Computational Electromagnetics and Applications
Professor Krish Sankaran
Indian Institute of Technology Bombay
Lecture No. 19
Finite Element Method-I

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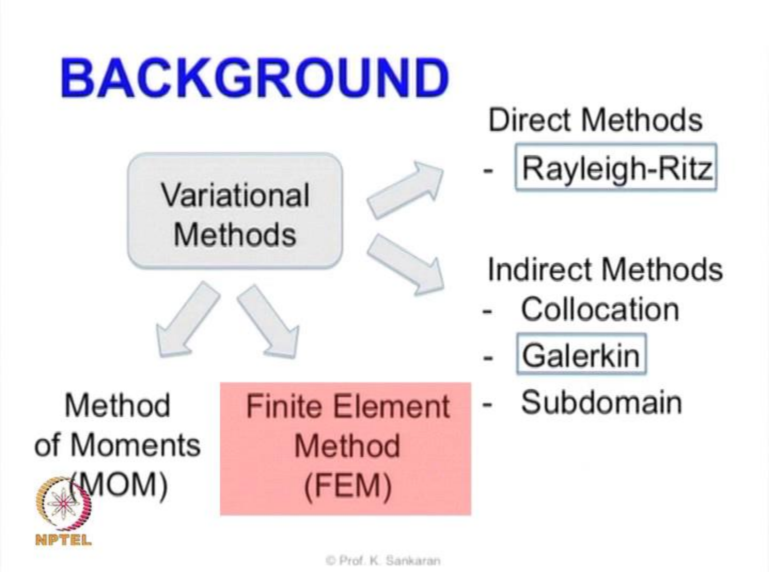

Welcome back! Where are we now? So we introduced the basics of variational methods and we showed some examples that we discussed some of the mathematical aspects of variational methods with some steps which are important for us to know. Today we are going to look into one of the most widely used method which is called as the Finite element method. The method has captured the attention of quite a lot of scientist in various domains particularly Electromagnetics. Although its origin is in the domain of mechanics and Mechanical Engineering.

So we might be still using some of the terms which are quite used in Mechanical Engineering like the stiffness matrix or material matrix so on and so forth. But the idea is quite widely used also in electromagnetics and we will see the counter part of what we have been doing in the mechanical engineering. So here the entire attention will be on problems related to electromagnetics particularly the Maxwell equation.

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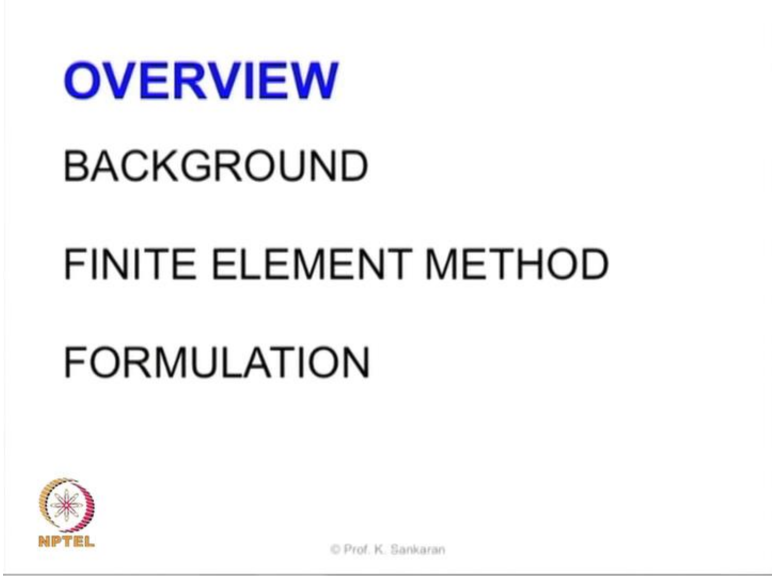

So we will start with the basics background of the Maxwell equation and in the Finite Element form. We will start with the background explaining the motivation for going into this. And then we will also explain the finite element method itself and then we will look into one type of formulation.

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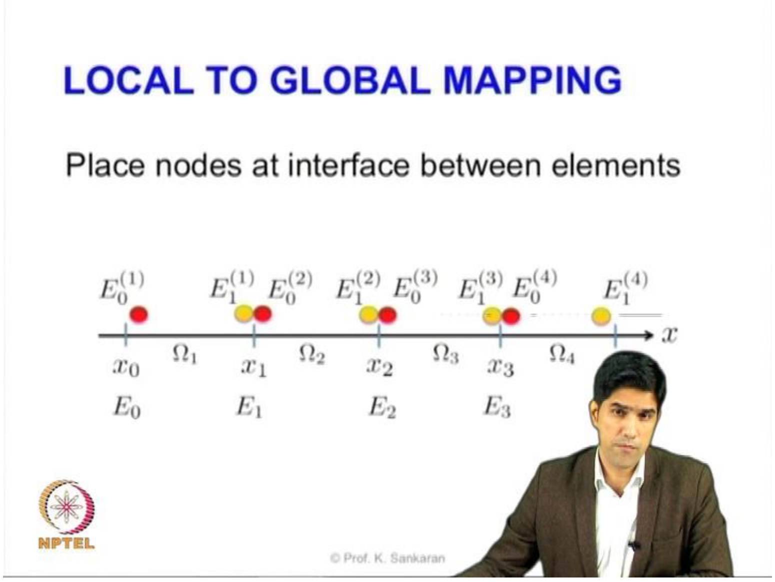

I say one type of formulation because Finite element itself can be modelled either using the nodal element or edge elements. In this module we will be looking into the nodal element. With that being said I will not leave you just like that, so I will also introduce the edge element aspect which is quite important to avoid some of the problems that we face in nodal method. We will talk about it as we come towards that point and the lecture will not end without mathematical aspect. So we will discuss one example step by step deriving the Finite element solution or Finite element formulation for Poisson Equation.

So with this let us start with the background itself.

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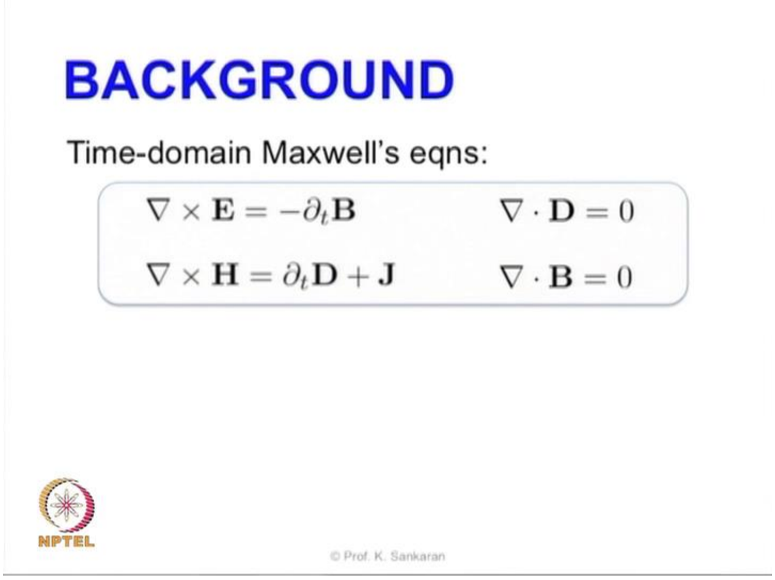

So the Time domain Maxwell equation essentially consists of the main Curl equations representing the behaviour of the magnetic field and the electric excitation. As a curl of the corresponding electric and magnetic fields. And you also have the diversions conditions. Here we talk about diversions conditions set to 0 because we do not have any charges in the model which we are considering and obviously the diversions of B will always be 0.

So when we look at the formulation of finite element method. Most often people use the frequency domain approach. And obviously it has its merits and it has its role in various applications. But one can also think about modelling finite element formulation in Time domain and that is something we will look at it briefly over the next lecture. For now it is good to know that we are going to model the basic problem of Maxwell's using the Frequency domain approach.

