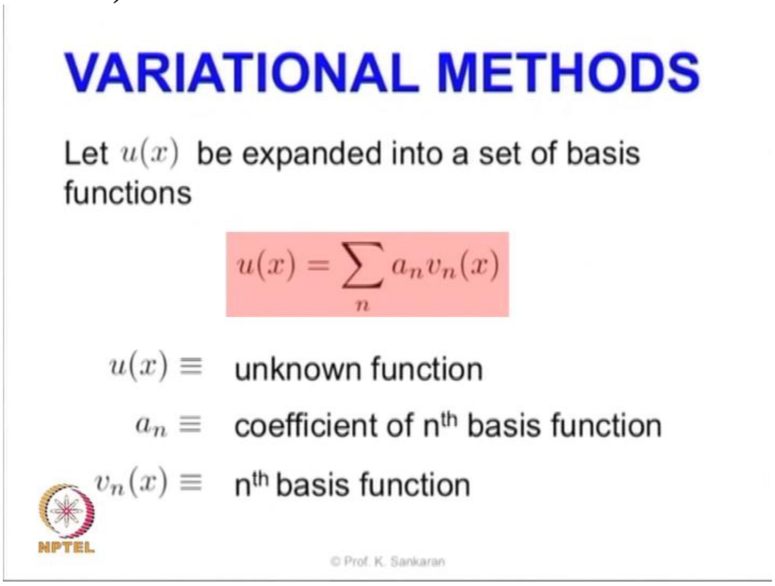
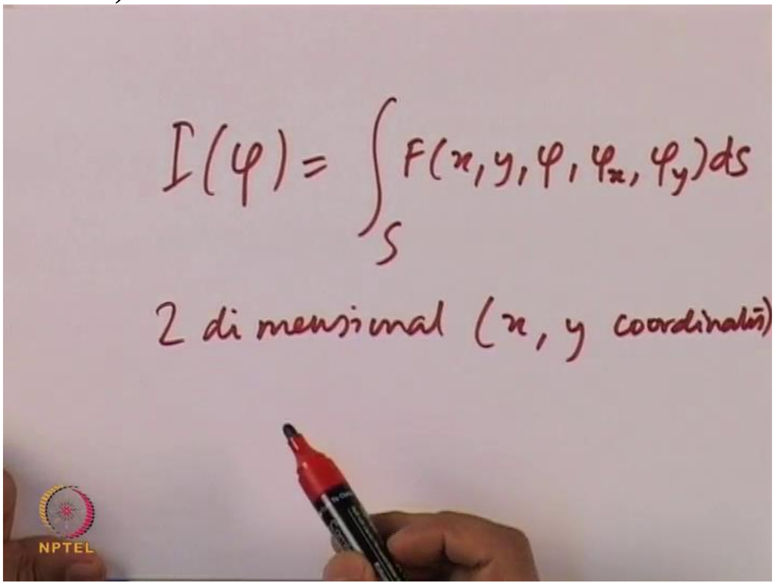
Computational Electromagnetics and Applications
Professor Krish Sankaran
Indian Institute of Technology Bombay
Lecture No.17
Variational Methods

(Refer Slide Time: 00: 33)



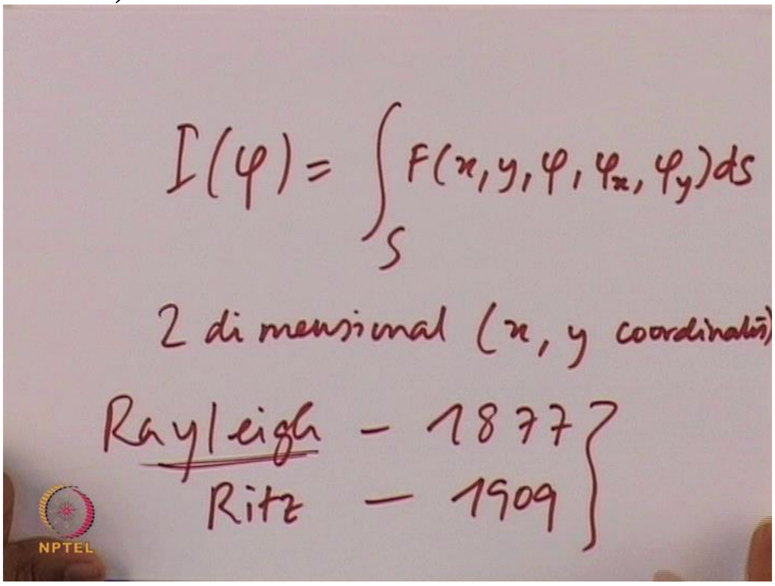
So we look into the basics of ironing the action principle or the functional unknown that we are looking into. And also we looked into how to expand an unknown function using a set of basis function and coefficients. Now what we will do is we will look into the one of the most widely known direct method of applying the variational principle, the method of Rayleigh Ritz. So what we are going to do now is the Rayleigh Ritz method directly starts with the action principle or the functional itself. So do not need to go into the aspects of doing variation and things of that sort. But we directly start with the action principle and then we see how to derive at the solution using the Rayleigh Ritz method.

(Refer Slide Time: 01: 10)



So let us say we are interested in the function for which we know the action principle is given by the equation

(Refer Slide Time: 01: 19)

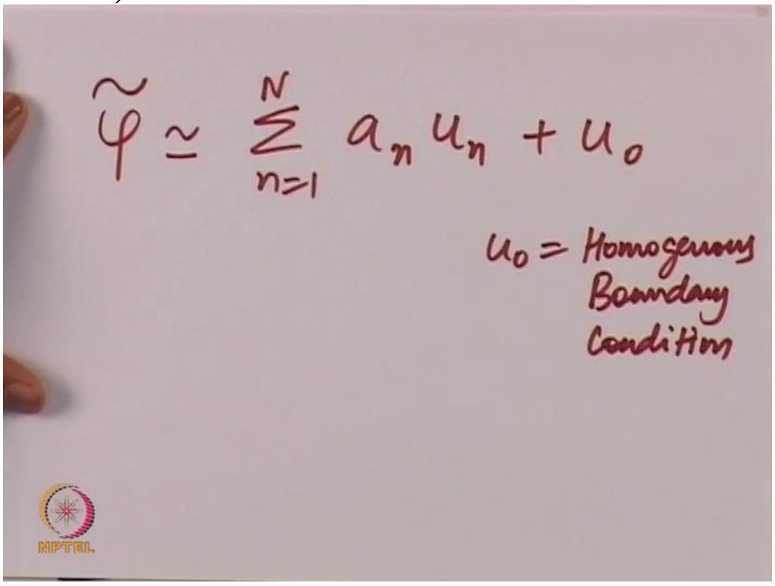


I(Phi) is equal to over some domain S F(x,y,Phi, Phi x , Phi y) ds. So here we are having a two dimensional problem so we take the domain as a two dimensional domain. So we have x and y coordinates. In the previous examples y was dependent variable on x but here in this case we are talking about situation where we have x and y coordinate. So the unknown function which is Phi here varies a s a function of x and y and what we have is an action principle or the variational principle that is a function of x, y, phi and the partial differentiation with respect to x and y and it is given by this integral equation.

