Computational Electromagnetics and Applications
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Indian Institute of Technology Bombay
Lecture No 16
Variational Methods

So we looked into some of the motivations background and also some physical understanding behind what is the meaning of action or functional that is going to minimize or maximize integral.

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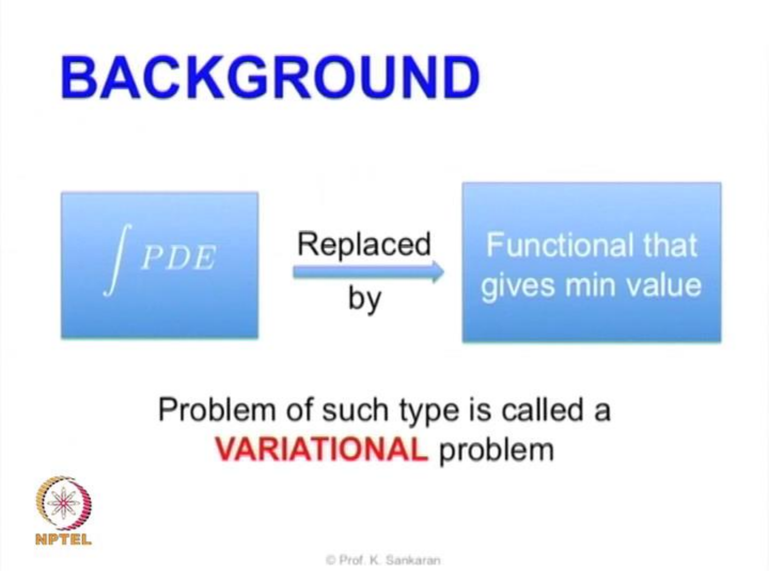

So we will now look into an example to get more understanding into what this functional is how it behaves. And how can we look into it more deeply.

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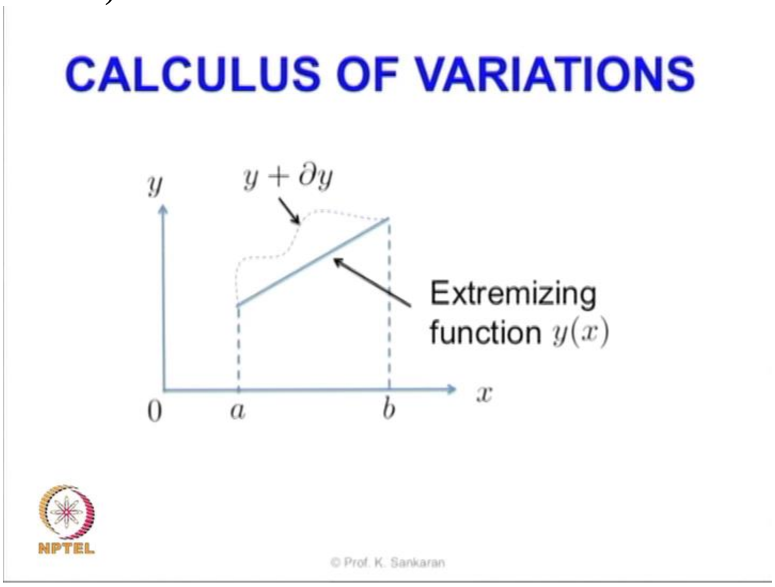

So with that starting step let us take the particular example where we have a function, let us say y which we are interested in finding is going to vary as a function of let us say x. So x is the independent variable here and y is the dependent variable. And the starting point and ending points are fixed. So let us say x equals to a and x equal to b are the starting and the ending points. And then let us assume that the function is going to change by a small amount.

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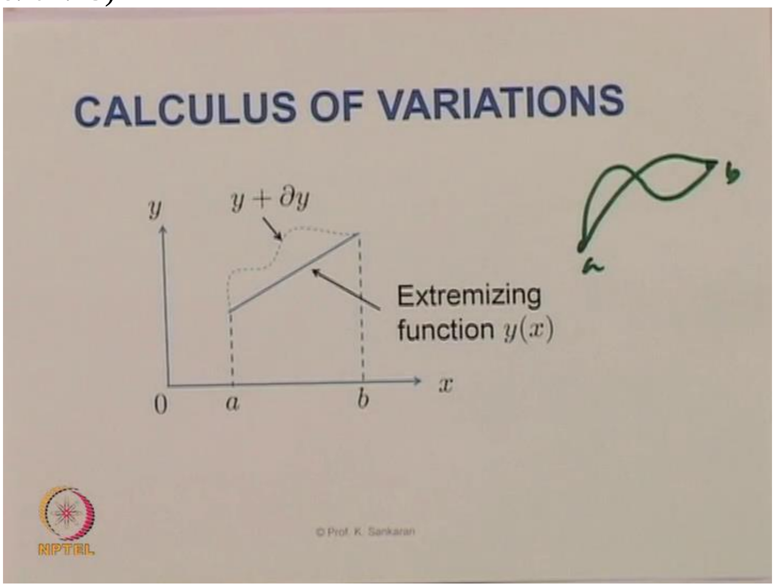

So what we are saying here is let us say this is the real function we have , and then we say that there is going to be a small change in the functions not in the starting and ending point but in the intermediate point. Remember in our old example where we talked about throwing a stone from point a to point b. So we set the actual path and we also talked about other paths. So the starting point and the ending points are always fixed. And we talked about the variation in this particular function.So this is something similar. So what we are doing here is we keep the starting point and ending point to be fixed. But we say that the function itself is going to change.

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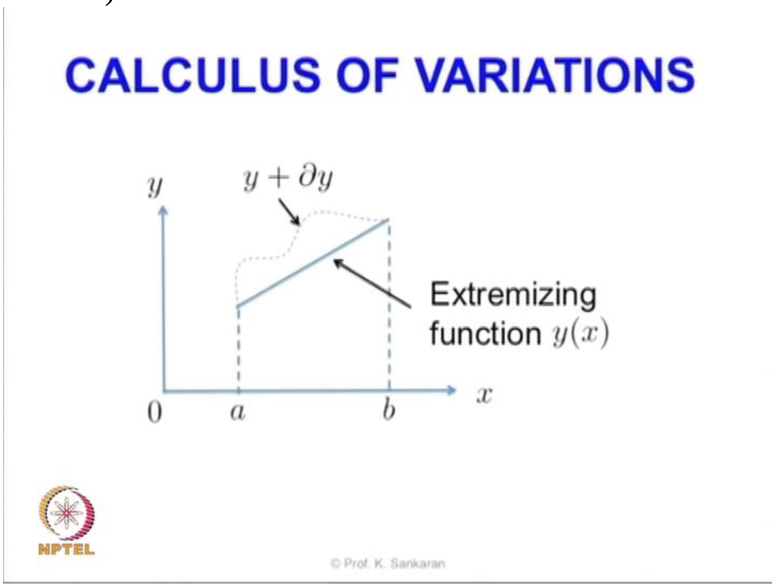

So as I told you in the earlier part of the description about calculus of variation. We are interested in the necessary condition for a functional to achieve a stationary value.

