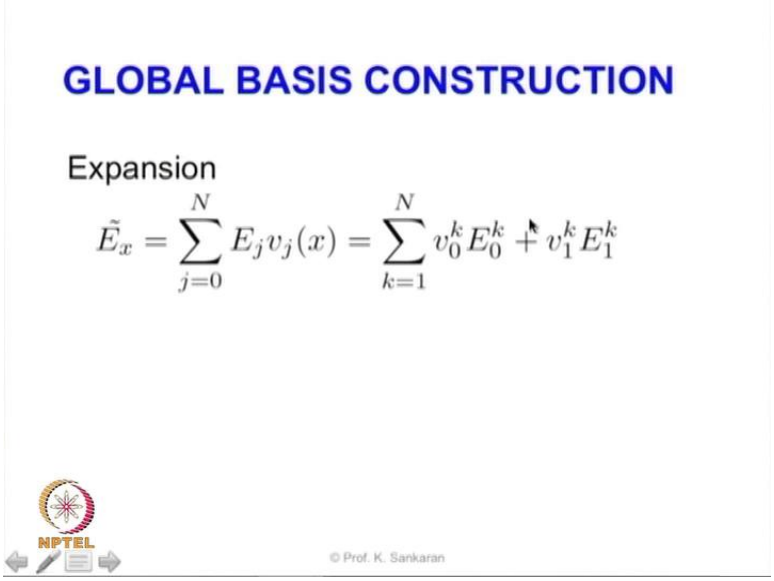

So now we are going to say the value inside each of the node is going to depend on the value that I assign to each of the nodes which is going to be the expansion coefficient E j multiplied by certain basis functions v j. So inside each of the elements, this is like this for the entire domain it will be j equal to 0 to N. So I can split this into two things one which contains all the bulk element or which has all the elements plus the left hand side and then one which has the right hand side is separated here.

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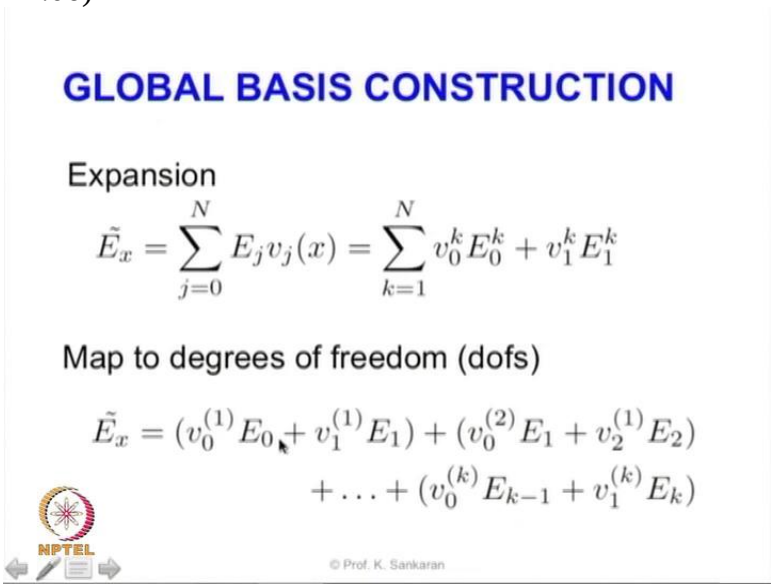

So what I am going to do here is I am going to map to the degrees of freedom. So inside each of the element I said there are two degrees of freedom in the one dimensional case, so we have the left hand side we have the right hand side, we have the left hand side we have the right hand side, so on and so forth. And we are going to collect all the terms that are assigned to a particular degree of freedom global degree of freedom. So I can collect all the E 0 in the equation, all E1 in the equation, all E 2 in the equations.

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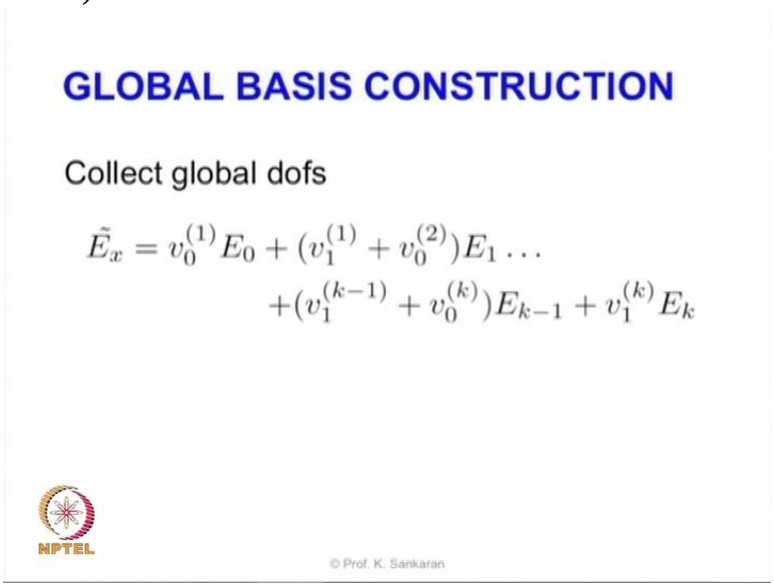

And then I can write them in this form. So the reason for doing that is I can map the elements such that I can write them in a matrix form later on.