Our objective here is to minimize this particular integral. So what we do is we directly start with the method of Rayleigh Ritz. The method was first presented by Rayleigh in the year

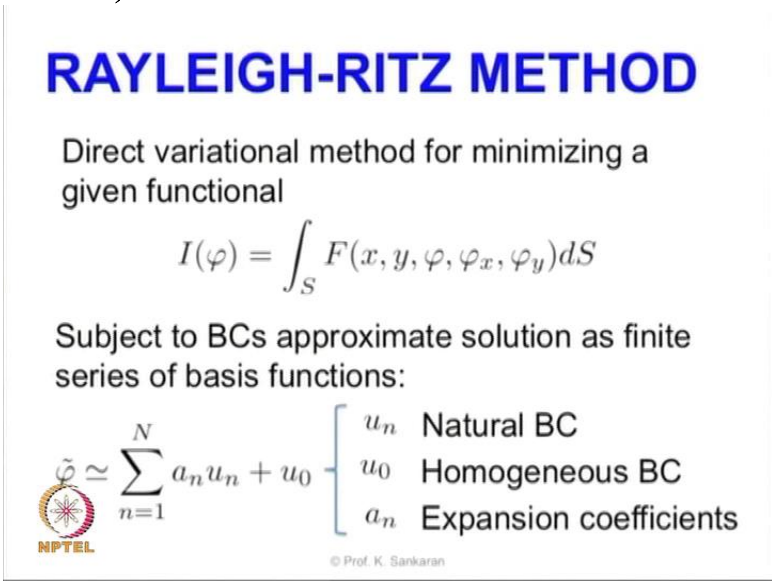
1877 and then Ritz actually expanded or extended this method in 1909. So this is a little bit historical note on this particular method. The person who was very primely responsible for this was Rayleigh and then Ritz expanded this. And what we are going to do is what we have said before our objective is to minimize this integral and we are going to start with the process now.

(Refer Slide Time: 03: 36)



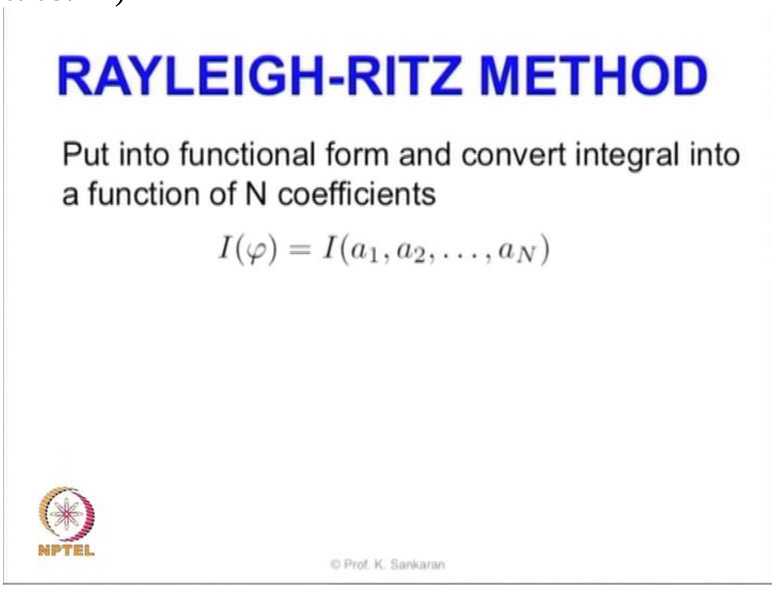
So what we will is we will say let us say this unknown we are calling as Phi can be written as a set of basis function. So Phi is equal to Sigma n equal to 1 to n a n u n plus some boundary values which is represented by u 0. So here we say it is approximately equal to our N is limited. If N goes to infinity it will be equal but since we have a limited number of basis functions and coefficients we say it is approximately equal to. So what we are doing here is in fact a simplification and we say this is equal to not the actual Phi itself but Phi with a tilde on the top. So here we have u 0 is equal to the homogeneous boundary conditions and u n and a n are the set of basis function and the expansion coefficients. So with that being said let us say we are going to take this particular thing forward.

(Refer Slide Time: 05: 05)



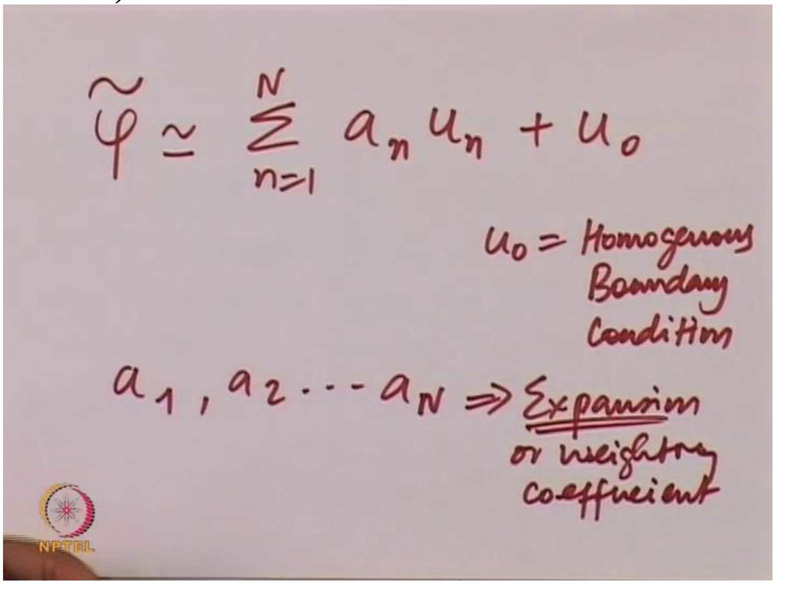
So let us look into the slide, so what we have is this is something that we have expanded. So now what we do is we are putting this into a functional form and convert the integral into a function of N coefficients.

(Refer Slide Time: 05: 22)



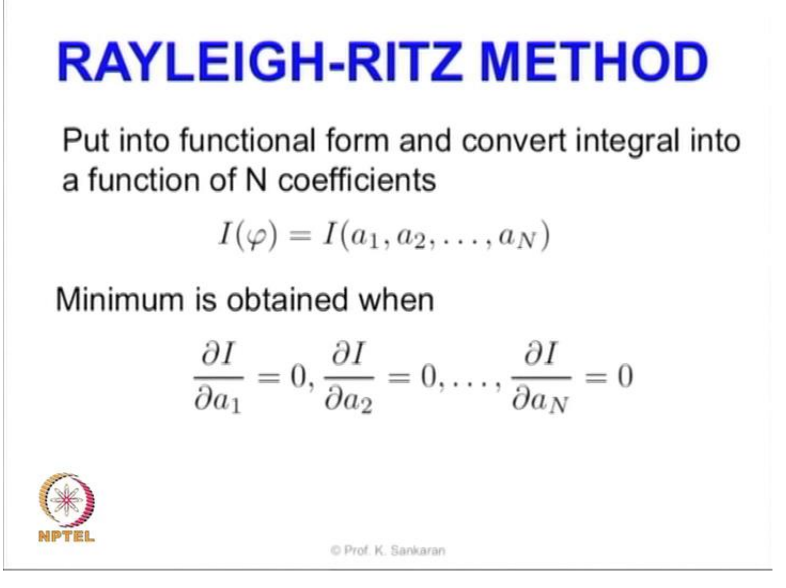
What we do here is we say I(Phi) is equal to I(a1,a2,a3.........so on and so forth until aN)

(Refer Slide Time: 05: 35)



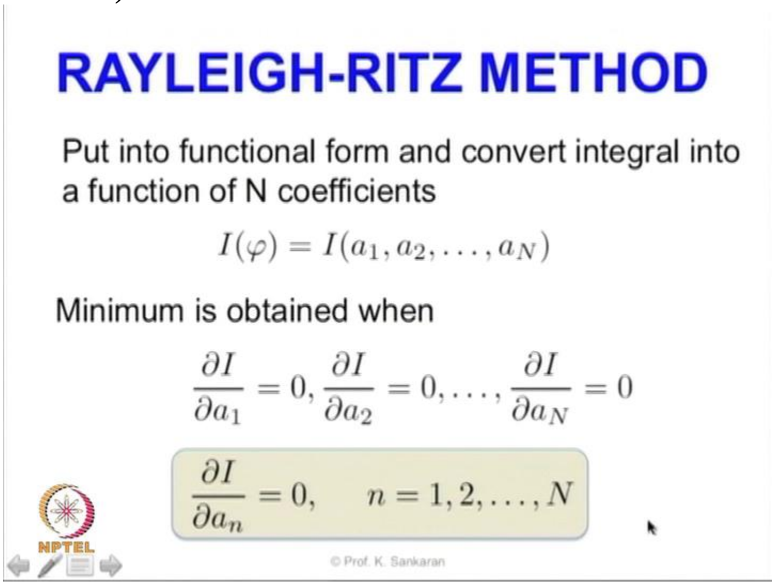
So what is here a means so this a1, a2, so on and so forth until aN they are nothing but the expansion or weighting coefficients. So I tend to call them expansion coefficients because they are used here as a variable we have set here in this particular equation.