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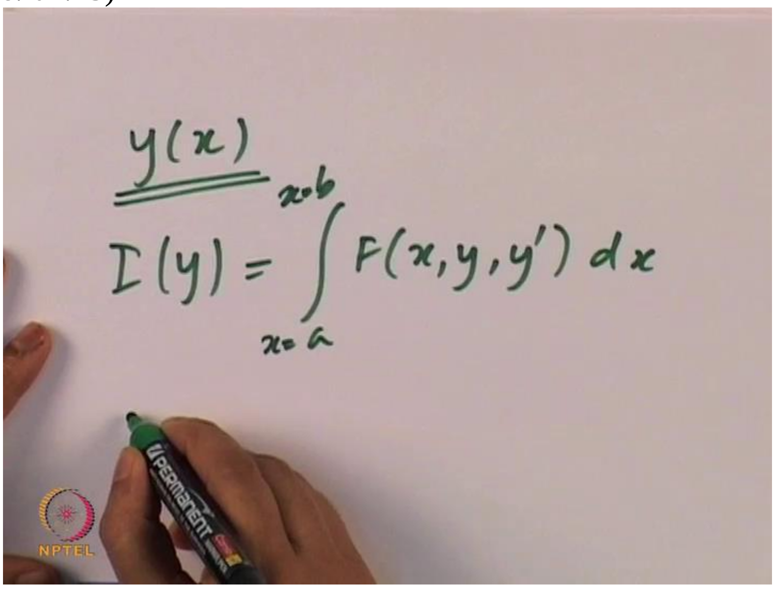

So let us say we are interested in the function y (x) and now instead of finding function itself we are going and saying that there is another function which is let us say as a function of y.So it is going to be an integral between a to b. F (x,y,y dash)dx.

So what I say here is this function what I am interested is y(x). But instead of directly going about finding this y(x), I say there is an integral which I call I (y) which is of the form integral a to b. So x equal to a to x equal to b F (x,y,y dash)dx and it is integrated over the dx. For simplicity we consider only a one dimensional problem where we set the value is going to change from a to b.

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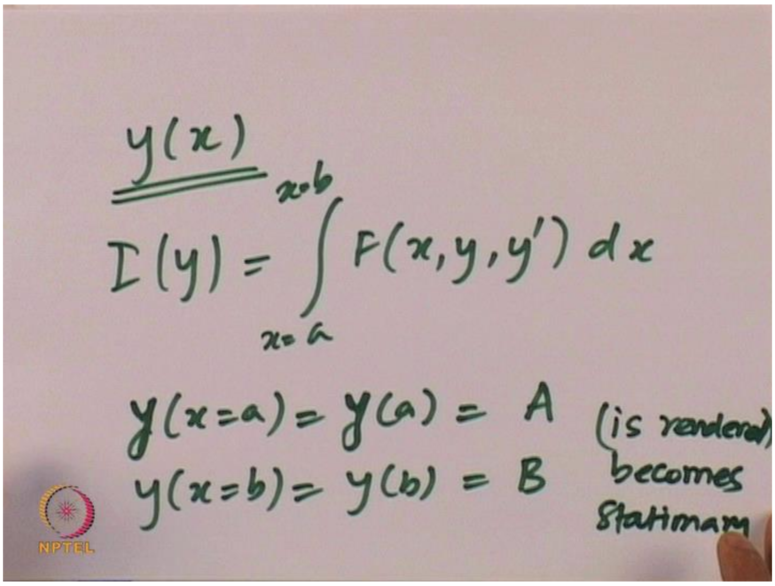

So here what we are interested is we are subjecting the boundary conditions. The boundary condition is we say y (x) equal to a is equal to y(a) is equal to a value A and then we say y (x equal to b) that is equal to y (b) is equal to some number B. So the entire problem here is consider the problem of finding a function y(x) such that the functional represented by I(y) given by this particular functional subjected to these two boundary conditions is rendered or becomes stationary. So this is the way we are transforming the entire problem from finding the function itself to the particular integral.

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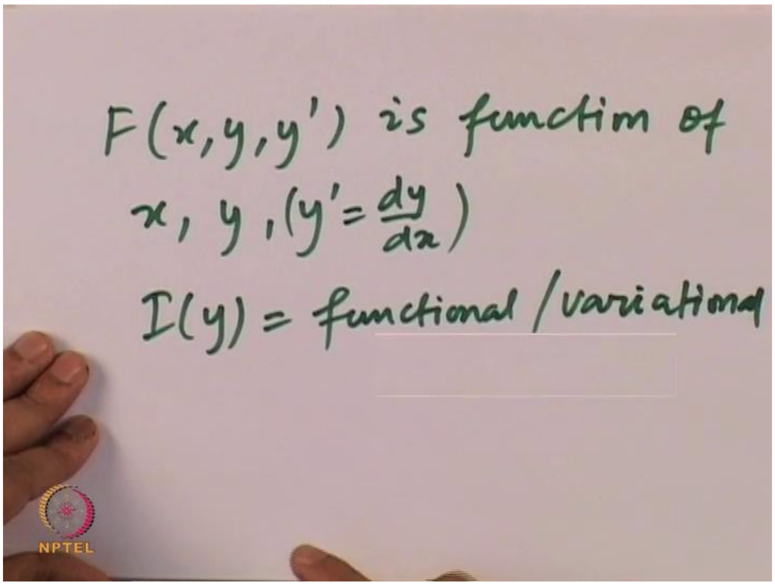

So let us look what is going to be the consequences of this problem statement. So the integrant F that is equal to (x.y, y dash)is a function of x,y and y dash. Here y dash is nothing but dy by dx. So the I(y) is a functional or the variational principle. So we say that the problem here is finding an extremizing function. So the function is y(x). So we say that is going to be having an extreme value. Whether it is highest or minimum going to depend on the physical problem itself. So in most of the applications in electromagnetics we are talking about minimum but it could also be a maximum.

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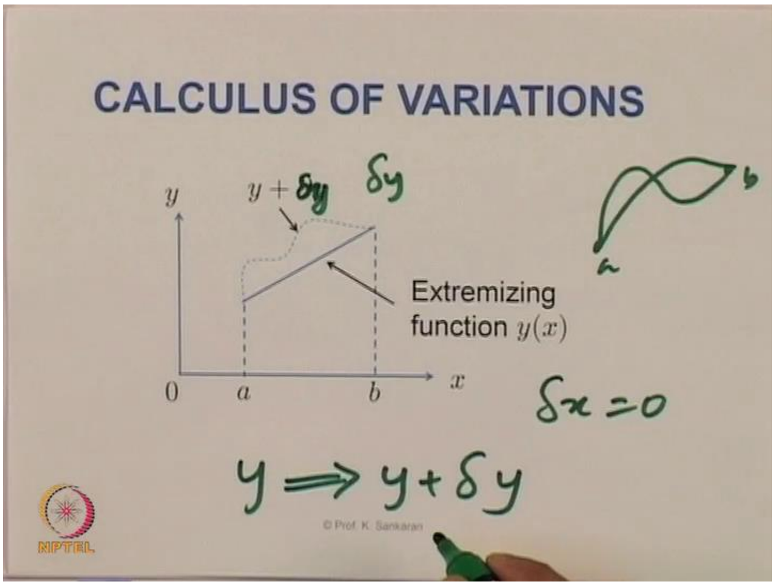

So now let us say, how we treat this problem further. So we said in the earlier case this is the way it is going to vary and we say that the variation is the value given by a small variation y and this variation is going to be 0 at the end points. And the variation with respect to x, in other words is equal to 0 as well. So the function is going to vary only with respect to the y axis or the y variable itself it is not going to vary with respect to x. So due to this we are talking about y going to y plus delta y.