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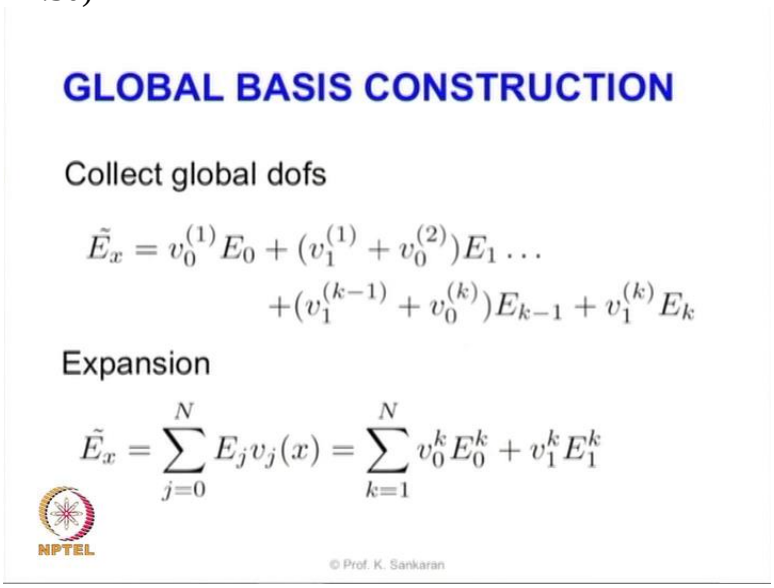

So this is the process which we have showed like before. We have all the elements plus the right hand side element which is the boundary element. So with the local matrices at hand we can construct global matrix.

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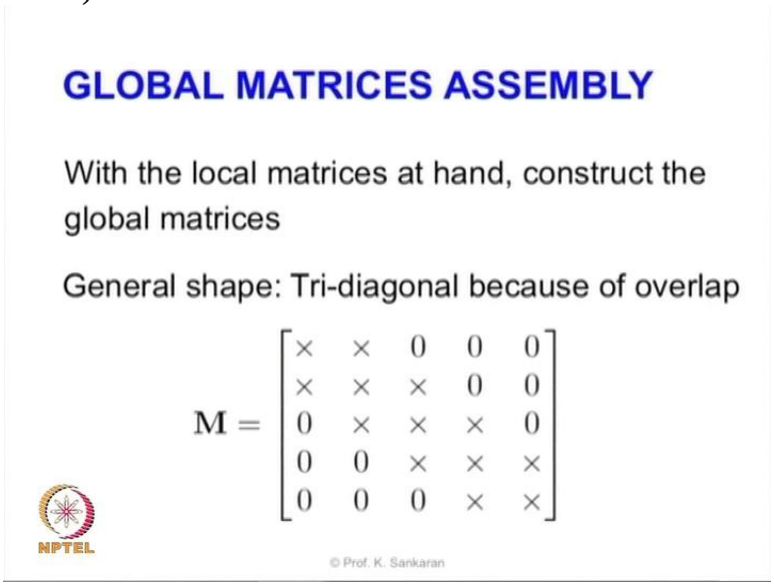

So we have found inside each of the local matrix we have certain values and we are going to substitute that values inside the global matrix. So the global matrix has a tri diagonal because there are going to be overlaps of the nodes. So what you are basically having is inside each of the element there is going to be contributions coming from a particular node. So let me explain this with the slide.

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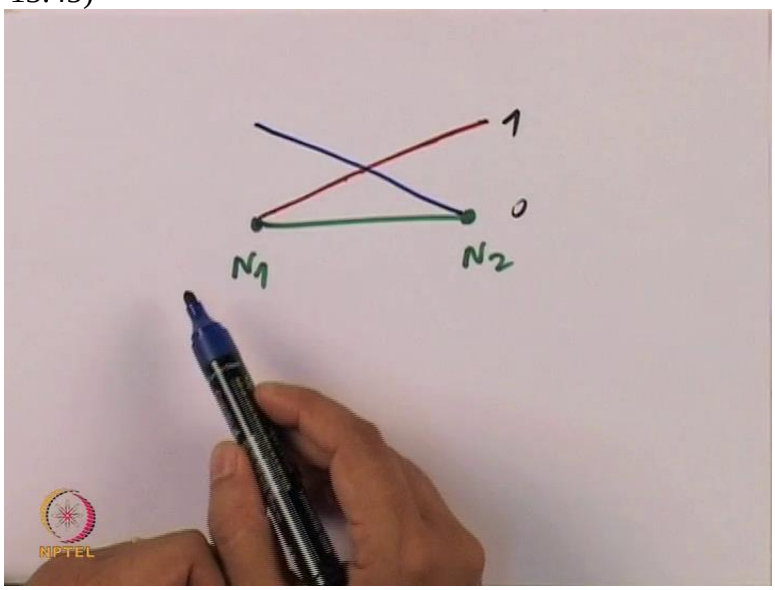