(Refer Slide Time: 06: 07)



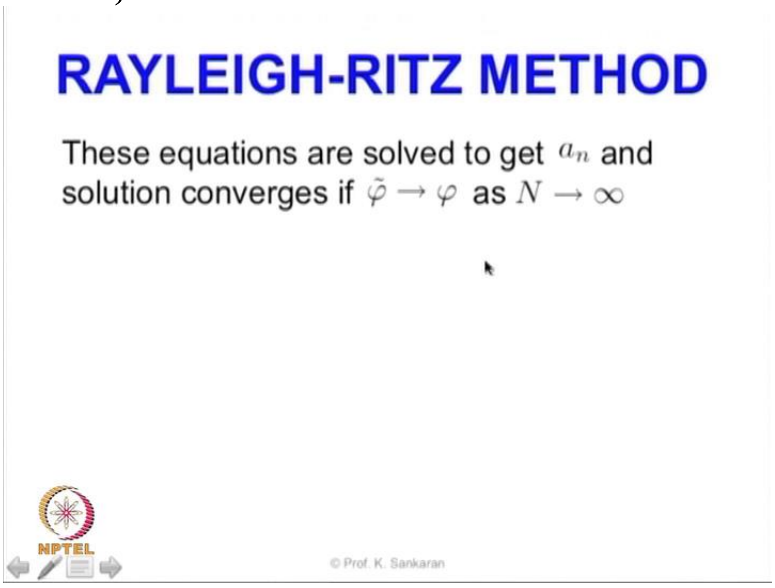
So what we get now is we are getting an equation si instead of being a function of Phi xy partial differentiation with respect to ay we simply say it is going to be a function of those expansion coefficients only.

(Refer Slide Time: 06: 22)



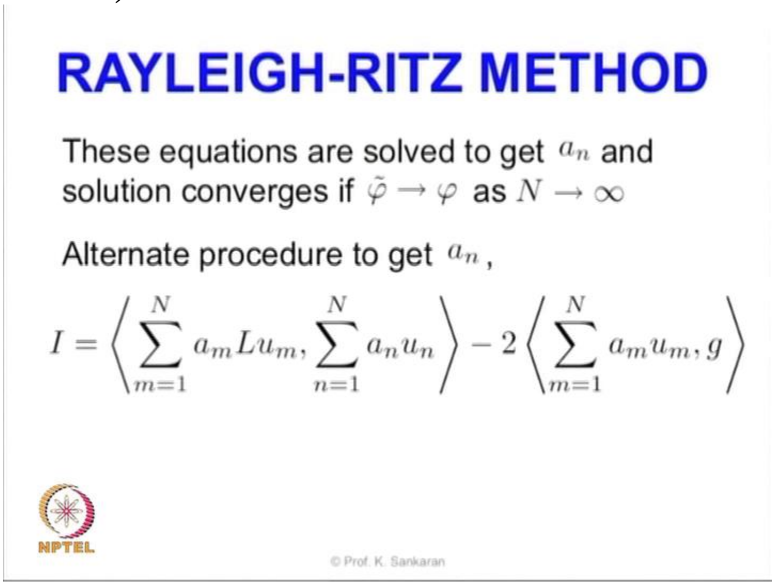
And now wecan say the minimum is obtained when this particular integral when they are partially differentiated with those expansion coefficients we get the value as 0. So for example we say in general we can write this particular expression as the partial differentiation of I with respect to a n is equal to 0, for n goes from 1 to N.

(Refer Slide Time: 06: 52)



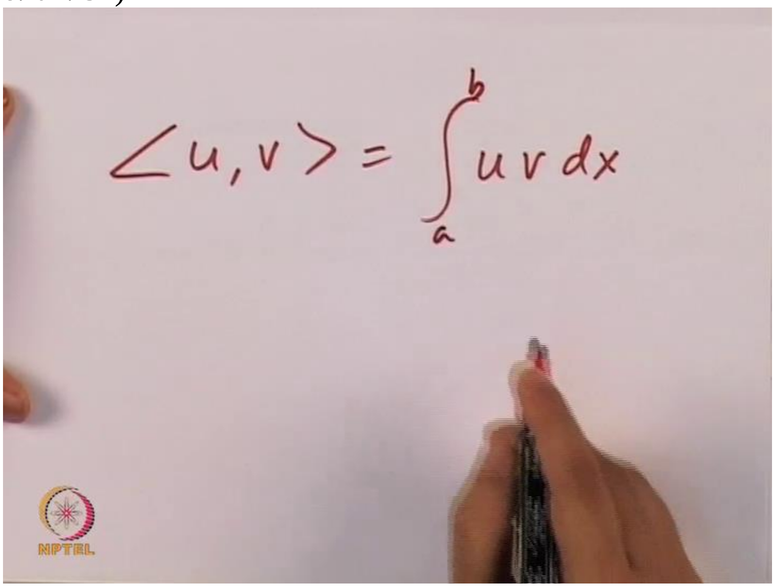
So what is happening in a sense here is the solution is going to converge to the actual solution itself when N goes to infinity. So that means we will have a n which goes from N to very large number of a n.

(Refer Slide Time: 07: 13)



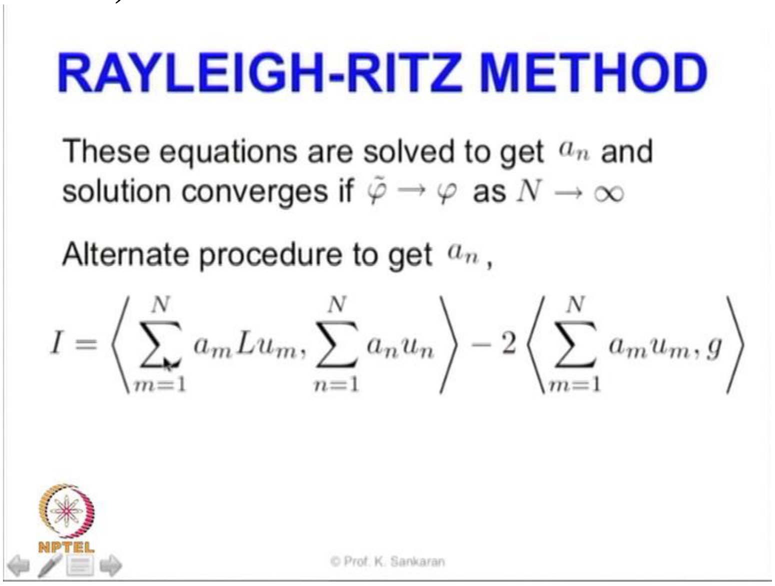
So instead of going this path we can say we can start with the different path what we can do is we can start with a different path by directly putting the value of the functional in this form. Remember we said this is a inner product term here, and this inner product term has variable u, v. remember that we introduced when we introduced the inner product we said the inner product has the integral of the two functions.

(Refer Slide Time: 07: 52)



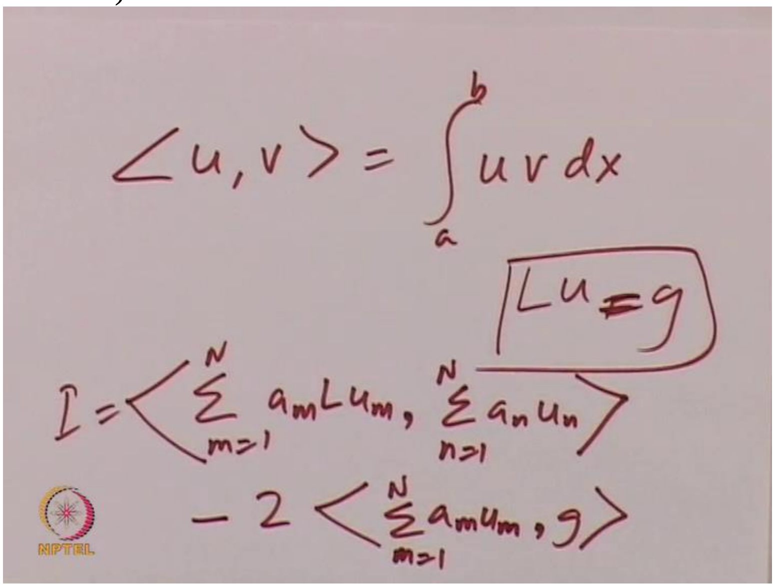
So we said u,v which is the inner product for two functions u and v is equal to integral u vmultiplied by dx. So for a problem where it is a one dimensional problem a to b so we have defined this as the inner product.