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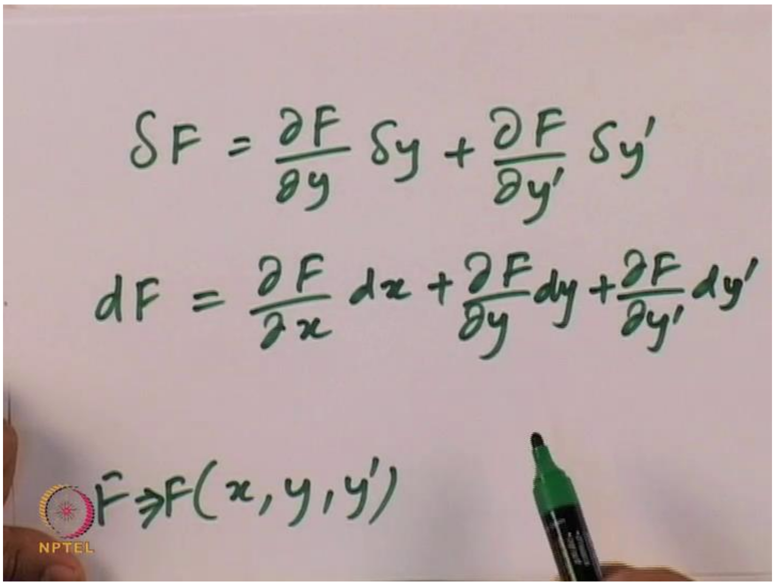

So we can write the entire equation as Doe(F)is equal to the partial differentiation of F with respect to y multiplied by the small change in y plus partial differentiation with respect to y dash and the small change in the y dash So this is the basic assumption here. So you can compare this let us say with respect to the basic df also. So when you take a function which is given by letus say the function is given by the value that is going to depend on x. So f is a function going to depend on x , y and y dash.

So when we do the small variation in d. So the differential of f is equal to Doe f by Doe x of Doe x plus Doe f by Doe y of Doe y plus Foe f by Doe y dash of Doe y dash. So here what you see is we taking the partial differentiation with x y and y dash for which the function is dependent on. And we are multiplying with the differential and this thing and this thing are more or less the same and we know that in our case there is no variation with respect to x axis so we can put them into 0. So in other words you see that we see a similarity between this equation and this equation. In that sense the small change whatever we are talking about could be analogous to the differential of F itself.

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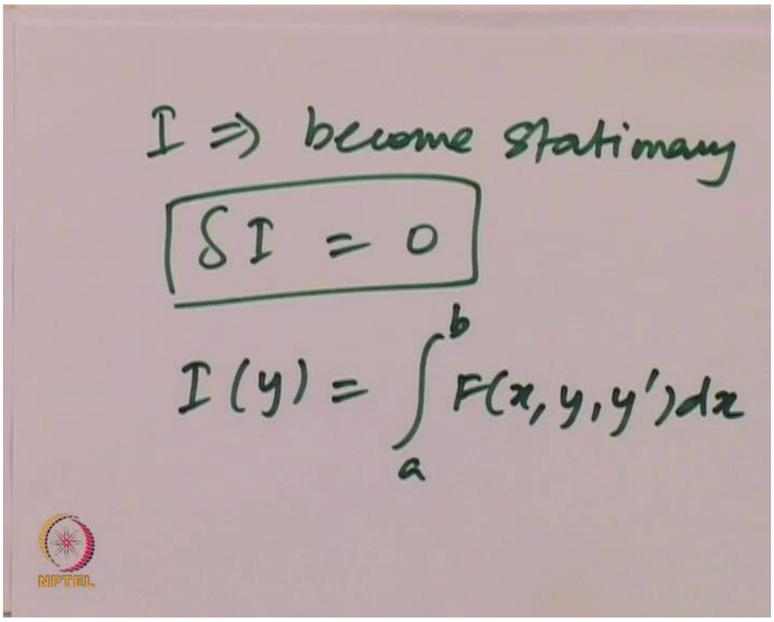

So let us now look into the necessary condition itself, the necessary condition we said is going to be the condition in which the integral i will become stationary. In other words the small should be equal to 0. So for as to apply this condition we must be able to find the variation so we have the variation in the equation what we have initially assumed. The equation what we have assumed is the initial equation that I(y) is equal to integral a to b F(x,y,y dash)dx. So what we are going to do now is we are going to apply this particular pin into the main equation.

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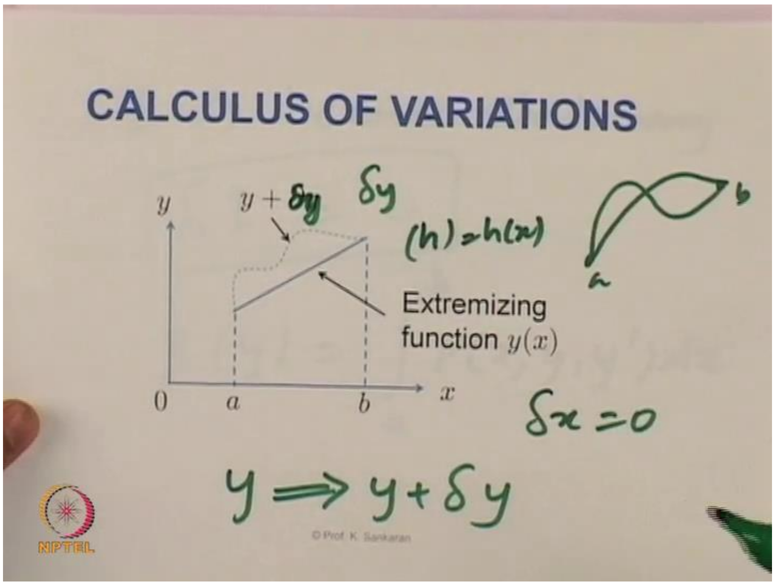

So let us say we are saying the change what we are interested in. So let us say this change in the y axis is given by h. So whatever is the change at each of these points is going to be different values of h which is going to be a function of h of x at h (a)the h is equal to 0 and h (b) the value is also equal to 0. So the variation in the y is represented here by h.