So inside each of the element. Let us say this is node 1 this is node 2. And I am going to compute the value of fields at node 1 and node 2. And I said for simplicity case let us say I have two basis functions that are defined. One is going from 0 to 1, the other one is going from 1 to 0. And let me write this one as the scale is, this is 1, this is 0. So when I am here in this particular point, the contribution because of this basis function is going to be the maximum. Whereas the contribution because of this basis function is going to be the minimum. So inside an element we are always going to have a contribution from both the basis function that are defined on a particular finite element. So in this case inside each of the finite element I have two linear basis functions that are defined. And those basis functions are going to impact the value.

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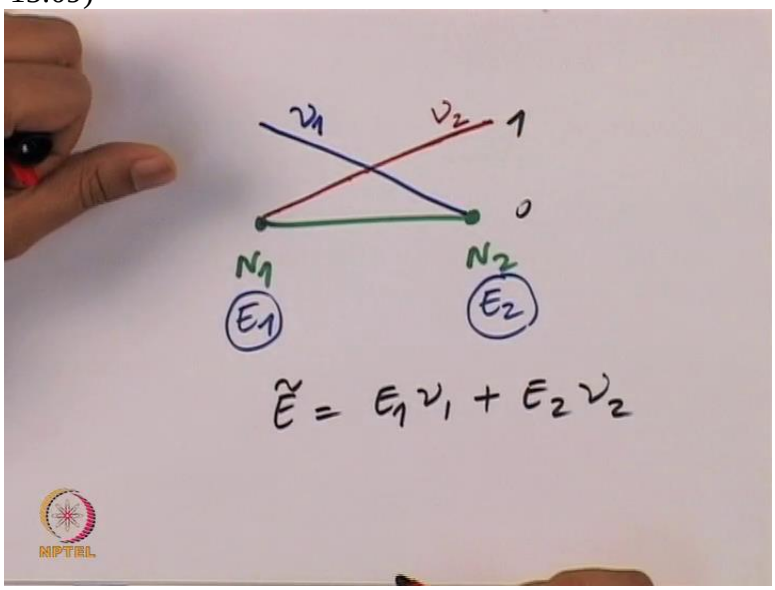

So I say here the field value is E 1, here the field value is E 2. And these are the expansion coefficients that I am going to use. And this is going to be my, so this is one which is here is going to be my v 1 and this is going to be my v 2 So I said inside each of the element my value is going to be E approximation is going to be E 1 v 1 plus E 2 v 2. So how do I know? What is happening here is there is a kind of a contribution that is going to come from this basis function and this basis function. So when I am at this point at this particular node the contribution of this basis function is going to be the maximum. When I am at this point the contribution for this basis function is going to be the maximum. And the other basis function is going to change accordingly. At inside each of the element the sum of the basis functions contribution should always be equal to 1.

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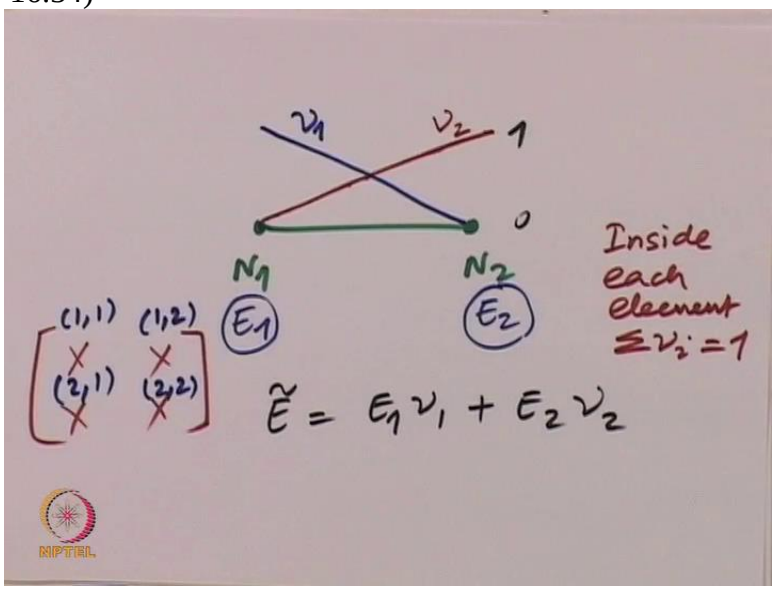

So inside each element so sum of v i should be equal to 1. So what is happening here is I have a contribution of this node on this node itself. And I will have a contribution of this node's potential on this node or in the case of the field what we are computing the value of this field on this node, the value of this field on this node. Similarly the value of this field on this node, the value of this field on this node. So totally inside each of the element you are going to have a 2 by 2 matrix. So you have a value that are represented by 2 by 2 matrix. The 2 by 2 matrix are made such that the contribution will be (1, 1) so the node 1 on 1 and this will be (2, 2). And this will be (1, 2) and this will be (2, 1).