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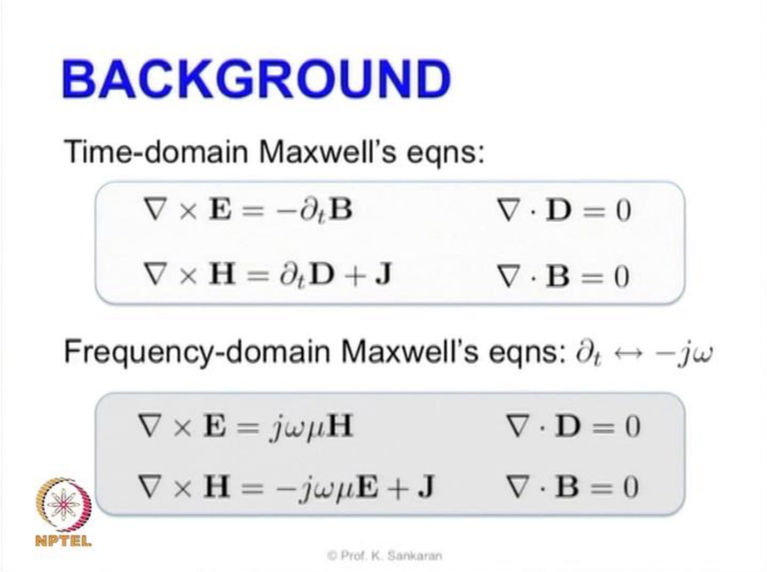

For that we have to go through the transformation where the frequency domain terms can be applied by having the partial derivative with respect to time represented as (minus j omega) where omega is the angular frequency 2Pif.

And the corresponding Maxwell equation in Frequency domain will be represented by these four equations. So one can basically look at this set of equation with certain boundary conditions, because without boundary conditions I said before the Maxwell equation by itself is not enough one has to look at boundary conditions both within the computational domain and also at the boundary of the computational domain.

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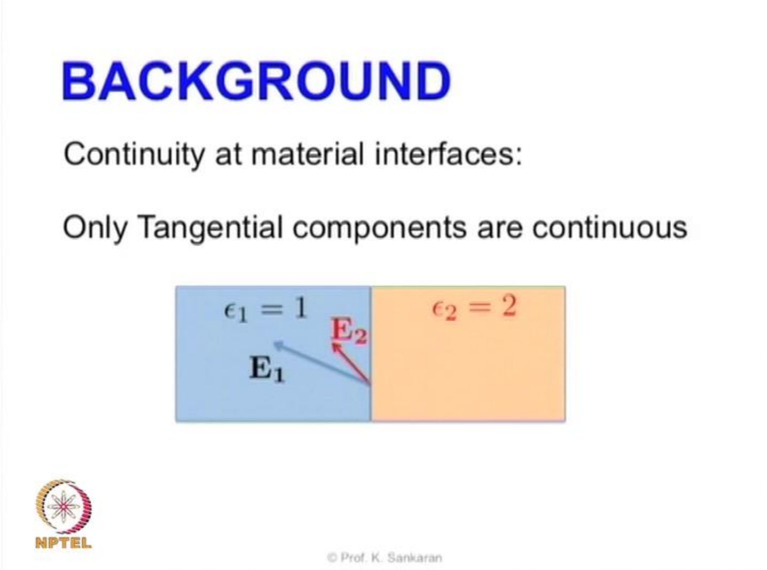

So one of the main boundary condition is the material condition. So the tangential components of electric field and the tangential components of the magnetic field. So one has to look at their tangential components as continuous both the electric and magnetic field. So here we have represented just an example using a material with two different dielectric materials and the tangential component of the electric field should be continuous. So that is the most important condition. Even when we have discontinuities between the material parameters. So here the material parameters permittivity is different for two materials and the tangential component should be satisfied.

So what are we going to do now is basically we are going to simplify the Maxwell equation. So we do not want to have this two curl equations. So basically we are going to combine them in a clever way.

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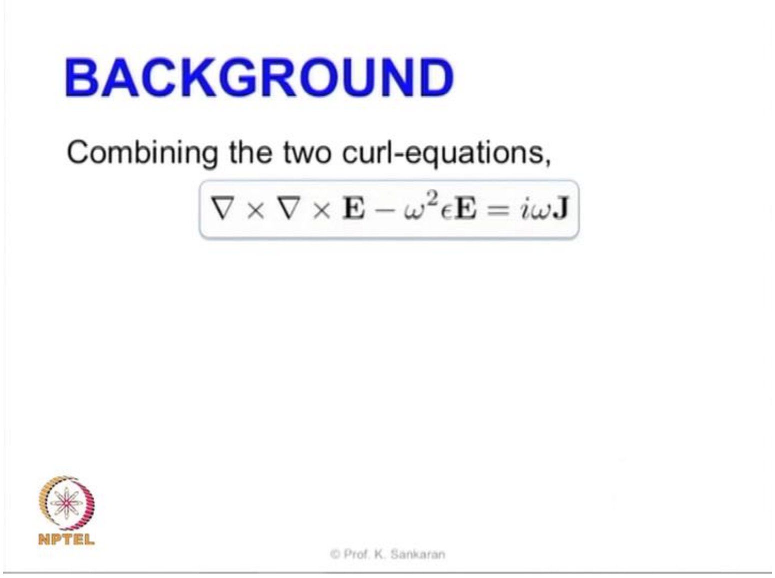

So what we have is we can take the curl of the first equation and we can substitute the value for the so the curl of H and then basically we are going to get this equation. Let us find out how we came to this expression.

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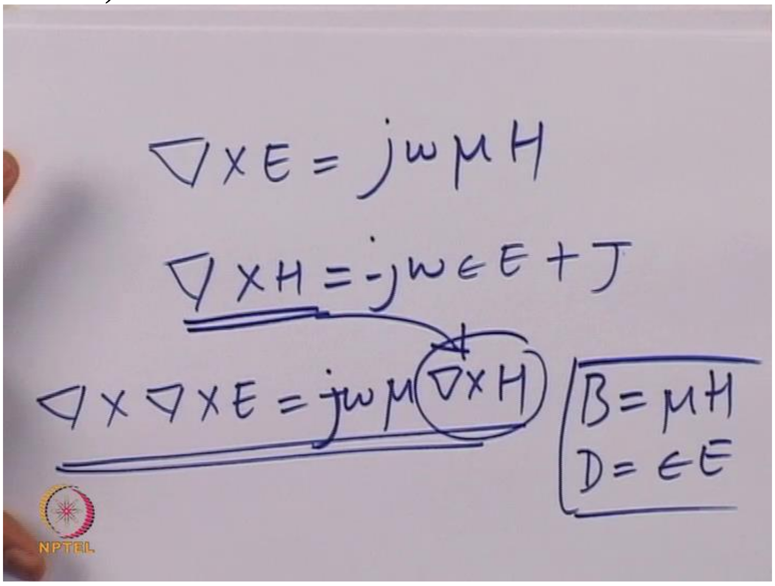

So initially we have the curl of E is equal to j omega Mu H and curl of H is equal to minus j omega Epsilon e plus J. The assumption is B is equal to Mu H. There is no magnetization in this material so it is simply Mu H. And D is equal to Epsilon e. So what I am doing now is I am going to take the curl of the first equation one more time. So I am going to do curl of curl of E is equal to j omega Mu curl of H. So when I have this I can substitute the value of this one in here and then I will get an expression only in terms of e and j.

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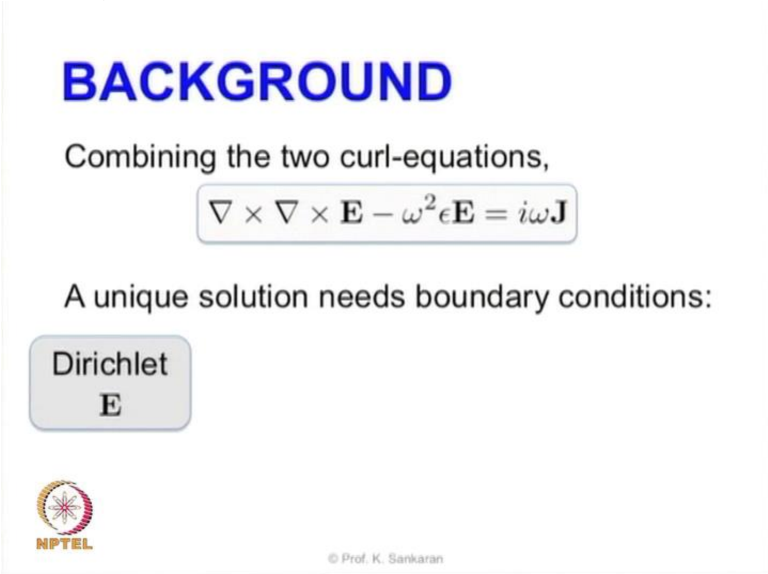

And that is what we have got in this particular case where our expressions are only in terms of e and j. And one more thing that one should know is the value of omega contains the information about the frequency we are considering. And like before we have to talk about the boundary conditions we can have the Dirichlet boundary condition which talks about hard boundary condition forcing the value of the fields. Let us say the tangential component or a tangential component as a step or is equal to 0 things of that sort.