(Refer Slide Time: 08: 17)



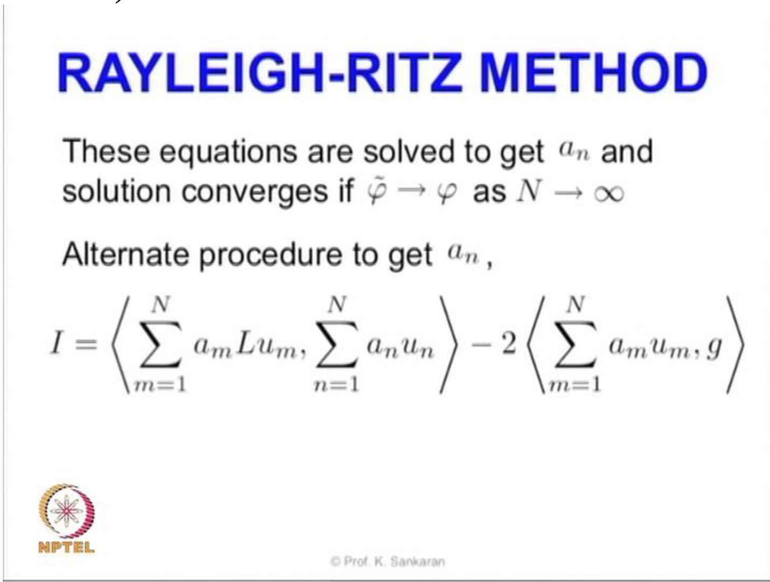
So similar way so here in this particular expression what we have is variable which is given by this particular term is the u function and this particular term is a v function. Assume that this entire thing is a u, this entire thing is v minus 2 times a particular variable, g. So this is an alternative approach where we can directly go to the action principle or value of i with this particular expression. And this particular expression is something that we need to look more deeply because you see here there are two particular summation terms minus 2 times a particular summation term, g.

(Refer Slide Time: 09: 12)



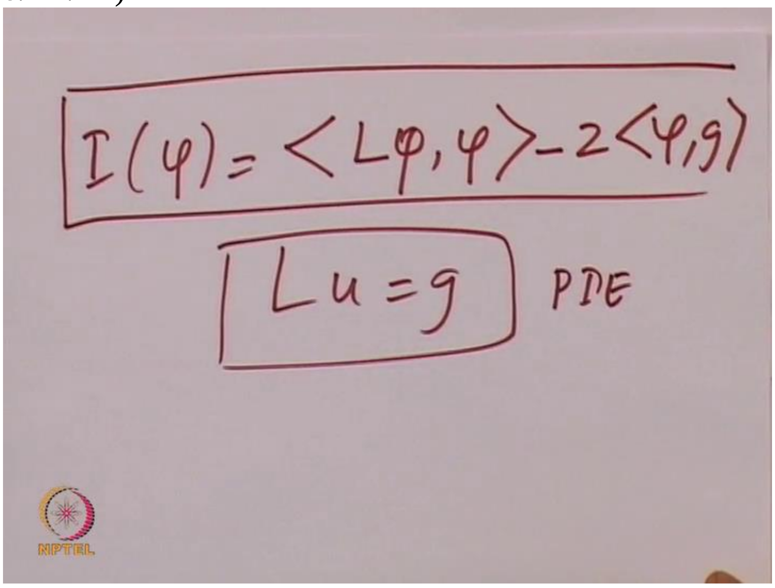
Remember so we had this equation where L u is equal to g is a initial expression. And the term what we are having on the right hand side is a summation term that goes from m equal to 1 to N a m L u m, we have a second summation term that goes from n equal to 1 to N a n u n minus 2 times again in a product sigma n equal to 1 to N a m u m , g. So what we have got now is nothing but a simplified form of writing the action principle or the variational principle which is due to the paper of (())(10: 24) where we say that the action principle itself can be written as the inner product.

(Refer Slide Time: 10: 31)



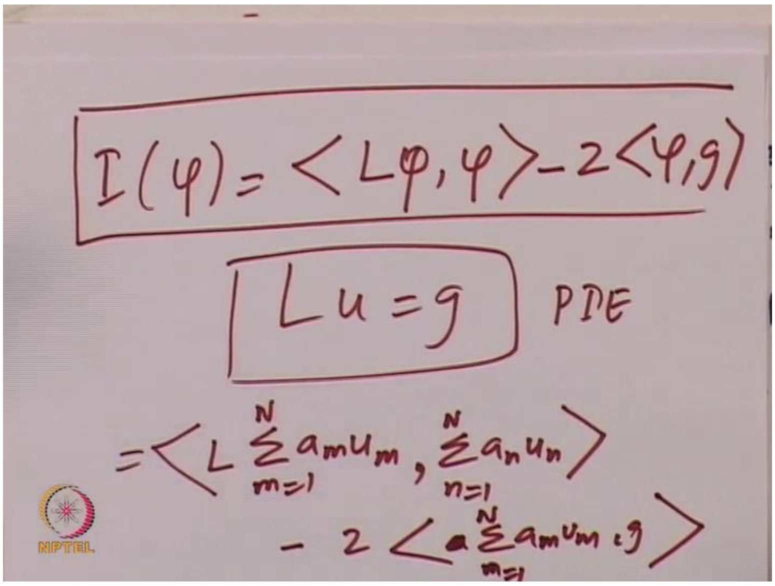
So we can write the action principle itself is equal to <L Phi , Phi> minus 2 Phi , g> so if we have an equation where L u equal to g. So this is our partial differential equation. The action principle for this particular PDE will have the form given here.

(Refer Slide Time: 11: 14)



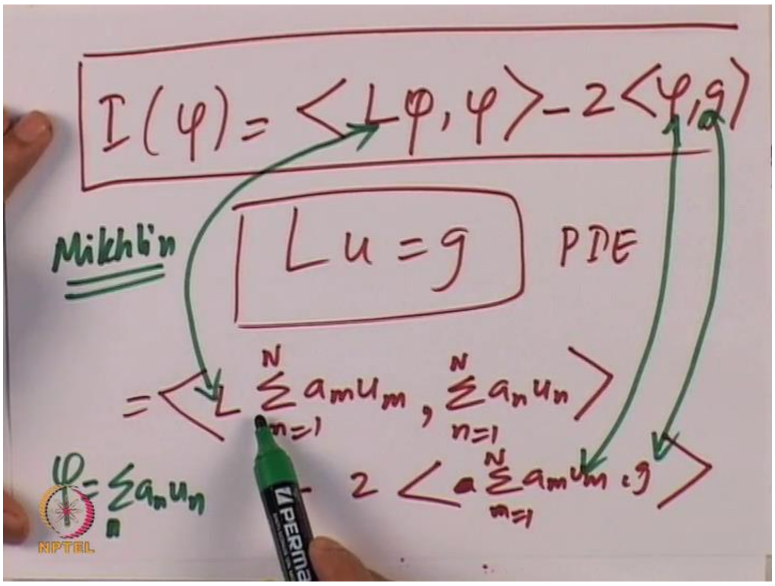
If you see this particular form you will see whatever we have got in this particular expression is very similar, so we have got some function which is given here. So I am going to rewrite this particular expression in the form that we have got now then it will become clear to you.