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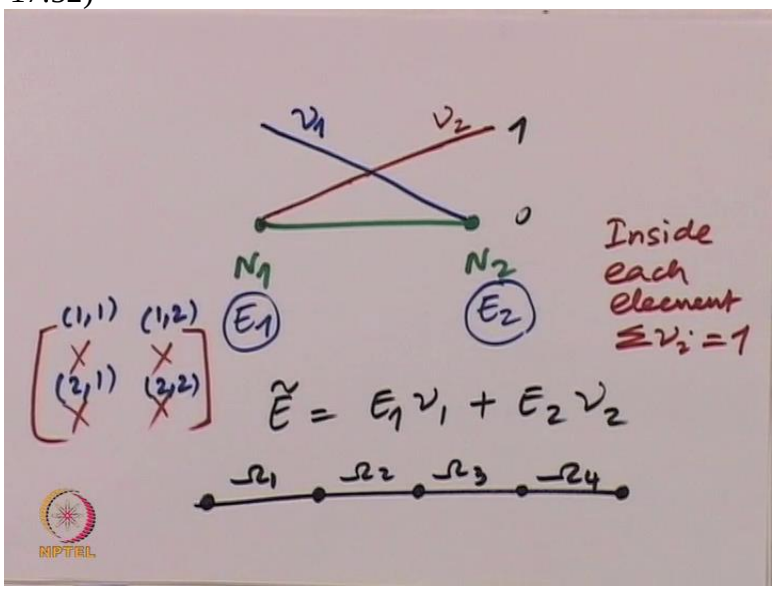

So that is what we are going to see when we are going to populate this particular matrix. And it is tridaigonal because the reason is when I am in a domain which has totally 4 in this case we have 4 elements, so there are totally 5 points. So what I am going to have here is 1, 2,3,4,5. So this is sigma 1, sigma 2, sigma 3, and sigma 4. And in this point this node is the same node that is for both this element and this element. Similarly this node is the same node for this element and this element. In other words this node is overlapping both for this element and this element. That is what we are saying here.

And since we have a 2 by 2 matrix and the nodes are overlapping on each other for each of the elements. We will have a tridaigonal matrix.

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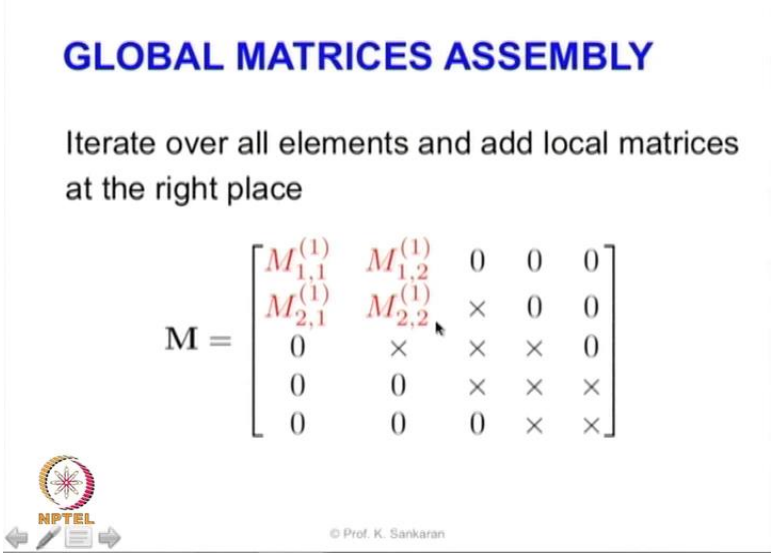

And now we are going to populate it as I explained the first combination is because of this element which is the first element. This is the first node's contribution on the first node itself. The first node's contribution on the second node and the contribution from the second node on the first node and the second node on the second node itself.