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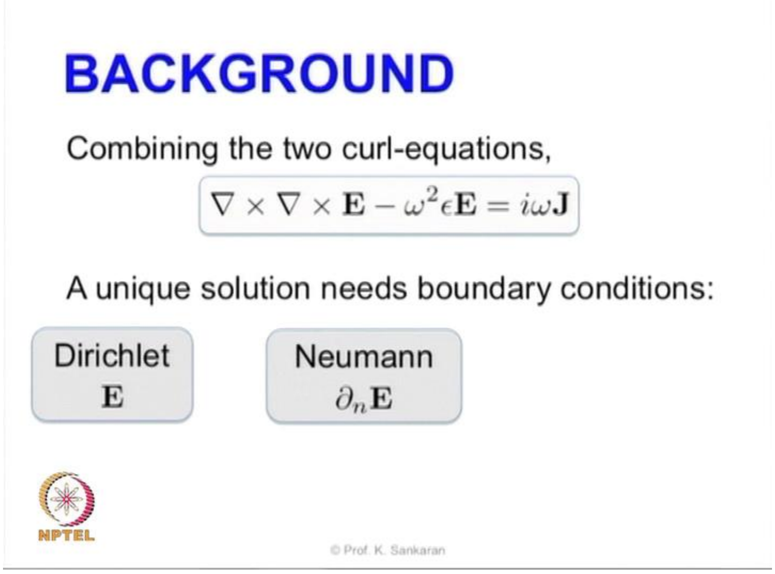

Or you can have a Neumann condition which basically tells the normal component of certain flux values.

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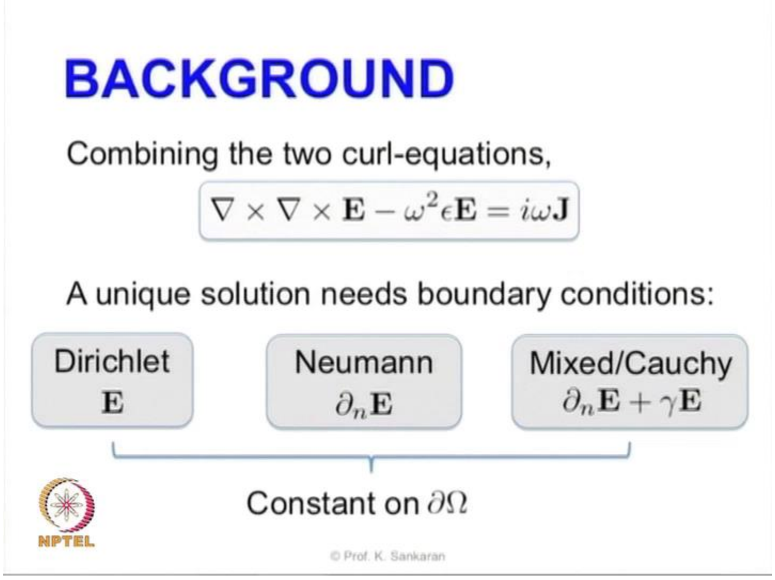

And it could be a mixed combination of both the Dirichlet and Neumann condition. So it could be a mixed or Cauchy boundary condition. So with that being said the Dirichlet condition and Neumann condition are special cases of the mixed condition. So if you say the normal component the gamma value here is equal to 0, it will be a Neumann condition, when the normal derivative components become 0 it will be a Dirichlet condition things of that sort. So these conditions are forced on the boundaries that are boundary of the domain sigma.

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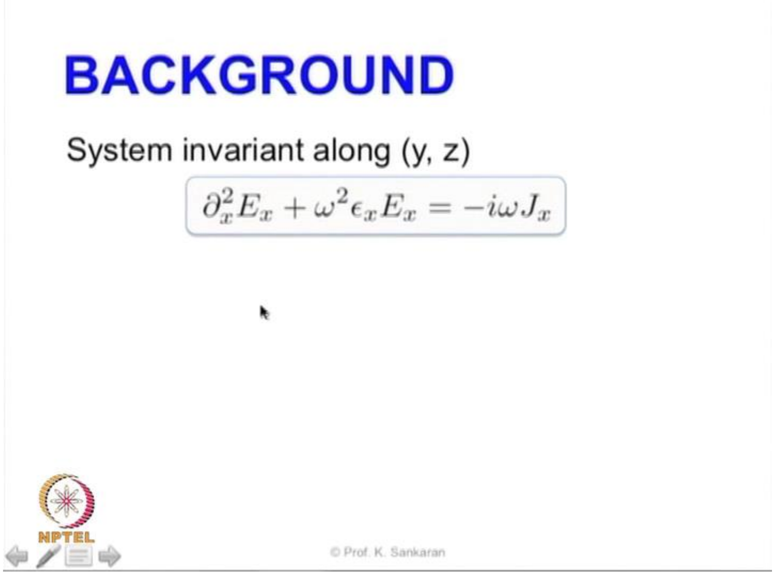

So now let us look at the equation itself this system is invariant across let us say y and z plane. So what happens here is we have written them as a scalar equation.

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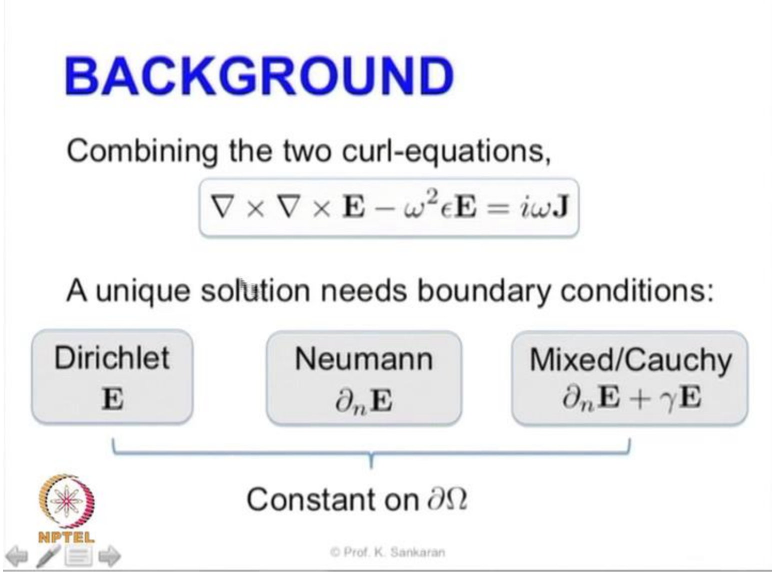

So what you have got here is basically a vector equation we have split them into their corresponding scalar components.

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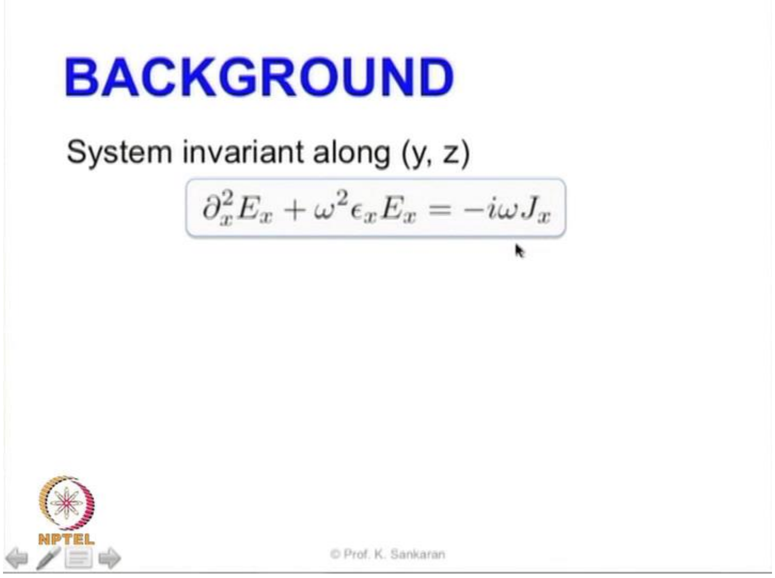

And we have got the E x component, so this system is invariant along the yz plane.