(Refer Slide Time: 11: 31)



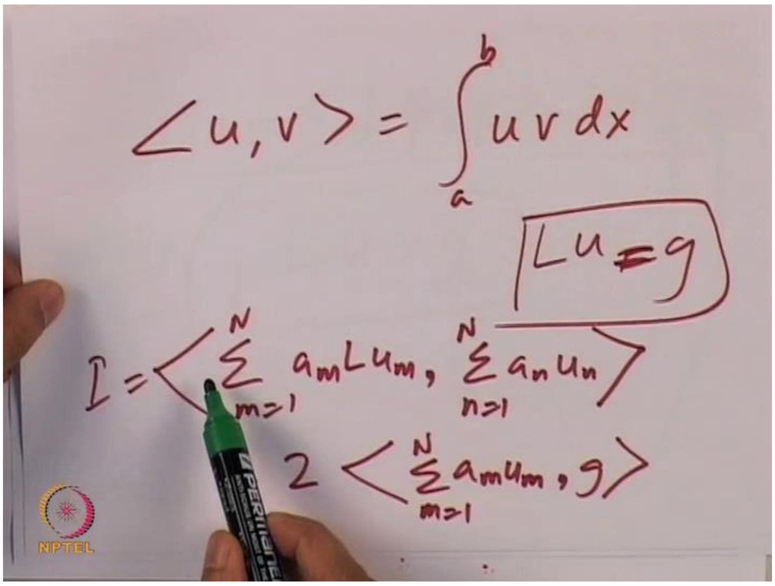
So what we have got is < L sigma m equal to 1 to N a m u m , sigma n equal to 1 to N a n u n minus two times in a product of summation a m u m so here m goes to 1 to m, g>.

(Refer Slide Time: 12: 28)



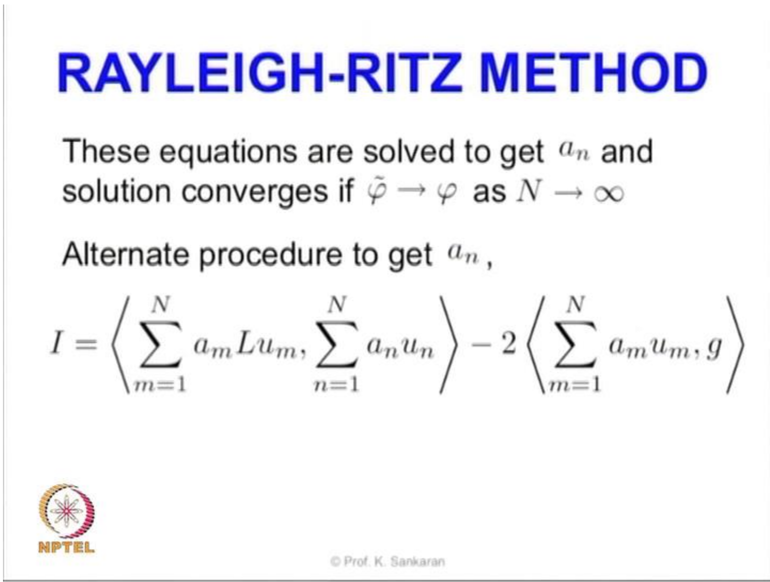
So what you see here is you have this L which is here you have the g which is here and you have the Phi which ishere and so these are the ways in which we have expanded our Phi. Remember we said the Phi is equal to summation over some n a n u n. That is what we have here. So what we have got now is we have directly deriving the actiion principle for a particular PDE knowing that this particular thing will be the action principle or the functional so it is due to the paper of Mikhlin who said that for a PDE we will have this is the form and then this form is enabling us to expand it. So what we have done is we have taken this particular form. We assume that the unknown function Phi is equal to over a particular domain we have summation over a particular number of basis functions and coefficients and we have substituted it. And obviously you can take this L inside the summation. Once you take it inside the summation what you will arrive is this particular form.

(Refer Slide Time: 13: 51)



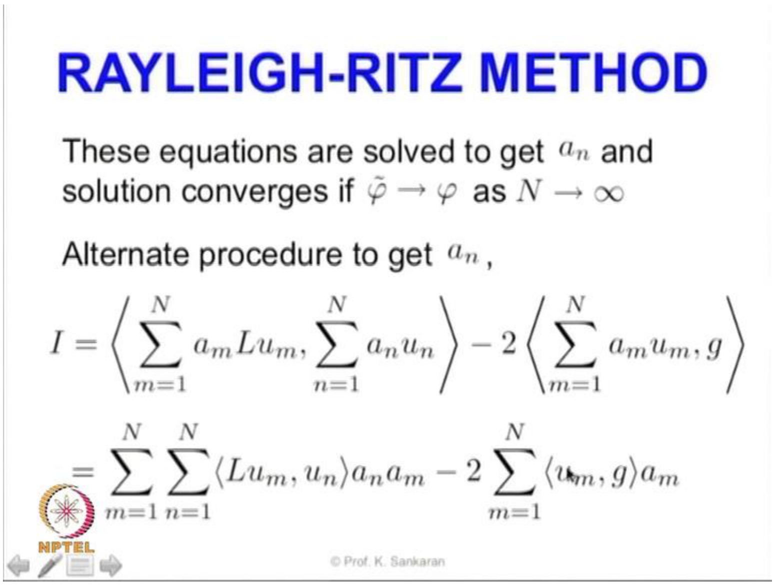
So I have taken the L from here I have taken it inside and the rest of the things are the same. So how I arrived to this particular expression is starting from here.

(Refer Slide Time: 14: 03)



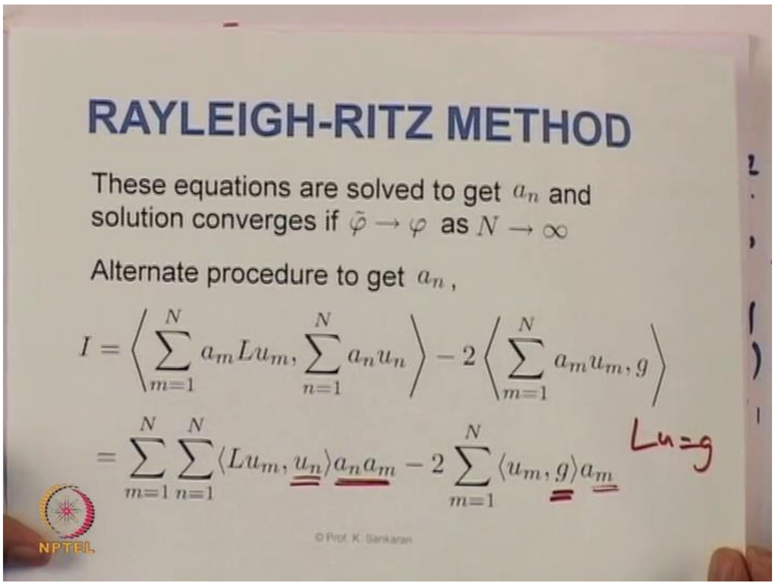
So with this let us look into the expression itself as a summary one more time. So what we have got here is this particular expression and now we know how this expression came into play. And now what I am going to do is I am going to take both of these summations outside which I can do.