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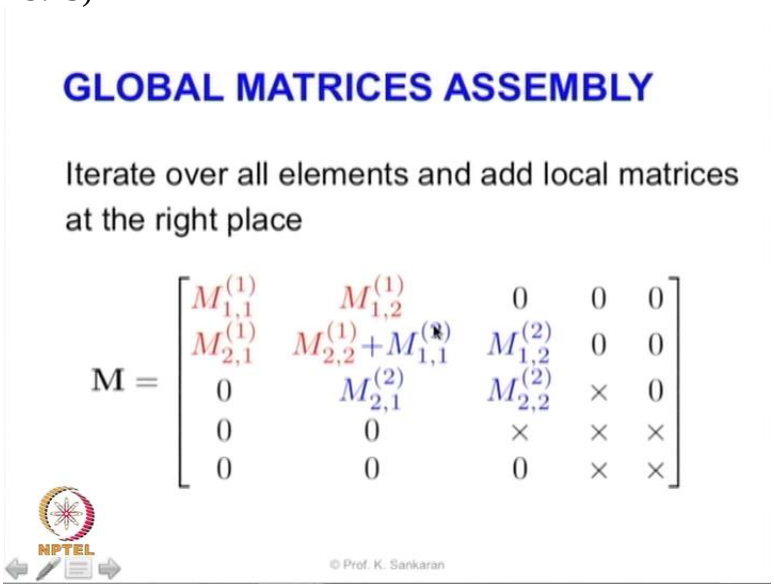

Similarly when I go to the second element there is a overlap, the overlap as we can see the second node of the first element and the first node of the second element they are the same. So we are going to have a overlap. However the subscripts are different because in this case it is belonging to the first element, in this case it belongs to the second element.

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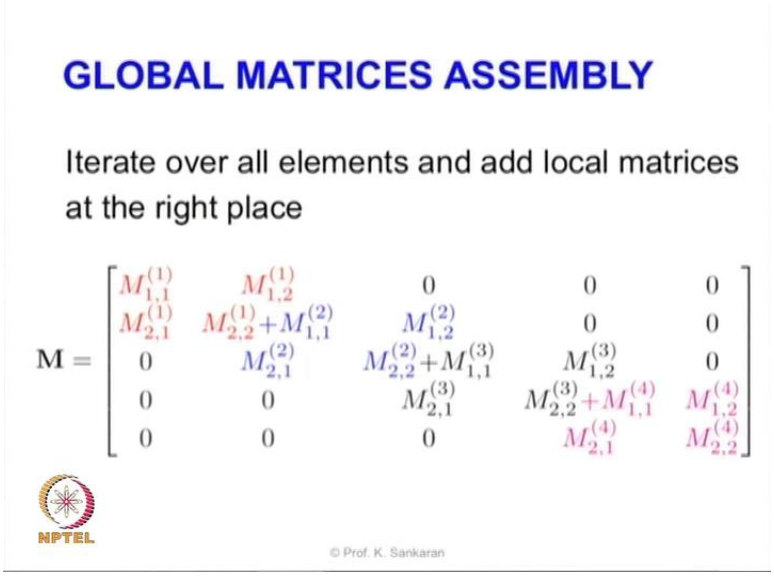

Likewise I will have the third element contribution and finally the fourth element's contribution. So with this we are going to now assemble in the same manner also the stiffness matrix. So what we have done now is for the material matrix the stiffness matrix will also have the same form.

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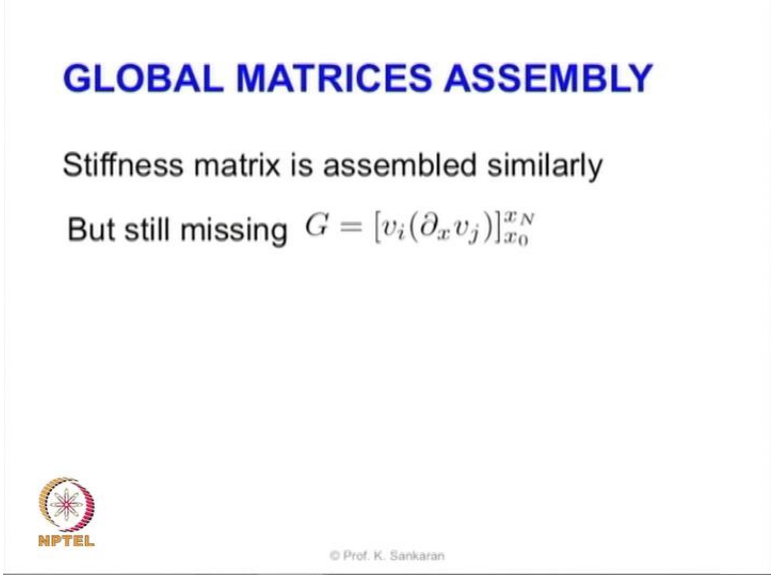

But one thing is missing; what is missing is the value of G. Remember we have given the value for G to the boundary conditions.