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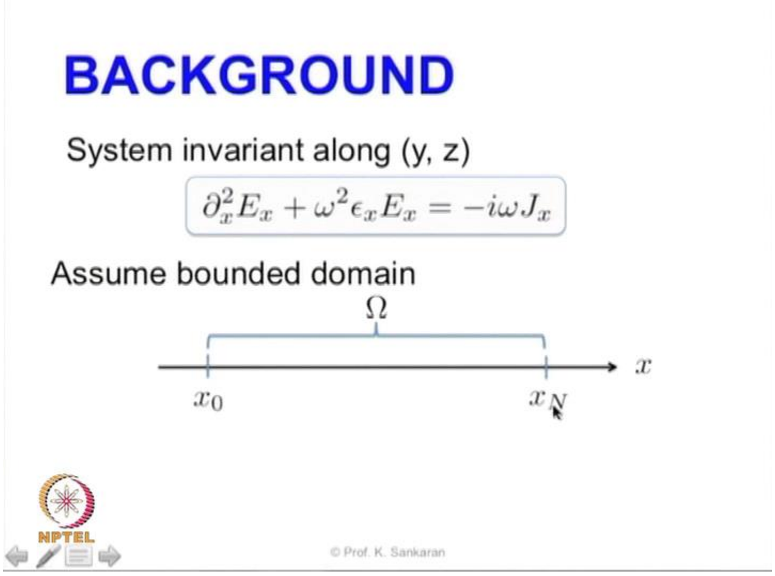

And we are talking about the computational domain to be one dimensional in x axis and has a starting point as x 0 and the ending point of the domain is x N. And whatever is in between is called as the domain and the boundary of this domain which we write it as Doe partial derivative form of omega in the previous slide are nothing but these two end points. So with that being said now we have got the formulation for setting up the basic equation and this domain can be forced to certain boundary condition.

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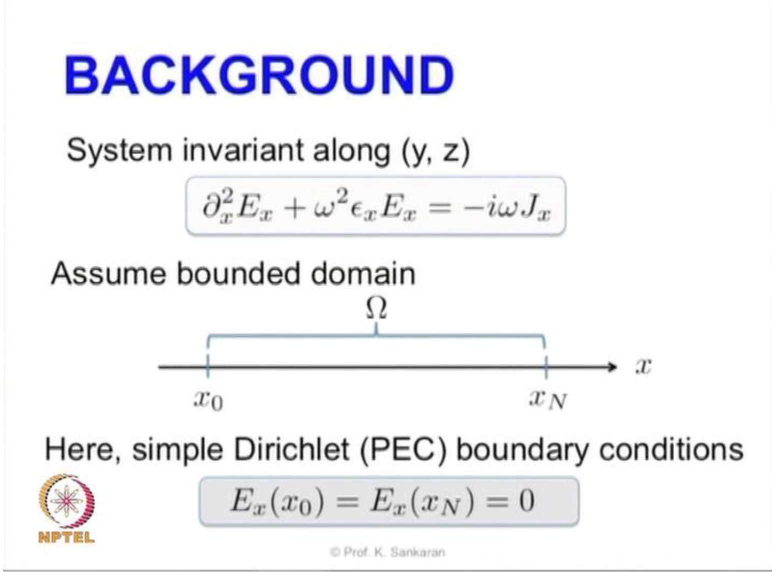

So here the boundary condition is simple PEC condition and the PEC condition says the tangential component will become 0. So in this case the value of the electric field will become 0 at those two points. So at the point x 0 and x N the value of the electric field will become 0.

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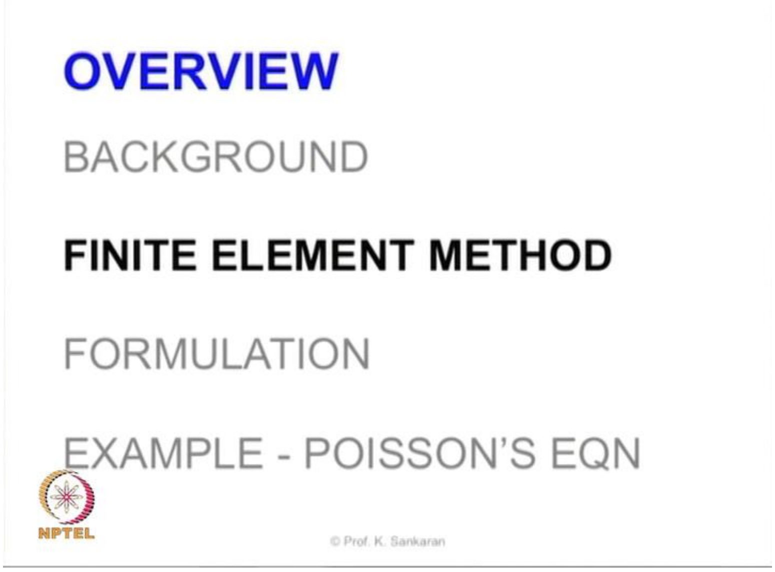

So now we are going into the finite element formulation itself. So as I said Finite element formulation is nothing but an extension of the variational method. And we saw already most of the important mathematical manipulation one has to know in the finite element formulation.

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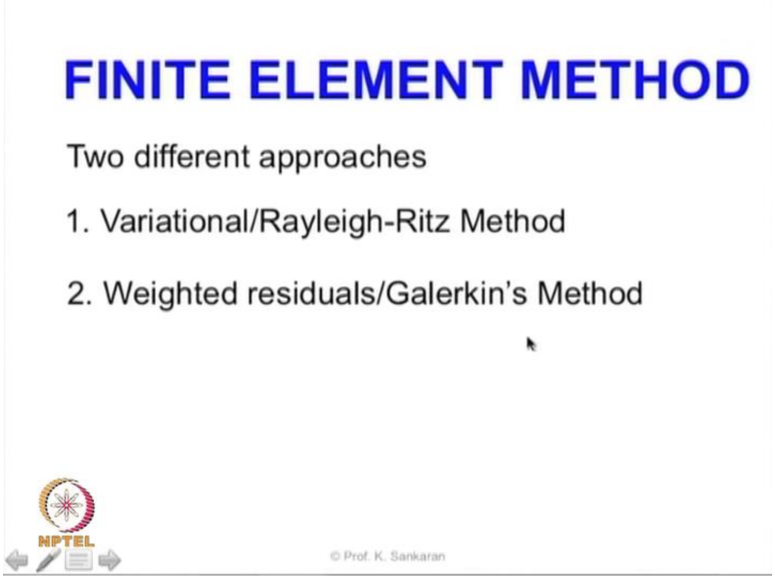

So the two approaches which we discussed in depth are those approaches related to variational method or the Rayleigh Ritz formulation. And we also discussed about the Weighted residual or Galerkin formulation.

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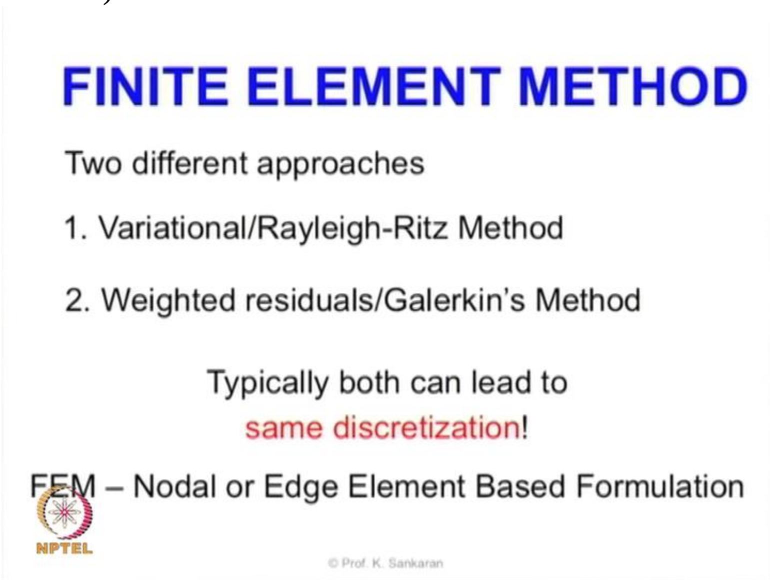

And the most important thing is typically both of them gives rise to the same discretization. So there won't be a big difference in the final form what we are getting. But the most important thing what I said is if you are talking about Finite element method. You can either start with a nodal based finite element approach or the edge element based finite element approach. In this particular module what we will be doing is we will be looking into the nodal approach. So we will look into the edge element at a later stage.

So let us say we are starting with the weighted residual or Galerkin method. So we will not start here about the Rayleigh Ritz or the varaitional method. This we discussed in depth already before. So we will start with the approach of Weighted Residual and Galerkin method.

So like in the earlier case we have to start with the equation itself and let us say the equation has certain assumptions so the assumptions are basically. Let us say my equation is perfect when the number of elements I am going to consider is going to be quite large. But it won't be the case so we will have always some errors.