(Refer Slide Time: 14: 21)



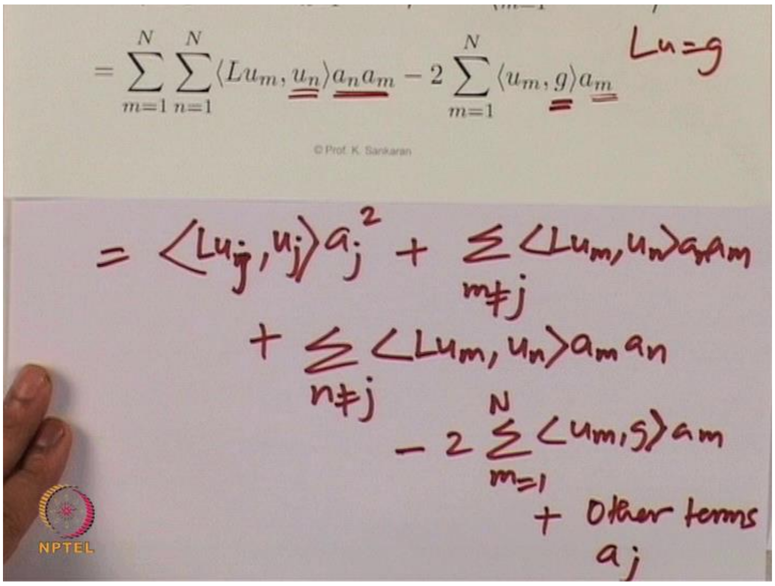
So what I am going to do is I have taken this summation which is going from m equal to 1 to N, and the second summation n equal to 1 to N completely outside and I kept this particular in a product which is L u m and I have taken out the coefficients expansion coefficients outside the inner product. So basically the inner product consists of L u m , u n minus the same way I have done is I have taken out this particular expansion coefficient out of the inner product and the inner product is only here as a function of u m and g. So what is important to notice here is i have got an inner product in a particular form. The reason why I have got it in this particular form is this particular inner product has certain convenience.

(Refer Slide Time: 15: 22)



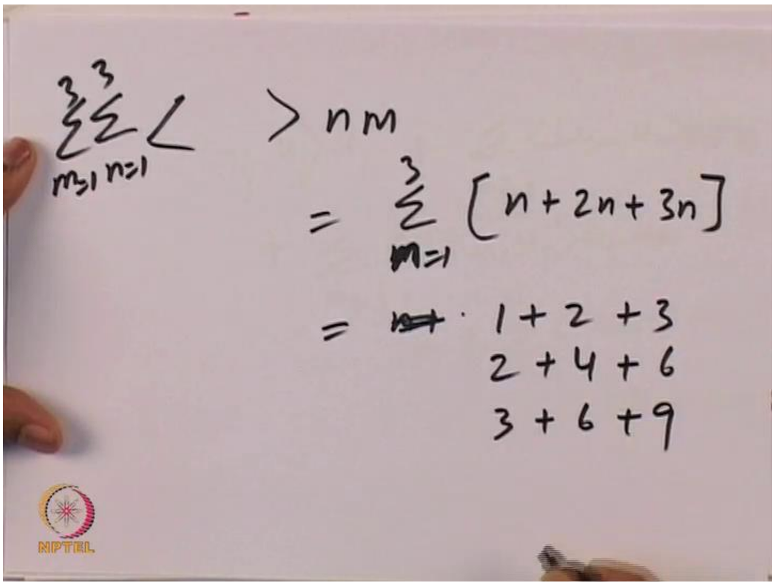
So the convenience is I know the g which I have given as the forcing function.Remember L u is equal to g. and g is known. So I know what will be the outcome of this because in a product is nothing but the projection of one particular function or the other and similarly here I have got the inner product on certain basis function about which I also know something and these are the unknowns that i need to compute so the unknowns are the a terms so the inner product consists of only the term which I already know. So the reason for moving around the summation out of the inner product is to make the inner product in such a way that we have something that we know about the inner product. So with this we have to now somehow expand this particular expression to get some useful form.

(Refer Slide Time: 16: 23)



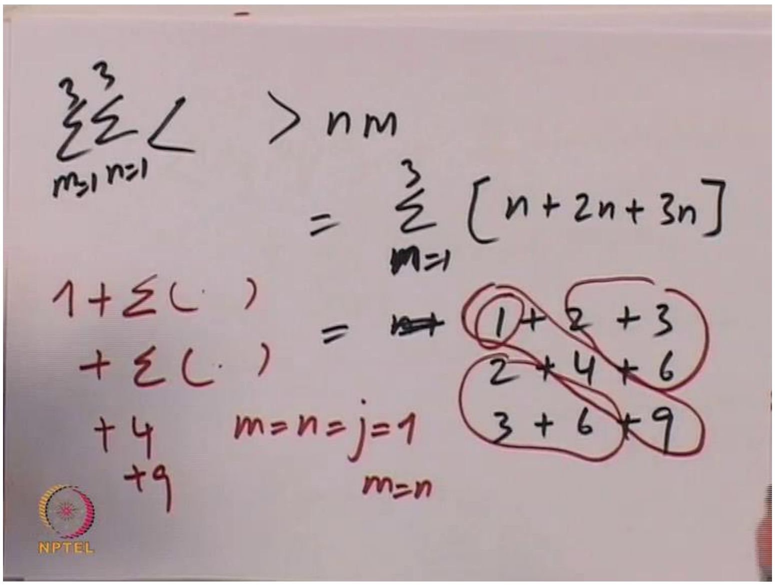
So what I am going to do now is I am going to manipulate this particular equation in such a way that I am able to separatethe a m out and to get an expansion in terms of a m. So what I will do is let us say when the m value and n value are going from 1 to N and I am interested in taking out those terms where a m and a n are equal to the same number. So what I will have is when n and m are equal to the same number. Let us say that number is a j. So when n is equal to m equal to j this term will become ajsquare. So I will have a term which is ajsquare and the inner product will stay the same u m, u j and then I will have the same terms. The other terms when both m and n are not equal to j. So I will have a term sigma m not equal to j < L u m , u n> a m a n plus sigma n not equal to j the same term< L u m , u n> a m a n and then the last term will be the same term which we will have two sigma m equal to 1 to N <u m , g a m>. So we can do this because we can expand it as a function of things. But we will also have higher order terms which is basically plus sigma other terms where there is no aj.

(Refer Slide Time: 18: 56)



So let me explain this what I have done in a very simple form. so what I have now is let us say I have a function which is given by let us say some inner product times <>m,n and I have two integration on it. And let us say this integration goes from m equal 1 to 3 , N equal to 1 to 3. So what I am going to get here is nothing but first summation 1 to 3 I will have [n plus 2 n plus 3n] so this is let us say n. And when I apply second summation what I get is n plus when n is equal to 1, So it willbe 1 plus 2 plus 3, when n equal to 2 it will be 2 plus 4 plus 6, when n equal to 3 it will be 3 plus 6 plus 9.

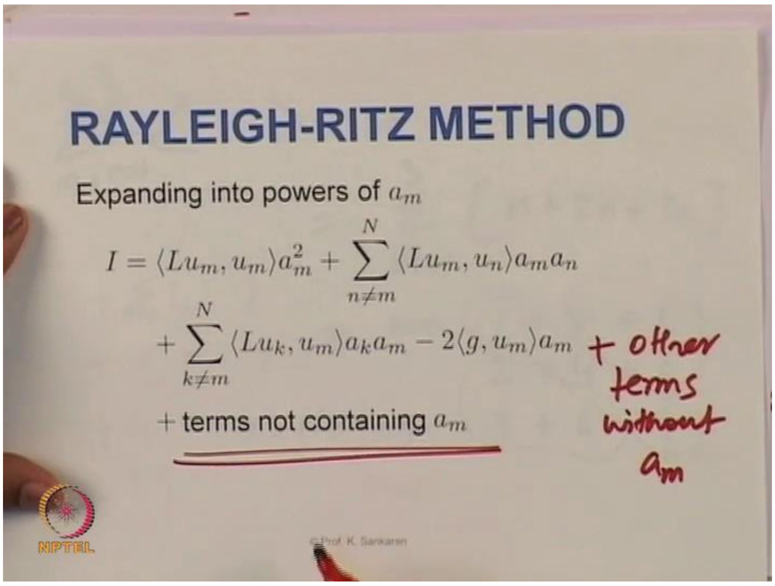
(Refer Slide Time: 20: 04)



And what I am interested is when both m equal to n equal to j that is equal to 1 let me say if that is the case I am interested in only this particular case. Where both m and n is equal to j that is equal to 1. So what I have is all these terms are when m equal to n. And all other terms are when m and n are not equal. So what I can do is I can separate only one of those things where I am interested is that is equal to j and I am bringing them as 1 plus then I have these three terms and then these three terms. So 2,3 ,6 2,3,6. So what I have is I have a summation which has these terms plus this summation and the term. So this 2,3,6 will be here and this other 2,3,6 will be here and those terms which are not consisting of m equal to n equal to j equal to 1 are these 4 and 9, so I will have those other terms 4 plus 9 terms.