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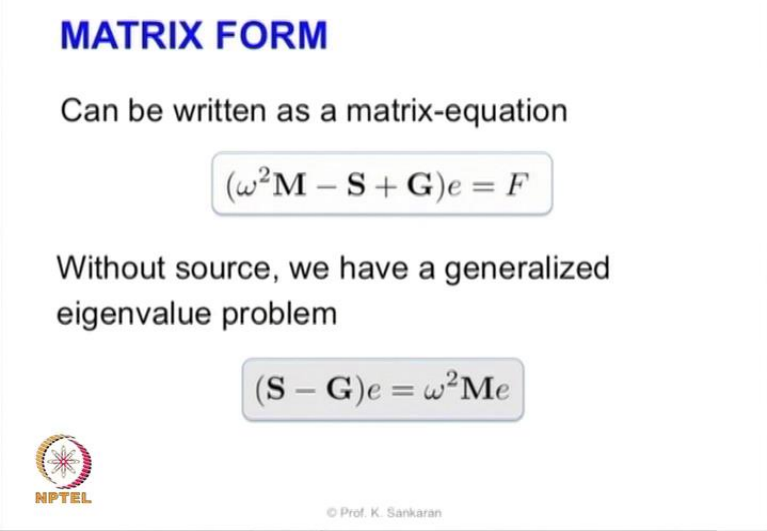

If you go back to the slide where we have discussed about the value of different elements what we see here is this component is going to be the contribution because of the boundary conditions.

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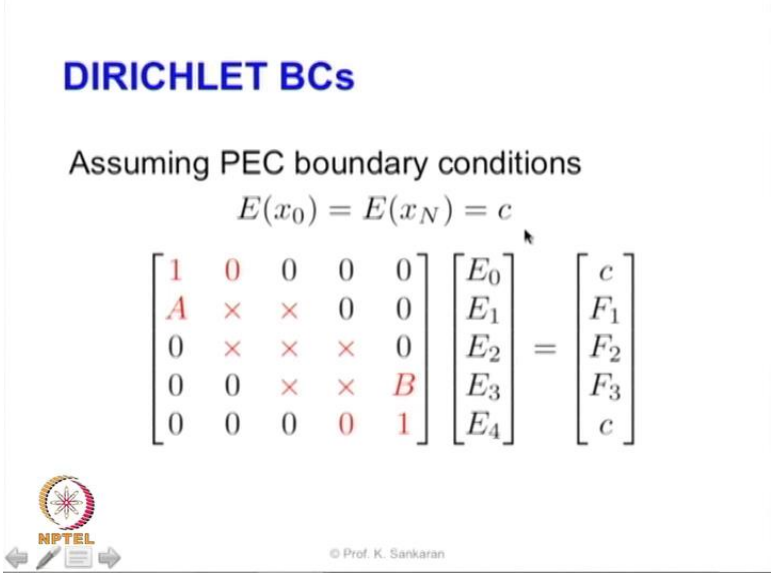

And that is what we are going to do in this particular matrix and it is going to be slightly different. Whereas in (())(20:37) it is going to be easy and straight forward. Let us say we are assigning certain PEC condition and the condition is we have the value of E (x 0) and E (x N) is equal to certain constant. If it is a perfect electric conductor, the tangential components will have 0, if it is a not perfect conductor if it is a jump condition you will have certain values. Ok let us assume that it is going to be certain values c. And we are now going to substitute this inside this matrix. So we know the value of E 0, E N equal to 4, will be equal to c.

So that is what we are doing now here, we are substituted the value on the right hand side as c. And similarly we have changed the values of E 0 and E 4 because this matrix will give the value of E when you multiply the first row you will get E 0 equal to c, similarly the last matrix will give you E 4 equal to c.

And this value and this value should be logically 0, this is also coming from the boundary condition itself. Assume that this second entry an element is A, first element of the second row is A and the last element of the fourth row is B. So we can manipulate this particular matrix in such a way that we can eliminate E 0 and E 4 because their values are given they are fixed Dirichlet Boundary condition.

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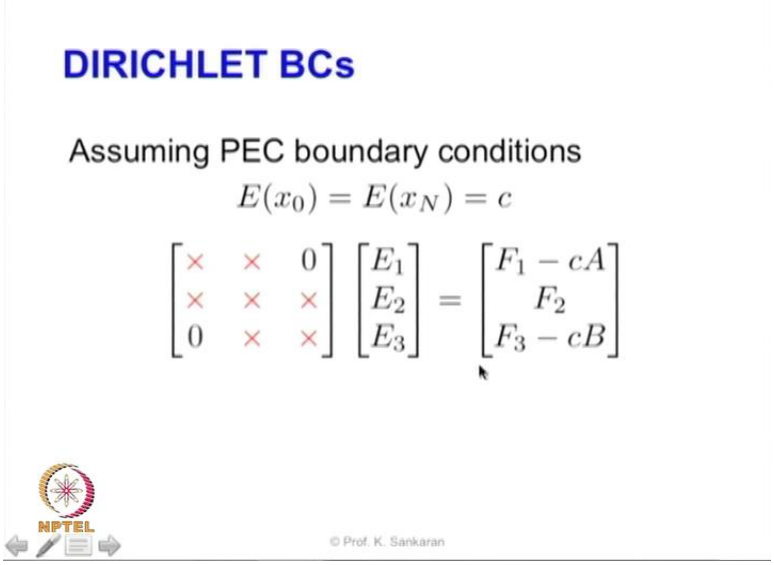

So we can only have a matrix linear algebraic form for E 1 E 2 E 3 whose values are unknown. So that is exactly what we are going to do now. So we are going to do the matrix manipulation where the value here a and b are coming from the previous slide where these are nothing but the first entry in the second row, the last entry in the fourth row and we are doing the matrix manipulation.