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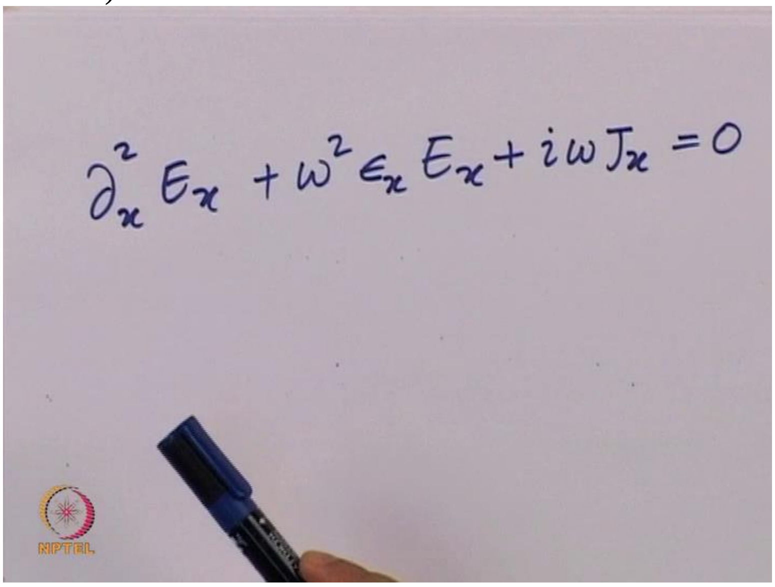

So what we have is when I have this differential equation Doe square x E x plus omega square Epsilon x E x plus i omega J x this is equal to 0. So this is the basic equation that we have got by taking the case of a scalar equation. We got this one from our earlier step. But here the value of E x are the exact value of E x. But in our case we are going to take certain approximation.

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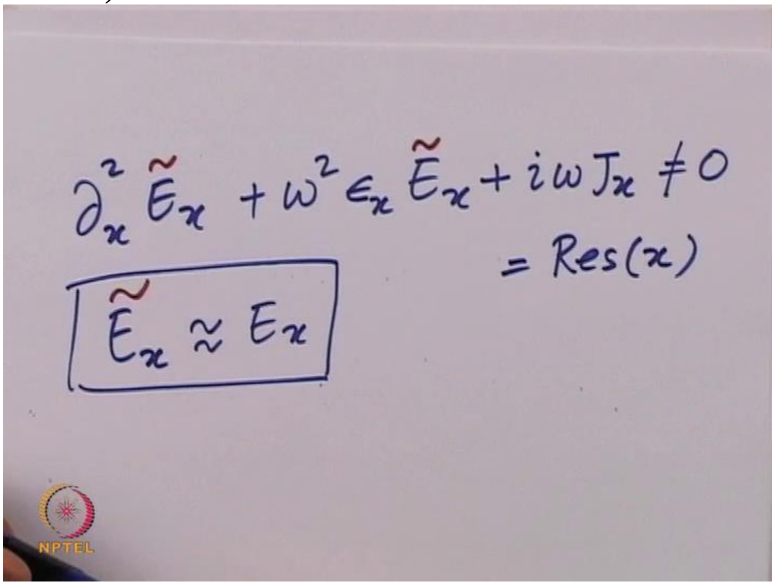

So the approximations are going to be the tilde values. So let us say we are going to have some approximations for E x. So the E x tilde whatever we have in the equation is going to be approximately equal to E x. But since it is only approximately equal to E x. This is equation will not be exactly equal to o. Will have some residuals. Let us say that residual I write it as Res(x). So since it is not exact value of E x I am only using an approximate value of E x. And I will explain you what is the approximate value. But it is enough to know this value will not be exactly equal to 0. But it will have some residuals. Residuals in the sense some errors.

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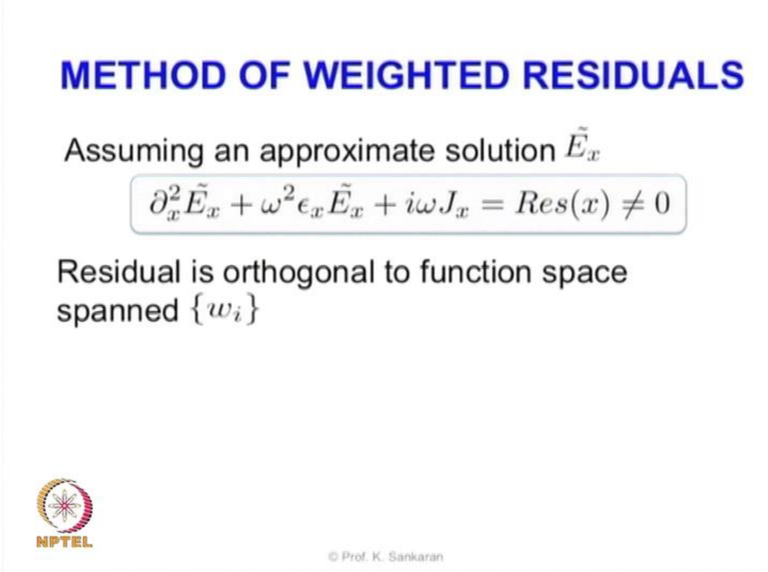

So now these residuals what we have is orthogonal to the function space spanned by this {w i}. Remember in our earlier case we had this {w i} as a testing function. The weights that we use for testing the value and this {w i} is something that we have already covered in the earlier module with respect to Galerkin method or Weighted Residual method. So we are having this condition that the residual is going to be orthogonal to the functional space of the {w i}. So {w i} are the weights that are corresponding to each of the i th element.

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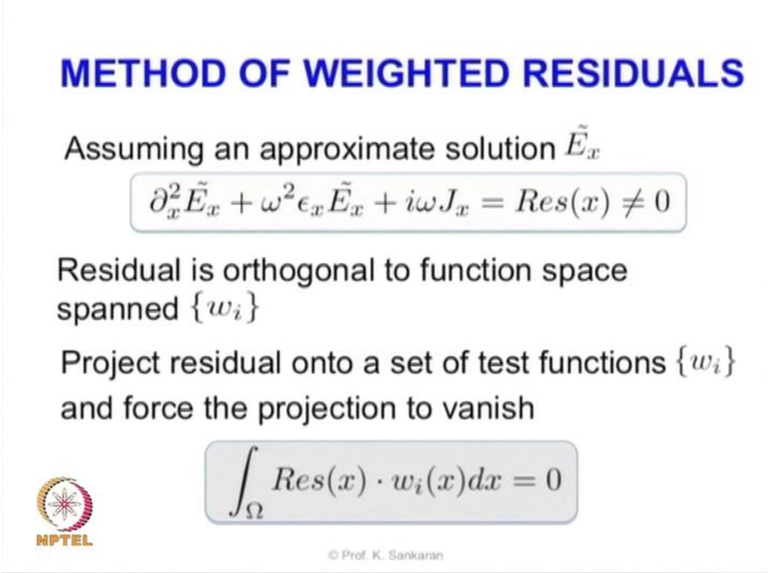

So now when we project the residual on to a set of test functions and force this projection to vanish is the key step when we do this Finite element method. So what I am saying here is we are starting with the assumption.

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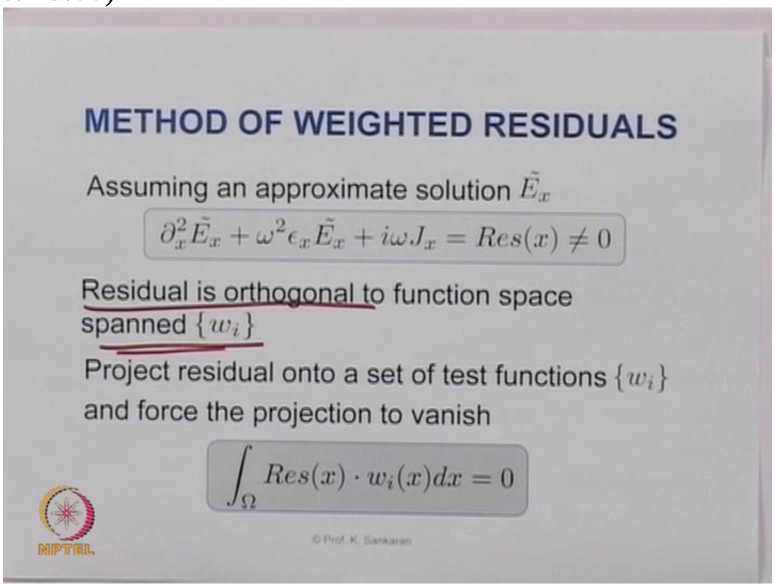

And then we are saying that this particular residual what we have , and we are taking the projection of this residual on this particular case is equal to 0. The reason why this is true is because residual is orthogonal to this particular thing. Since it is orthogonal projection is nothing but the dot product is going to be 0, this we know from basics.