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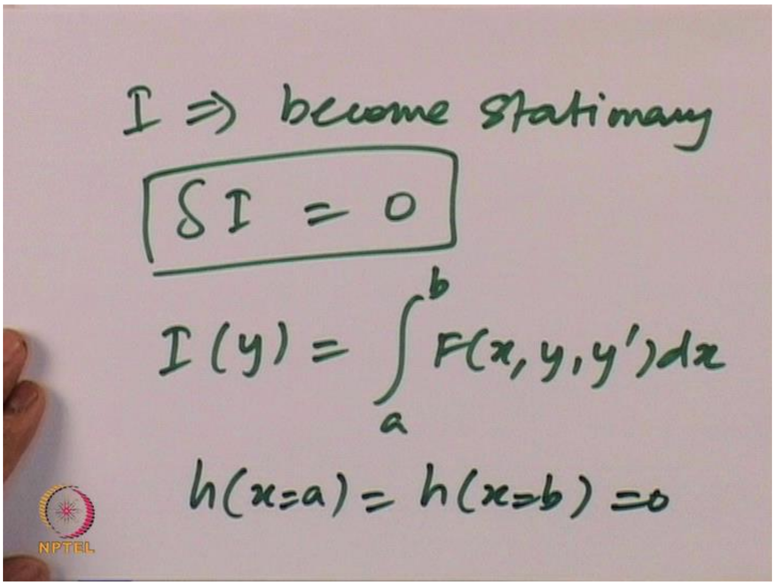

So in other words what we can say is h(x equal to a) is equal to h(x equal to b) is equal to 0. With that being said we can now write the thing what we are interested in to compute the integral itself.

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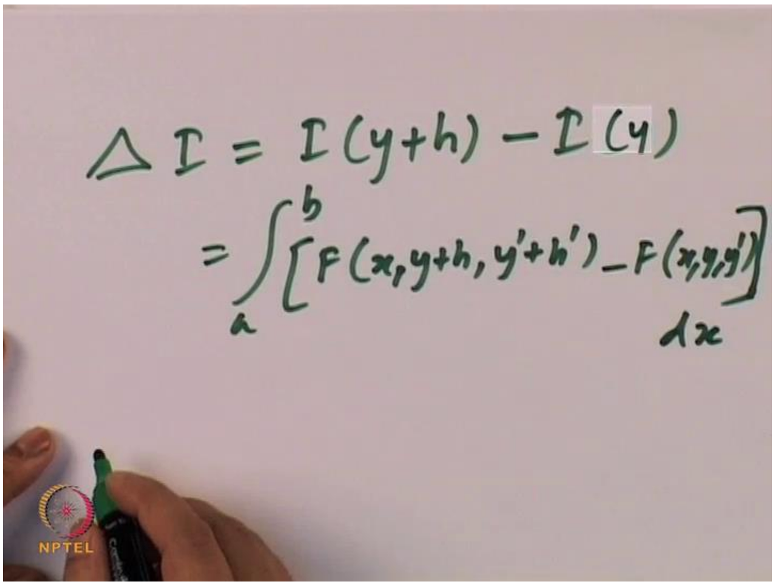

So what we are doing is we are doing an increment change in I that is going to be equal to I (y plus h) minus I (y). So we say that it is going to change only in the y and we are writing like this. So this is equal to integral a to b [f(x,y plus h , y dash plus h dash) so let us say there is also a variation with respect y dash. because it is changing with respect to y it is also going to change with y dash minus f (x,y,y dash)]the entire thing into dx.

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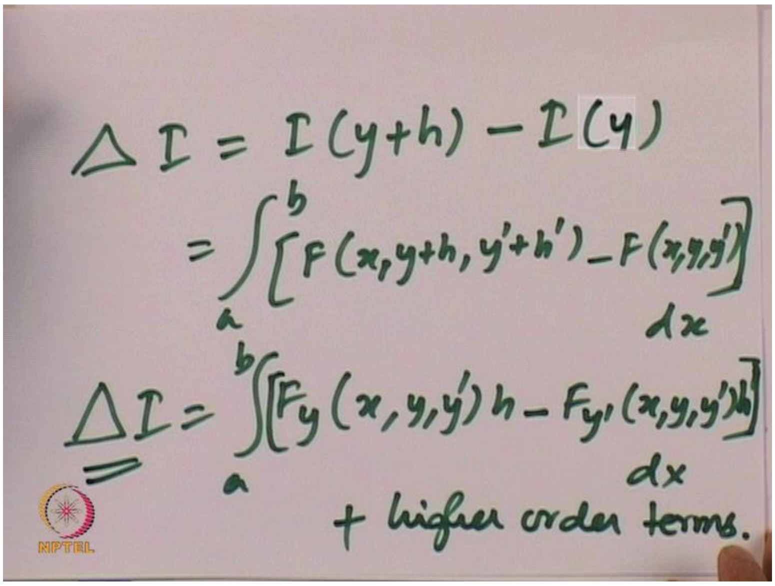

So we can apply Taylor series for this, so when we apply Taylor series what we get is integral a to b[F(y) so the differentiation with respect to y (x,y,y dash) h which is the variaton in the y minus the differentiation with respect to y dash (x,y,y dash)times h dash] the entire thing into dx plus some higher order terms.

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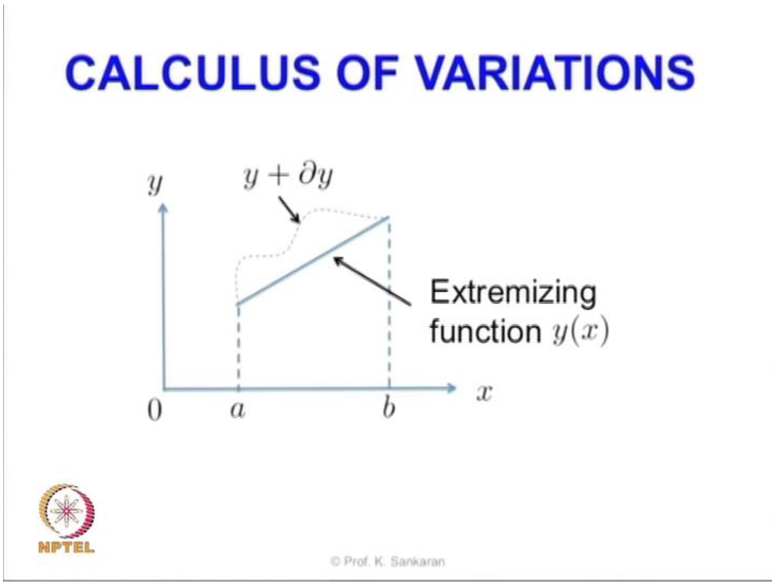

So let us take this further, so what we are going to do now is we are going to see how this entire equation is going to change.

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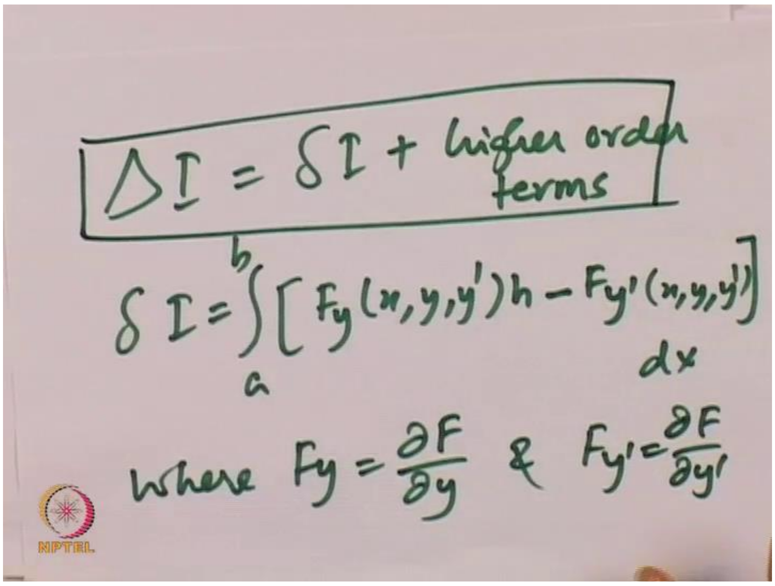

So what we have got is Delta I is equal to the value which is here given, and this is nothing but the value of small change in I itself. So the first term will be I plus some higher order terms. So here the value Delta I is equal to the value that we have derived in the initial case. What we have derived is integral a to b the [F partial differentiation with respect to y (x,y,y dash)multiplied by h minus F partial differentiation with respect to y dash (x,y,y dash)] dx, where F(y) is equal to dy and F (y dash) is equal to Doe f by Doe y dash.