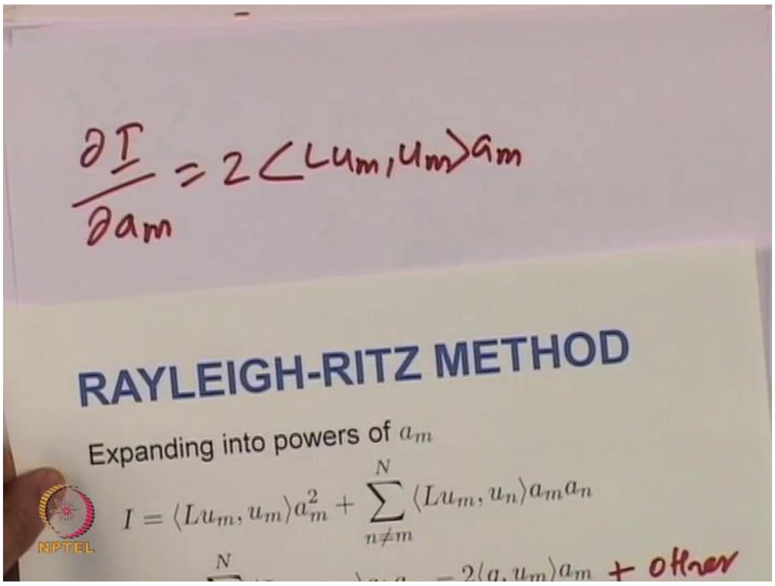
So this is a simple way to understand how I am separating the equation of this form.

(Refer Slide Time: 21: 29)



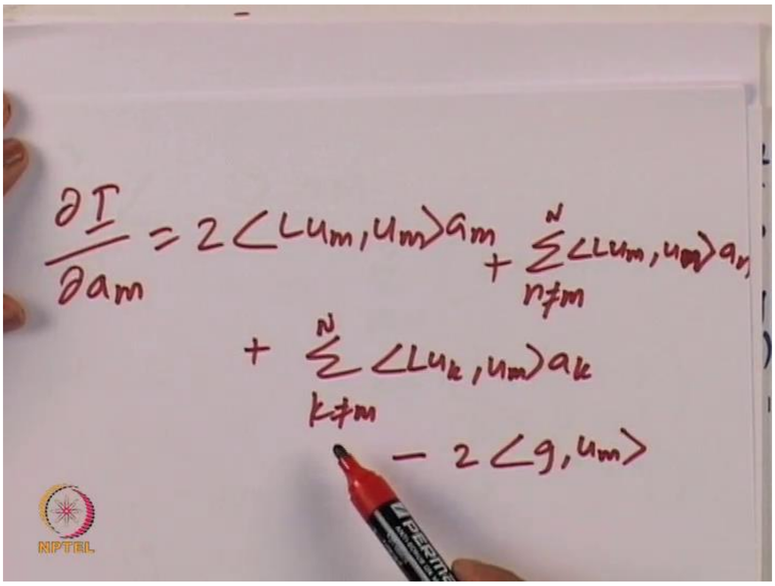
So I said this is the place where both the expressions are equal to 1. So where we have n equal to a particular number and then we have the second summation, third summation minus this and of course we said there will be other terms without a m. So that is we have particularly written here. So you are able to understand how we have come to this particular step starting from here. So what we have got is an expression we have expanded this and we have taken out only one of those conditions where we are interested in. which is a m and we have split them into two summations minus the original thing what we have here.

(Refer Slide Time: 22: 46)



So now we are going to differentiate this particular integral with respect to a m. So what we will get is nothing but let us keep this one here. So what we will get is Doe I by Doe a m is equal to the first term will have this 2<Lu m , u m> a m. So this is the first term contribution.

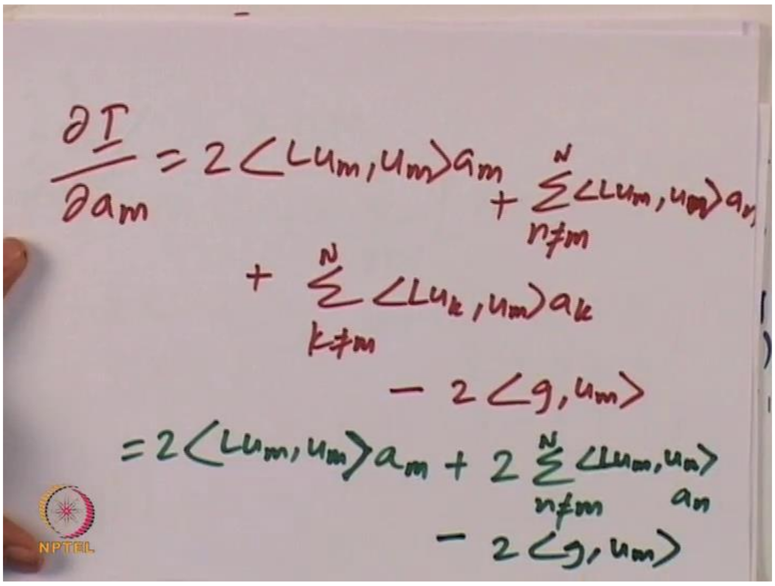
(Refer Slide Time: 23: 13)



The second term contribution will be this whole thing. So it will be plus sigma n not equal to m <Lu m , u n) a n. And then this particular term where we will have the same thing sigma k not equal to m to N the inner product <Lu k , u m > a k, and then the last term will be just this particular thing which is minus 2 < g, u m > here I have written it in the opposite way because this is real number you can swap the order if it is complex then you have to take the complex conjugate but since it is real it is fine to write it like this.

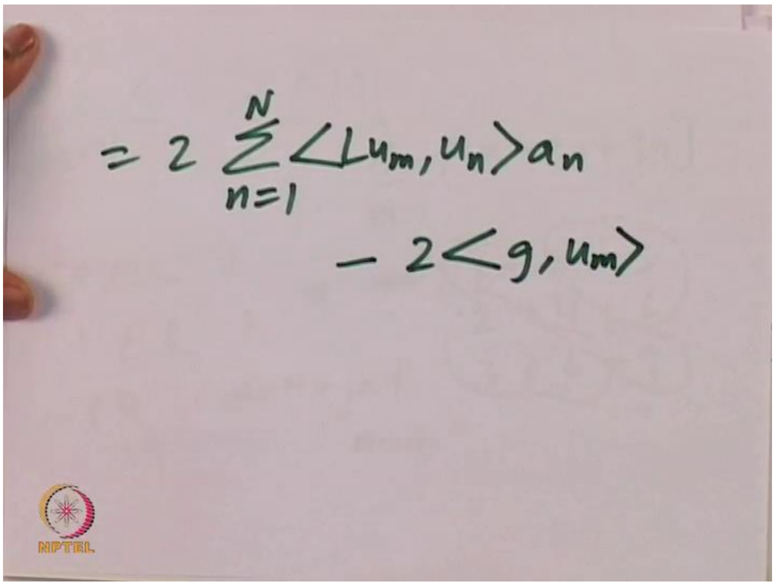
So now you are able to see I have takenm this particular term, I have differentiated it with respect to a m and what we have got is from here we have got to the point where we have got this expression.