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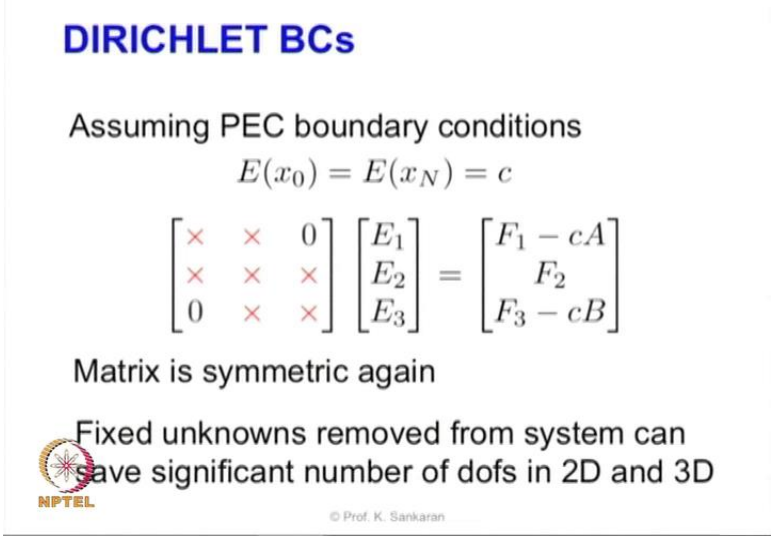

While doing that what we have done essentially is we have reduced the matrix from a asymmetric matrix to a symmetric matrix again. And we have also reduced the number of unknowns form 5 to 3. This will have a significant improvement in the case of a last scale

problem in 2D or 3D where we have to only focus on let us say n number of elements versus n plus x number of elements. So this is going to help you in faster computation and also reducing some of the unknowns to certain boundary conditions. This will help you to cleverly model the problem in such a way that you can get with least number of calculations the right answer or the approximate answer.

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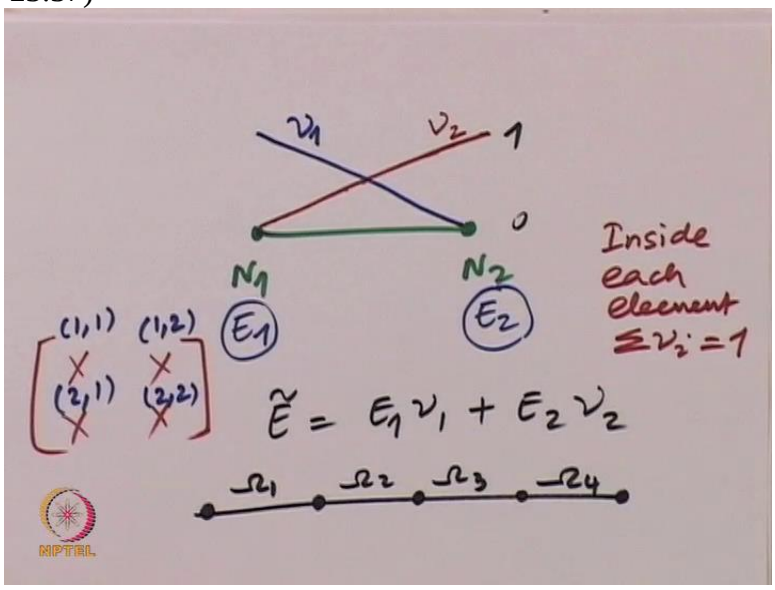

So now we have looked into a very simple case for the basis function in the one dimensional case. As I have explained here what you see here is a linear interpolating function whose values or in other words whose degrees of freedom inside the element are in two points.

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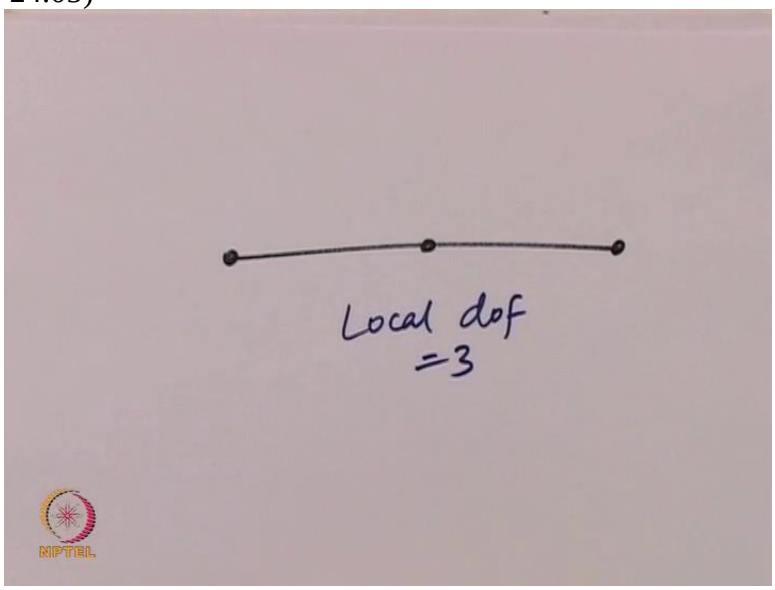

Assume that I have more than two degrees of freedom inside each of the element what do I mean by that for example, assume that instead of having two degrees we have three points inside each of the element. So the local degree of freedom is so what we are going to do now is how can we model such problems.