So we are able to get the form of this sort, so we are only interested in the first principle so we are kind of disregarding the higher order terms So we will say only the first aspect. The main thing is going to be equal to 0. So we will take only this part of the equation, we will not look into the higher order terms So what we will do is now we have a second derivative here because it is a Doe y dash and we want to have them as a function of the first derivative itself. So we are going to apply certain technique to get rid of this particular term and have the entire equation as a function of either F(y) or differentiation with respect to x.

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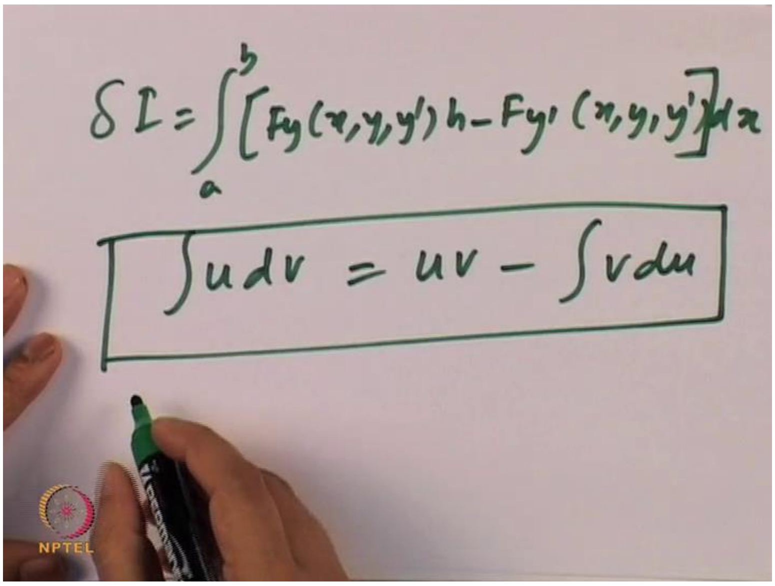

So what we will do now is we will apply the method of integration by parts so what we have we have is initially we have Doe I is equal to integral a to b [F(y) (x,y,y dash) multiplied by h minus F(y dash) (x,y,y dash)]dx. So we are going to apply integration by parts and we are going to change this equation so integration by parts in the sense you have integral udv is equal to uv minus integral vdu. So we wanted to make sure that using this particular rule we are going to transform this into a simplified equation.

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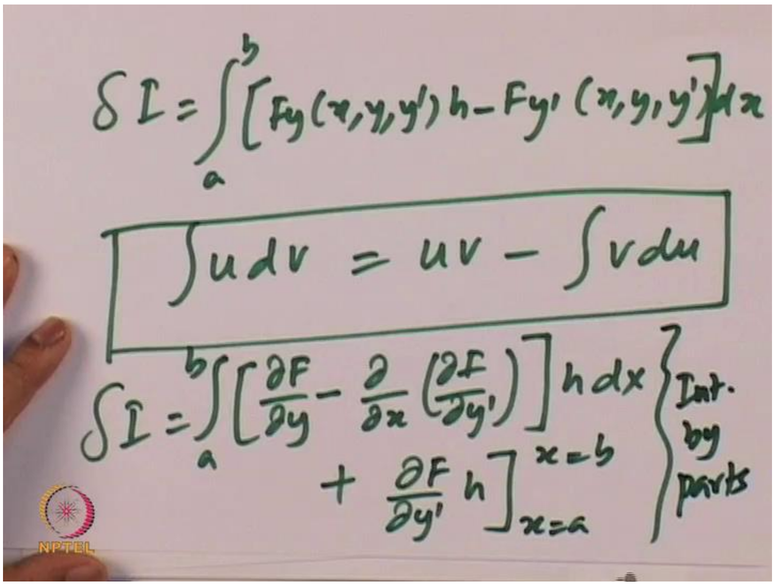

So what we will have is integral a to b which will be there we will have [df by dy which is the same term here minus d by dx of the function (df by dy dash)] multiplied by hdx plus so again integral a to b will not be here we will have only this particular term, so we have [df by dy dash multiplied by h] from x equal to a to x equal to b. So this is the way we are getting it, we are getting it through the integration by parts. So when we integrate by parts this particular equation gets transformed into this equation and yu will see this is this particular thing is a boundary thing. So you can apply the boundary condition and you will get the boundary condition at the boundary x equal to a and x equal to b, the variation along h will be 0. So this entire thing will become 0 and you will have only this particular equation.

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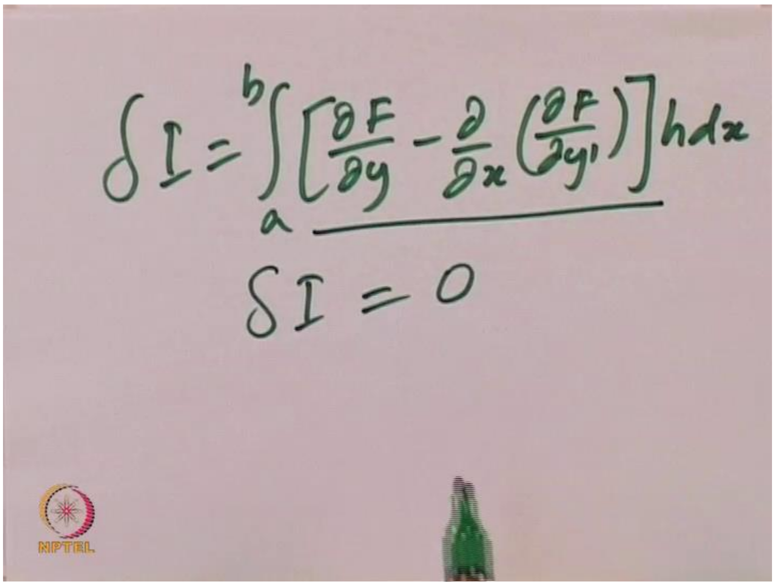

So what we will have is we will have an equation which is given by Delta I is equal to Integral a to b [Doe f by Doe y minus Doe by Doe x (Doe F by Doe y dash)] hdx. So we have applied the boundary condition to get the rid of the second term because at the boundaries the variation across the y which is h (x equal to a) and h (x equal to b) is equal to 0, which we saw before. So we will have only this particular thing. But for this particular thing to be stationary as in Delta I is equal to 0 means whatever we have here as the integrant this particular term should be equal to 0. Because we know h is not equal to 0, we need this particular term to be equal to 0.