(Refer Slide Time: 24: 37)



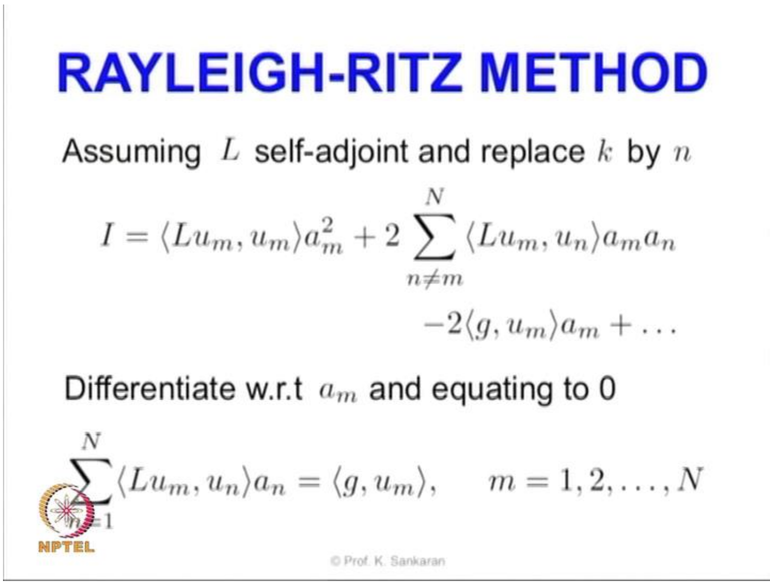
So from here there is something that we need to look into that this two particular integration has something unique about it. So if you swap k to n these two summations are nothing but the same. So these two summations are the same summations except for the fact that they have two different indices so we can write down as the same. So this will be equal to 2 sigma <L u m , u m > plus 2 summation n not equal to m, in both the cases n and k is not equal to m they go to N and then we have <L u m , u n> a n minus 2 <g, u m>. So there is something unique about this particular expression. The first expression contains where a m is there. the second expression contains everything except m. So basically you can join these two and write the entire equation goes from 1 to N. So that is what we are going to do we are going to combine this particular term and this particular term into one summation.

(Refer Slide Time: 26: 19)



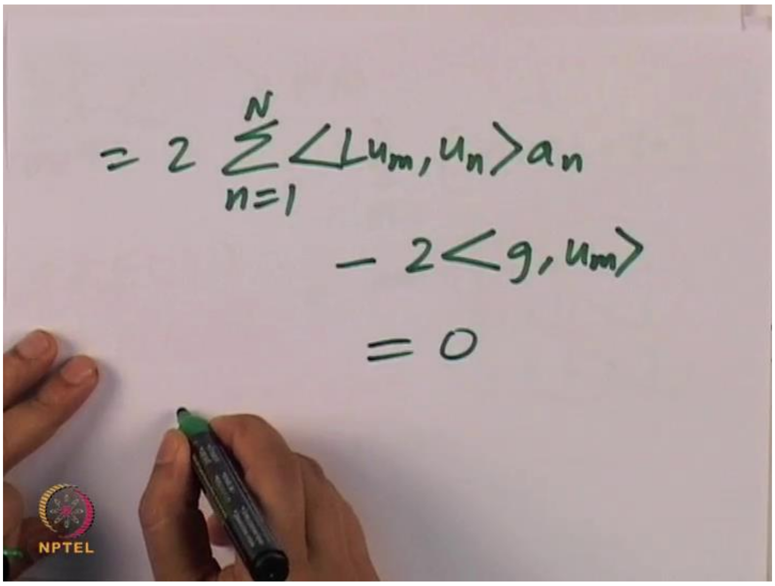
And we will have 2 sigma n goes from 1 to N, u m > a n minus 2 <g, u m >.

(Refer Slide Time: 26: 44)



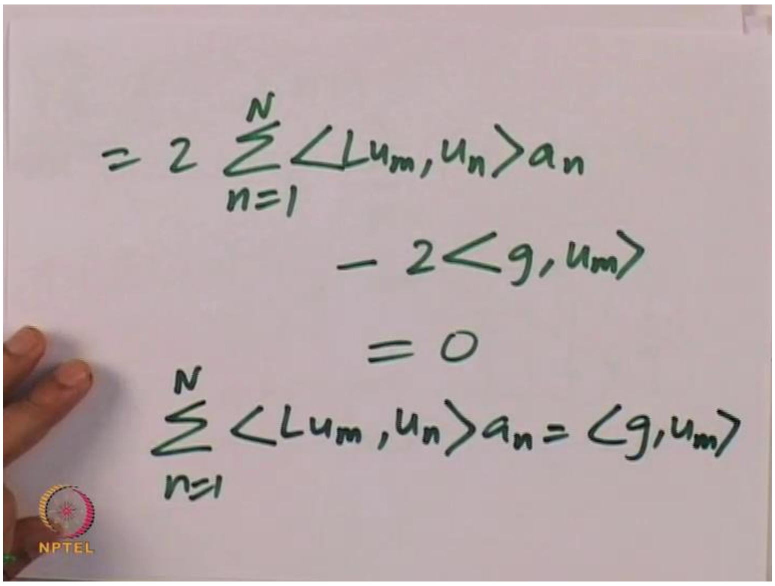
So what we have got is the final expression what we need and this you can revise it one more time in this graph what we have seen is we have differentiated it we got the initial expression we have differentiated the expression with respect to a m and we said this particular thing should be equal to 0 because we said that for that particular integral to be the action principle when we differentiated with respect to any of the coefficients it should be equal to 0. So when we are differentiating with respect to a m it should be equal to 0.

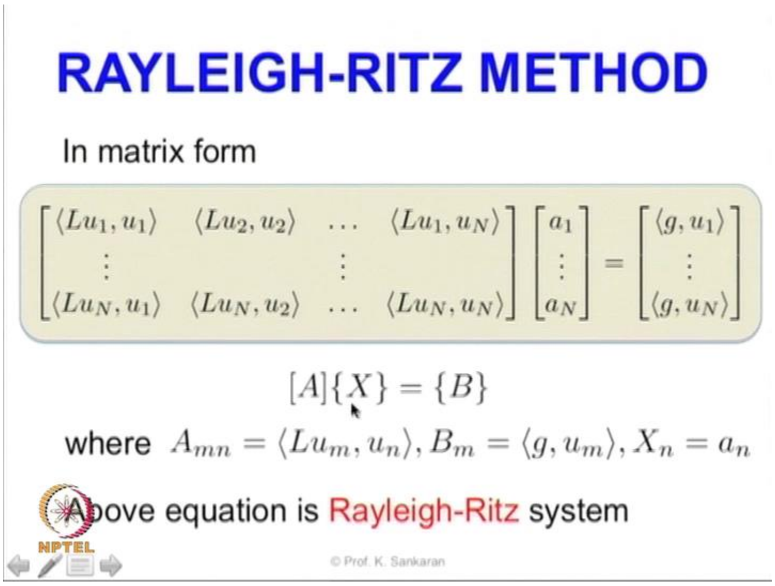
(Refer Slide Time: 27: 18)



So what we will get is this particular expression what we have here should be equal to 0. So when this is equal to 0 this term will become this term 2 2 will get cancelled. So we will have simply the expression n goes to 1 to N < Lu m , u n > a n is equal to g u m. So this is the final expression what you see. And this is what we will be using in the matrix form. So basically you have left hand matrix and right hand matrix. And the left hand side you have a matrix multiplied by a vector a n is a vector. And on the right hand side you have also a vector.

(Refer Slide Time: 28: 03)





So in the matrix form we can write them a matrix multiplied by a vector and is equal to another vector. So this is the basis for Rayleigh Ritz method and the above system is a Rayleigh Ritz system where we have arrived at the final expression starting with the assumption that we have done so far.

We have learned how to derive the Rayleigh Ritz equation starting from the action principle itself or the variation principle itself and we have used the simple expression given by Mikhlin which gives the value for the variation principle and we are able to use the logic of expansion functions and the coefficient functions whatever we have described to get the final expression from the system in the form AX is equal to B where we have got the matrix for the A which is the matrix that we have already known and we have the x that is alist of unknowns which are basically the expansion coefficients and the another vector which is basically given by the initial forcing function g.

We have learnt a lot of mathematics has been into your thoughts now so I would not take it so for now. in the coming modules we will pretty much use the ideas we have discussed so far to learn the method of weighted residuals and also the Galerkin method.

With that being said let us get back in next module until then see you. Bye Bye!