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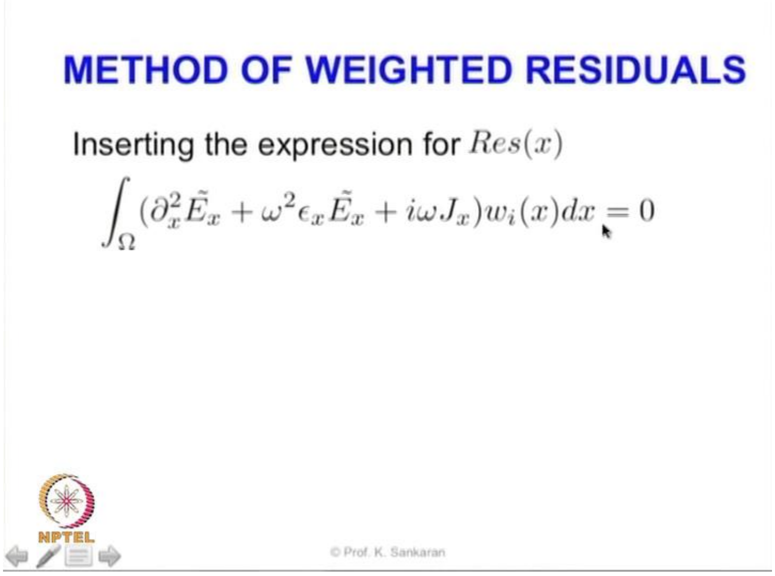

So when this is done inserting this residual into the main equation. Remember that the residual value what we have go is from this particular area. So now we are inserting this value here. When we insert the value here what we get is the actual equation should go to 0.

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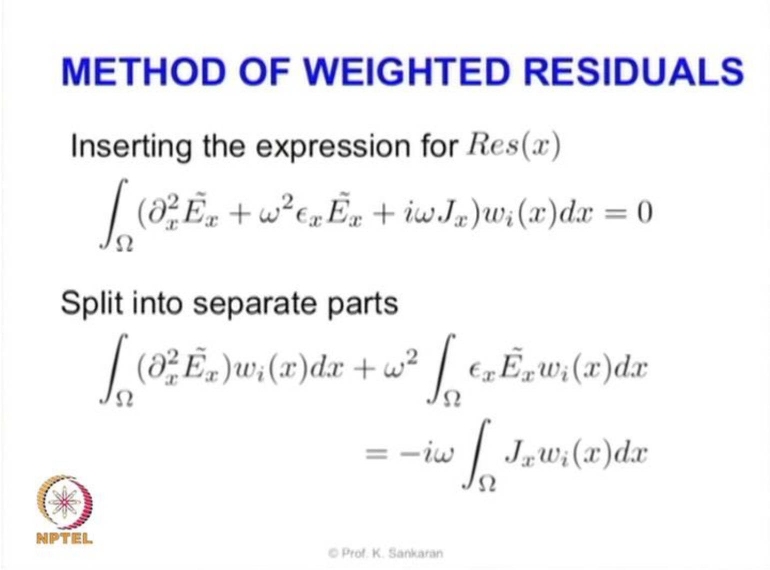

And now we are going to split them in two parts. So we are going to split them in the parts because we do not want to have this higher derivative sitting here. So we wanted to get rid of these higher derivatives. And then have a simplified form so that we can force the boundary conditions. So we are going to split this equation into parts. So what we have done here is we have taken this equation and then taken this forcing function where J x is the value that we know. We are taking it to the right hand side and now we are going to have the value on the right hand side.

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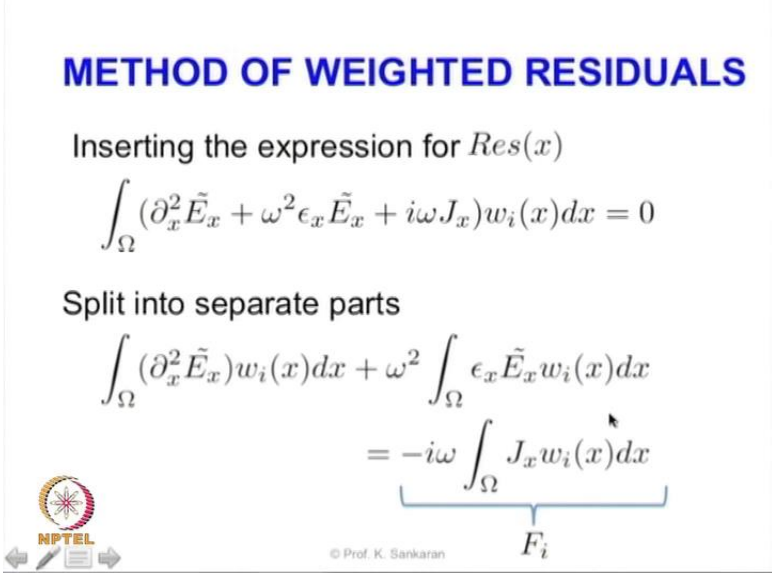

And this value is something that we call it as the F i which is on the right hand side.

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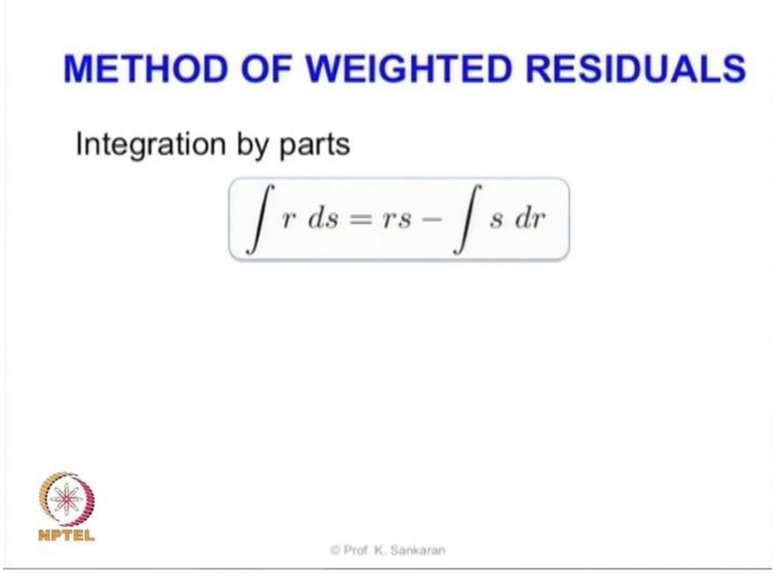

And now we are going to use the integration by parts. So what we mean by the integration parts is it is very familiar to us because we have been using it a lot in our earlier exercise.

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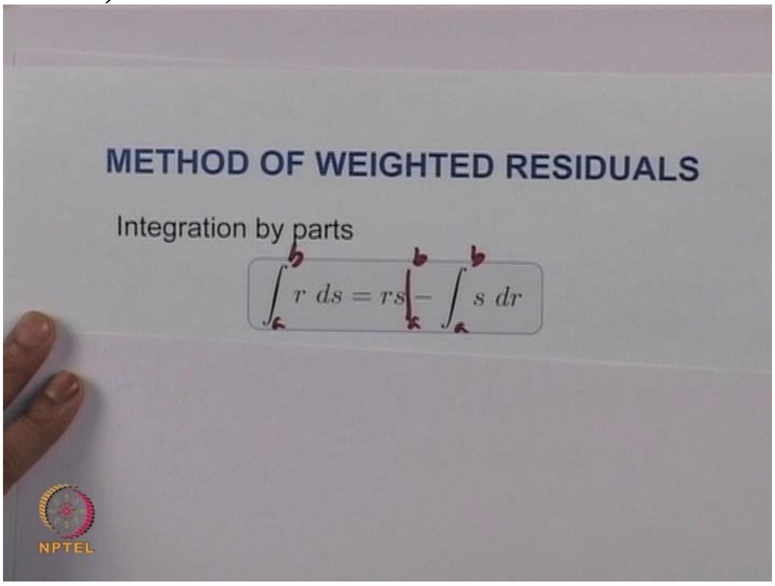

So what we are doing here is we are taking for any function given by integral r ds is equal to rs minus integral s dr. And of course the value here is let us say a to b , a to b, a to b.