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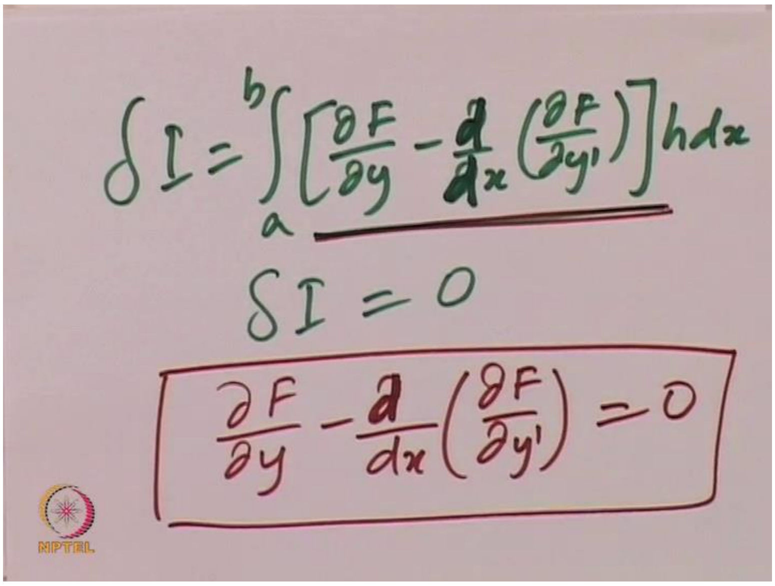

In other words what we have is the partial differentiation with respect to y minus d so this is the, so be careful that, this is the partial differentiation, so this is the partial differentiation but the main thing is not the partial differentiation it is a standard differentiation. Doe F by Doe y minus d by dx(Doe F by Doe y dash) is equal to 0. So we can write this in a short form where we can abbreviate the terms in a simplified notation what we had before.

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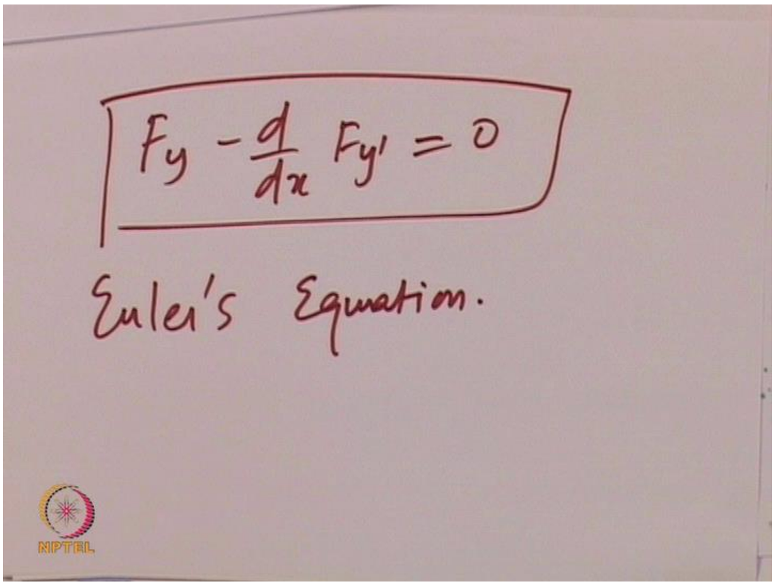

So what we will write is Fy minus d by dx Fy dash is equal to 0. So this is nothing but the Euler equation they call in the numerical methods. So this is the Eulers equation for this problem that we are trying where the I have a extrimum. So what we have done here we have started with a equation where we said the variation is going to be given by the expression so what I am talking about is the starting point of our entire assumption where we said it is going to have the change along the y. The axis are given so the starting point and ending point is given. So what we have found out is we found an integral which we call it as a action principle or variation principle and through that we claim to the point of finding the differential equation. Basically the differential equation is called as a Euler Equation. So this is the point where we say that we found the Eulers equation of that particular problem. So this isa very theoretical approach where we have shown the integrities of finding the Euler equation starting the integral itself.

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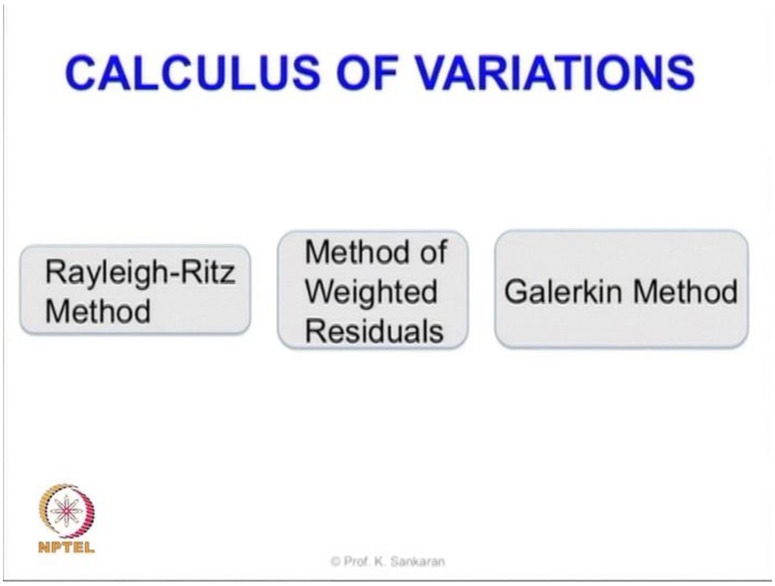

So let us go into the method itself there are several ways in achieving this for example there is a method which we call it as Rayleigh Ritz method and there is a method of Weighted Residual and there is a method of Galerkin. So what we will do now is we will introduce you some of the basics of the variational method where we will show how we start with the basic assumptions on the function itself and then that is common to all these methods whether you are doing a Rayleigh Ritz method or you are doing you are doing Galerkin method or you are in the method of Weighted Residual, the basics are the same so we will look into the basics and then we will see how we can apply that for each of the individual cases.

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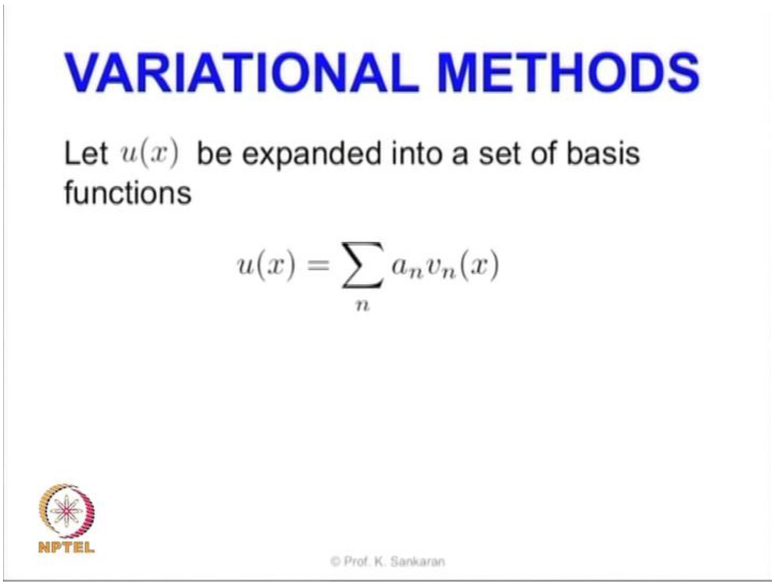

So let us assume that we are interested in a unknown function represented by u(x). So this u(x) can be expanded using the set of bases function which we call it as v(m). So now what we have done now is we have basically started with an assumption that the expression whatever we have for u the unknown function is going to be given through a mathematical expression it is a summation of certain variable.