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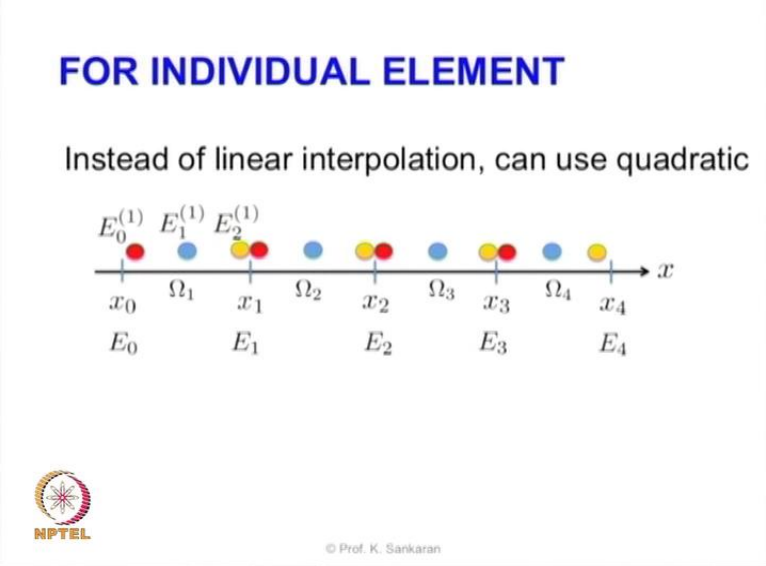

So that is what we are seeing here. So we can do instead of a linear interpolation we can use a quadratic interpolation where instead of two node s we will have 3 nodes. So as you see inside each of the element we have the left point the right point and also the middle point and likewise the numerical manipulation is quite straight forward so we will have the value going from 0 to 1 and also there is a point in between. So we can think about various interpolation functions or various basis functions.

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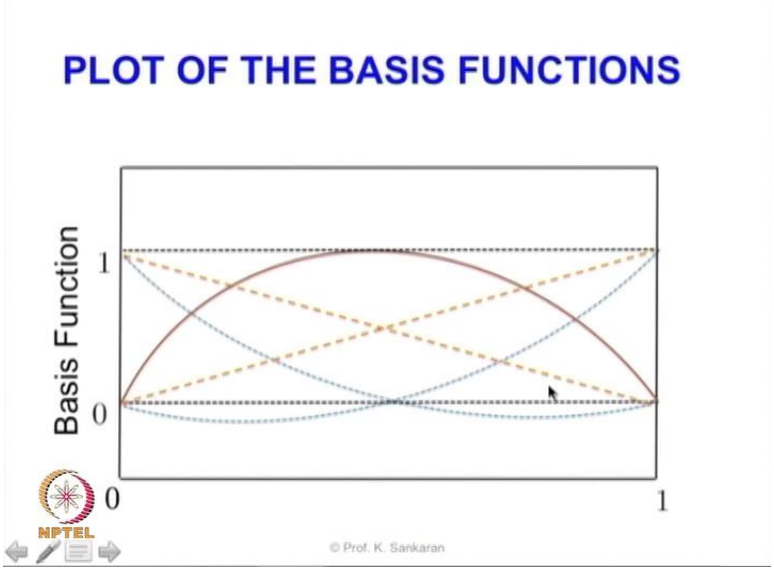

One option is to go for a kind of a quadratic function whose value is going from 1 to 0 and is quadratically changing or you can high a kind of a sync function whose value is having maximum at the middle and it is reducing on both sides to 0. And it can also have the linear interpolation which I discussed in this earlier case. So regardless of what you do the sum of the basis functions inside each of the element should be always equal to 1 at each of the points. So if I am in this point sum of the elements the contribution from this element, the contribution from this element should add up to 1. Similarly the contribution from this element, the contribution to this element should add up to 1so on and so forth.

So with that being said we have come to a stop what we will do now is we will look into a problem that is very common in electromagnetics which is Poisson equation, so now we have dealt with mostly one dimensional problem what I would like to do is I would like to do it in 2D show you the logic behind it on paper and also do it on a Matlab program to get a sense of what the solution will look like, how the Matlab can be used for solving such problems. We will do that while we come into the next module Thank You!