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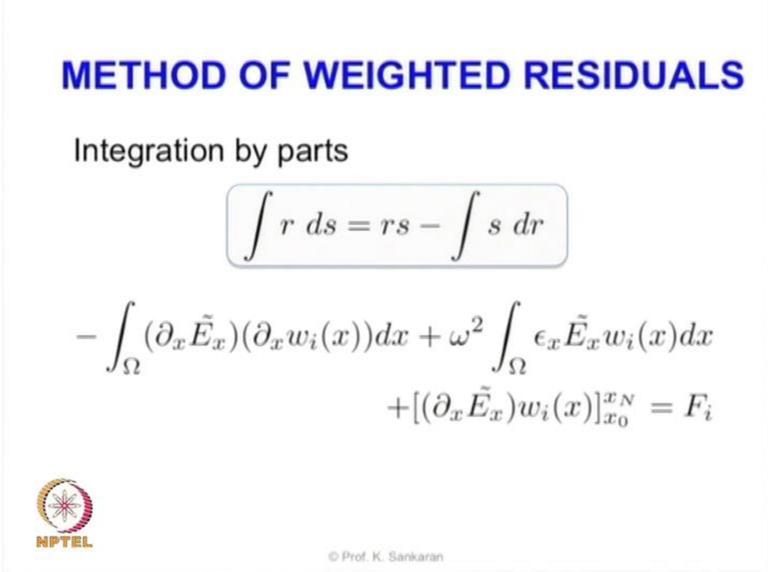

So this is exactly what we are going to do in the case of these higher derivatives that are sitting. So we split the previous equation so the equation which is here we are going to apply the integration by parts on this to arrive at this particular form. So as you can see the second order derivatives have gone away and we have got basically the main things and this particular point which is here this is actually the one term that is going to have the boundary aspect. So the boundary is physically x 0 and x N which we already set in our earlier case. And we have split them into parts using this particular equation. And this rs term is basically this term and we have these other terms sitting exactly as we expected.

And here the integration is done between x 0 to x N. And that is why you have limits here x 0 to x N. Now the thing is I told you there is some assumptions on the E x tilde. I did not tell a lot about the E x tilde we covered about it in the earlier modules.

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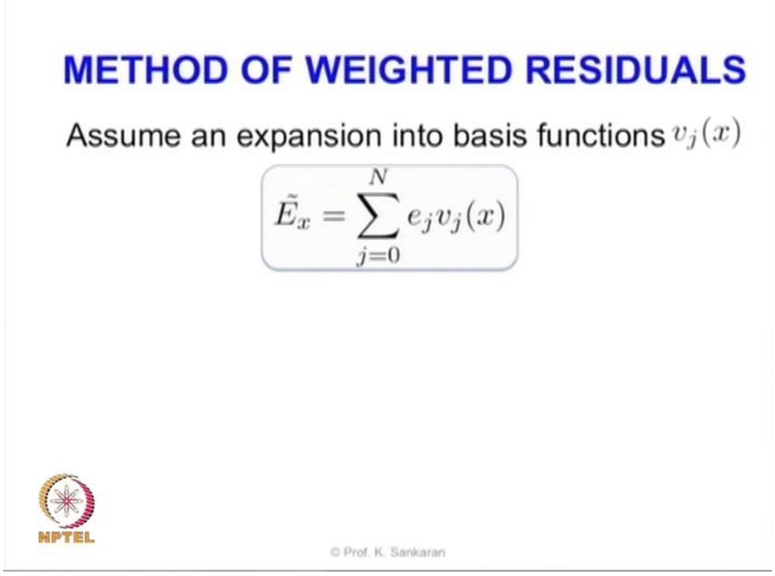

But let us look at it one more time. So we have certain set of basis functions which we call it as v j and we have certain coefficients we call them as expansion coefficients as e j. So we are assuming that the value of E x will become closer to the value of E x when the number capital N reaches a very large number. In other words when the value of capital N here is quite large.

So we have more number of basis functions and also more number of expansion coefficients and then this E x tilde will become closer and closer to E x. But for finite calculations the number will be limited which will create the errors that we have taken into account. So here the basic idea of E x is clear from the basic expansion and basis functions.

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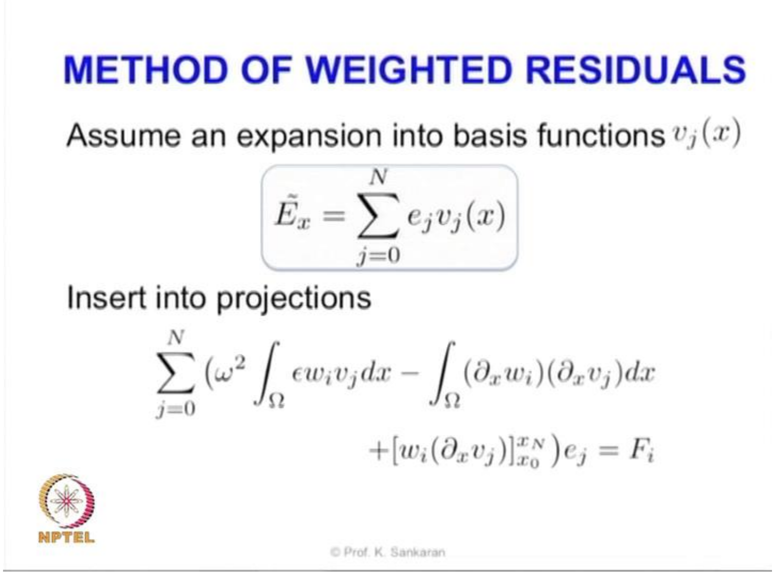

And now we are going to insert this E x into this particular expression. So wherever there are E x tilde we are going to substitute the m with this particular expansion and that will lead to this particular equation.

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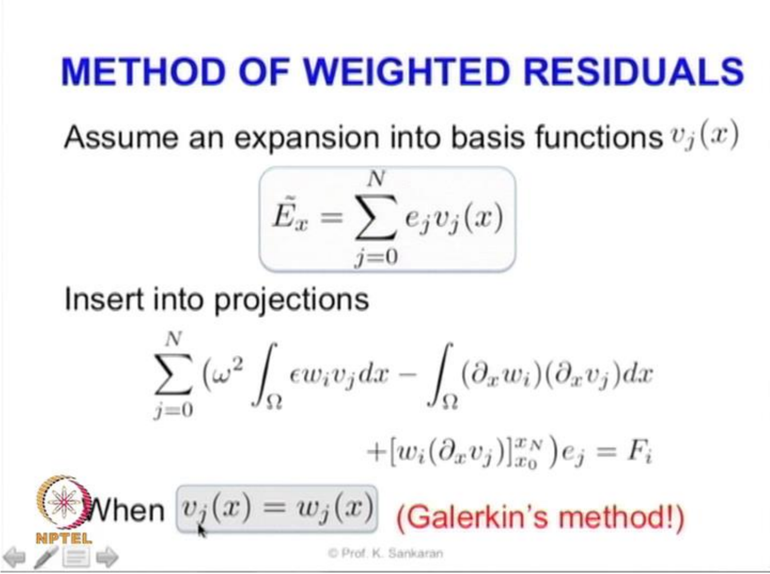

We saw in our earlier class that we can basically take out the summation out and the operator regardless if it is differential operator or integral operator so they can be moved in and out. So here what we have done is we have taken out the summation out and the rest are the same as before and of course the boundary condition are still to be applied and then the right hand side has this forcing function.

Now when we apply the method of Galerkin remember the method of Galerkin is due to the great Persian mathematician Galerkin who has set a lot of new techniques in numerical methods.

One of the methods is the Galerkin finite element method where he has assigned the weighting function and the basis function to be the same.

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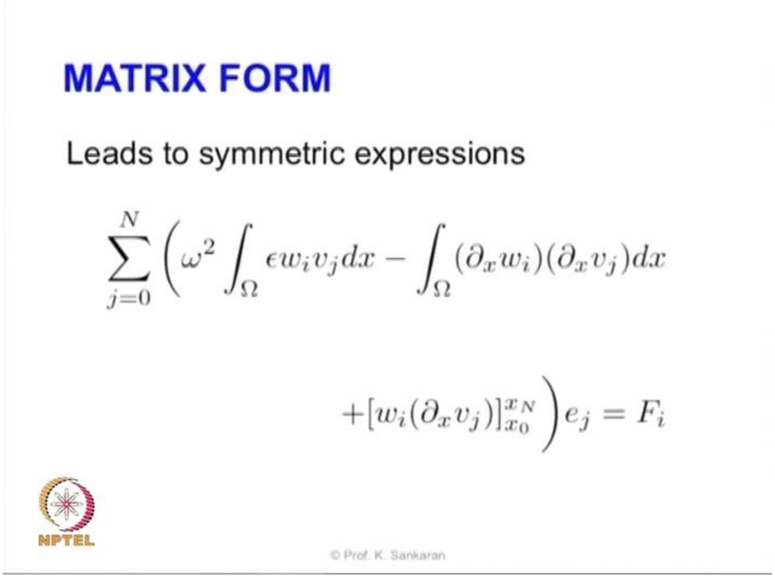

And then this particular expression will get further simplified where we will get a form that is without w i but only in terms of v i.