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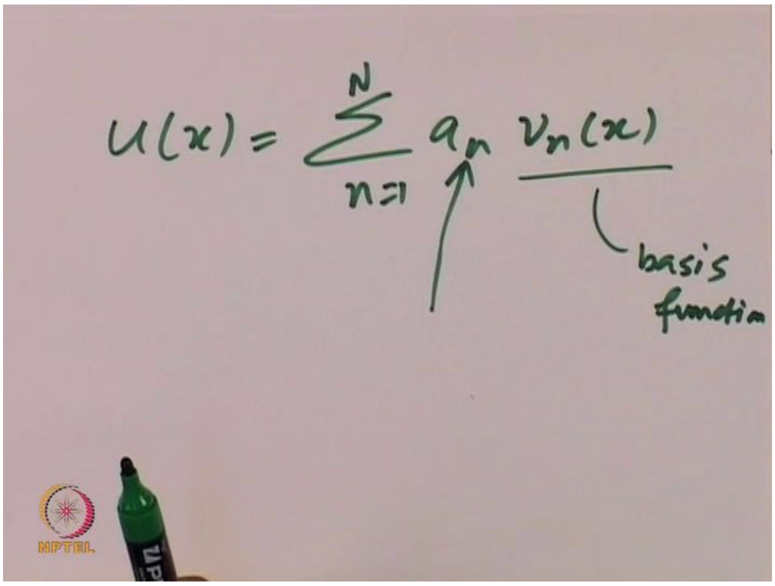

So let us say u(x) is equal to certain summation over a n v n (x). So we say n goes from certain number 1 to n. So what we have done here is we say that unknown function what we are interested is going to be a summation of certain function this is called as a basis function. so these are the basis functions. And we are waiting the basis function using some weights here. And these weights are going to be the weights that we need to find out at some point. So in other way the problem will go from finding the function to finding the weights that will be added to the particular basis function. So in a (())(22:30) we can say that the u (x) is going to be the unknown function.

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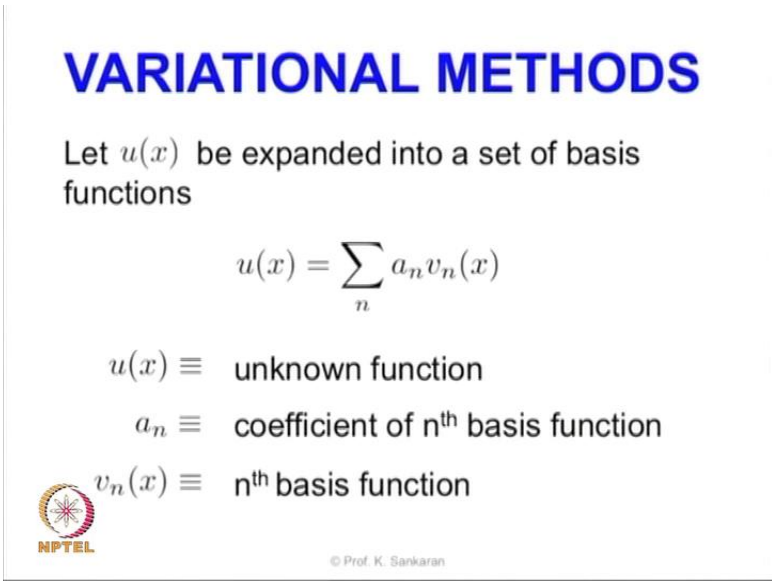

As you can see in the slide and a n are the coefficients of the basis function, and the v n are the nth basis function.

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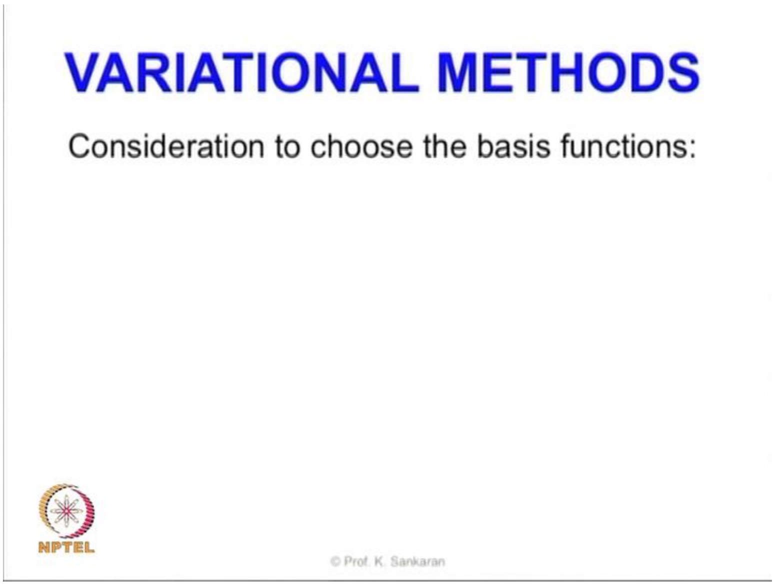

So let us say for the case of simplicity we have certain options for choosing certain basis function itself. We did not say much about the basis functions but you might ask what will be the shape or what will be the property of the basis function.

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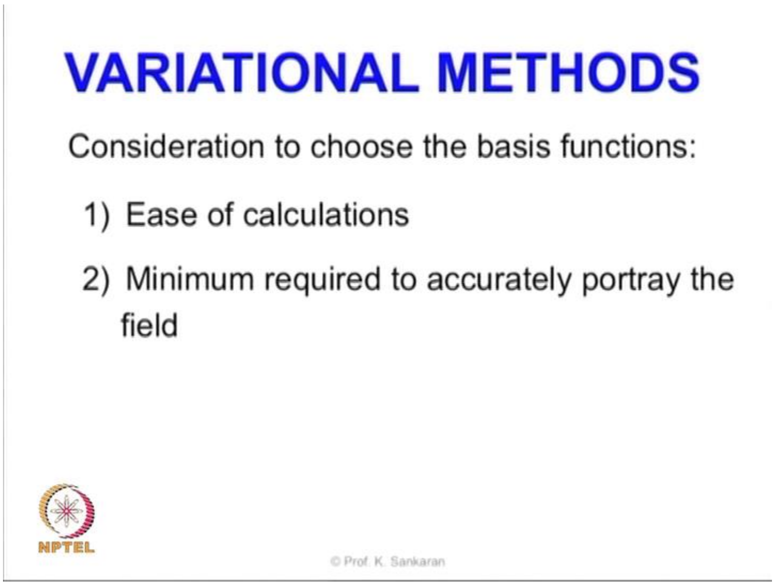

So let us look some options. So we want the basis function to be easy to calculate. And they have to be minimum so that we can accurately portrait as a summation.