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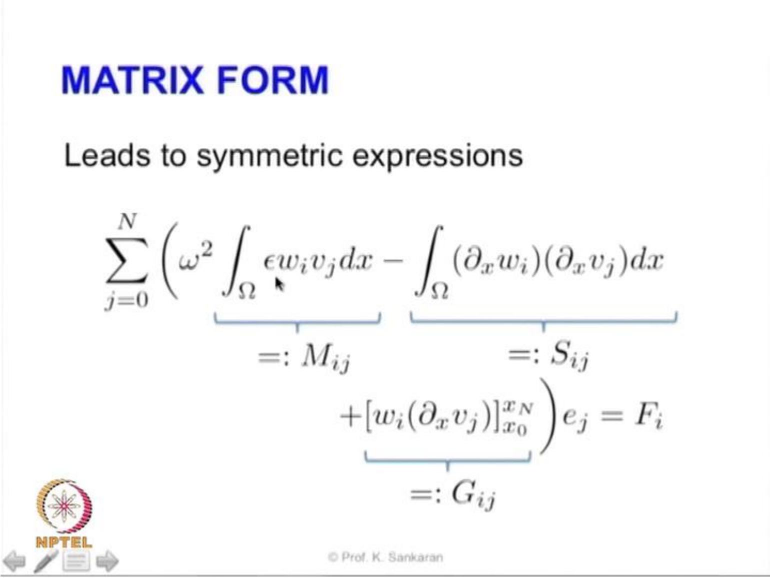

So in this particular expression there are certain terms that we need to look into. The first term is consisting of the material parameter Epsilon so it is called as the material matrix. The second term consist of only of the waiting and the basis function and their derivative with respect to the spatial derivative. So it is called as the stiffness matrix. And the last term here is due to the boundary condition which we have written as g matrix. The entire expression can be written in a much compact form. If we see the equation this expression can be written in matrix form because we have the summation here. So we will haven expression of this form.

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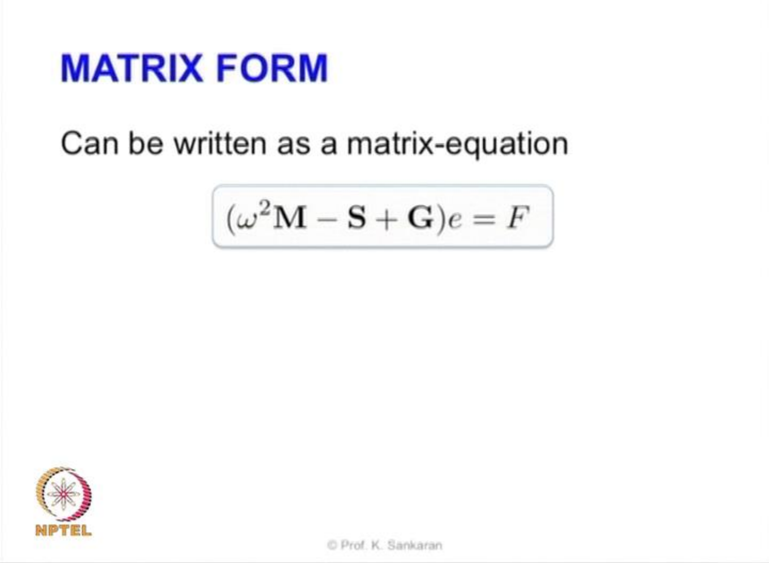

So the matrix the material matrix will be M minus the stiffness matrix plus the G and e is the matrix where all the values of the electric field is going to be there.

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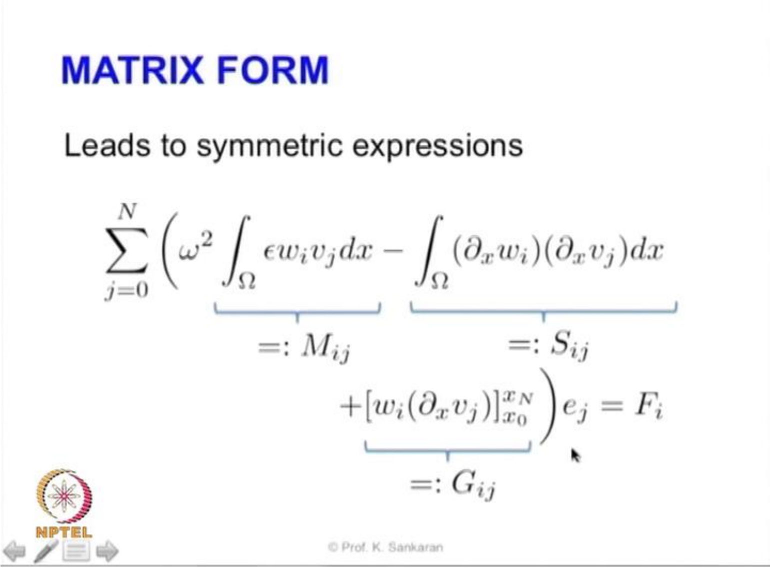

So I have taken out the electric field as a function of j. So all the e j are the coefficient matrix. e j are going to be the expansion coefficients.

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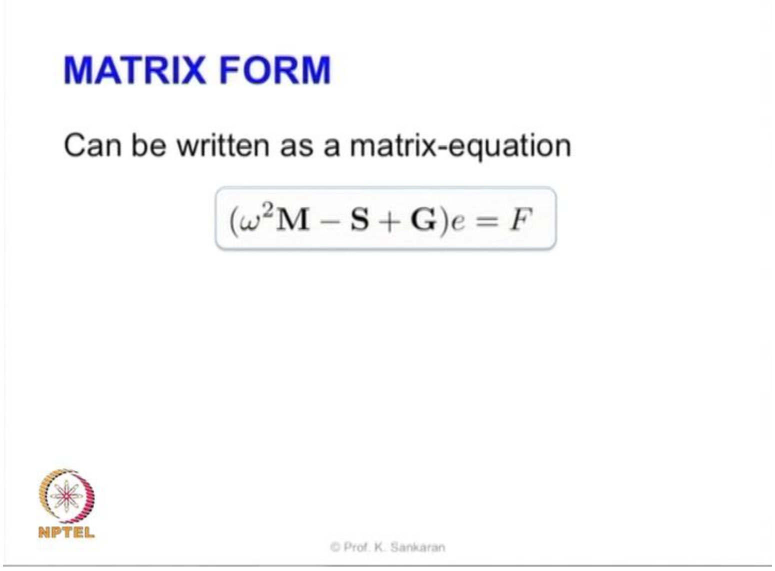

So the entire equation the unknowns are es we know the f and we know the matrix, equation which are represented by M, S and G.

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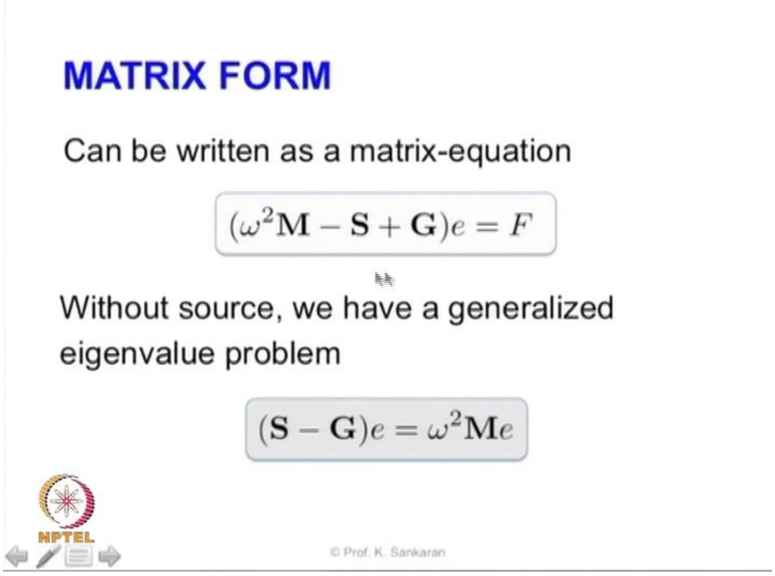

So when we do not have the source F. the entire equation can be simplified into a simpler form where we are bringing the S,M,G on one side and we have the M on the other side. So we have got to the point where we have got the entire expression for the finite element in Matrix form. In the next module we will look into the formulation itself. The devil is in the detail so we have to look into the entire formulation how the matrix are formed and how one can take the idea of a single domain from the entire domain.

So that will be the focus of the next module we will come and look at the formulation in the next module. Thank you!