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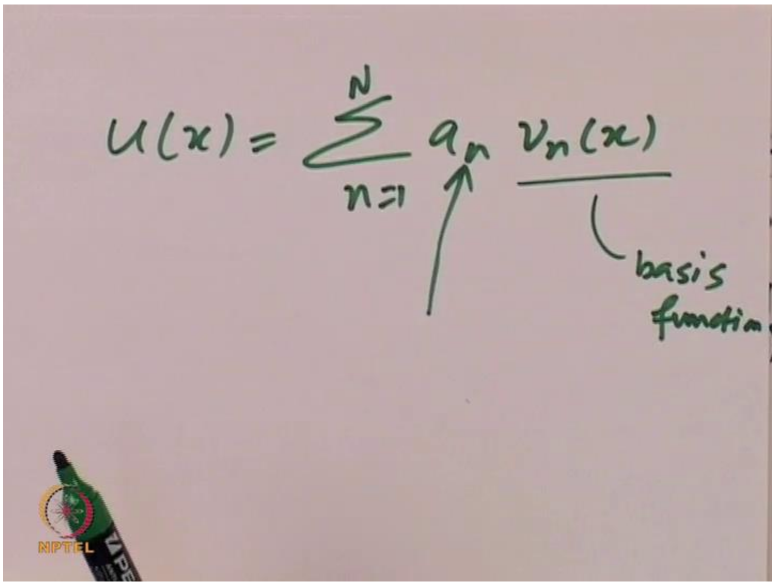

So we do not want the number n what we have in this particular slide to be large. So we want them to be small number so that we are able to have minimum number of basis functions to accurately calculate this particular value.

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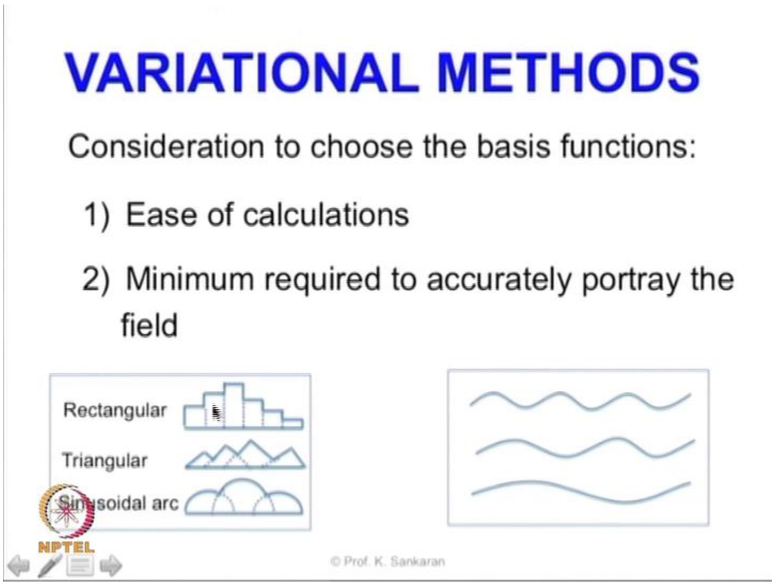

So the options what we have are quite interesting from these two conditions. We have these two conditions so we can say the basis function could be simple rectangular function like a step functions inside each of these individual units. So let us say this is one unit, this is second unit, this is third unit, or first element, second element, third element so on and so forth. We can also think of them being some kind of triangular functions. Or they can also be sinusoidal or they can have some functions within the particular thing. So we are we have quite a bit of options so most of them what we will do is we will stick to either a triangular function or a rectangular function but you can choose whatever you want depending on the calculation what you wanted to do and the time you have.

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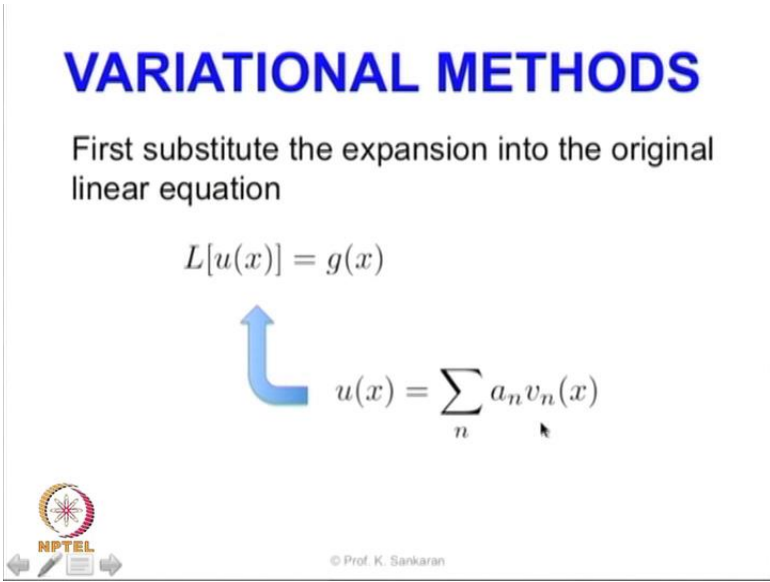

So when we have set what is going to be the basis function we can basically use that basis function inside the unknown web we have. For example if we say that this isour operator L, and it is operating on a unknown function. And it is equal to some forcing function which we know. We can substitute the value of u into set of coefficients multiplied by the basis functions which we have discussed so far.

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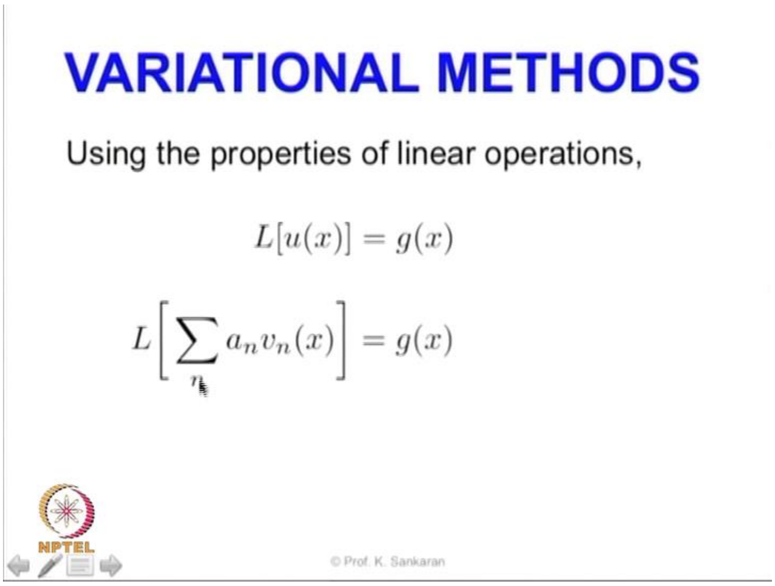

Once we do that so what we have is we have a problem that gets transformed into a simple mathematical change of u effects is equal to sigma from n to some value a n v n (x)] equal to g(x). So remember I told you when we have an operator we can basically change the operator's order

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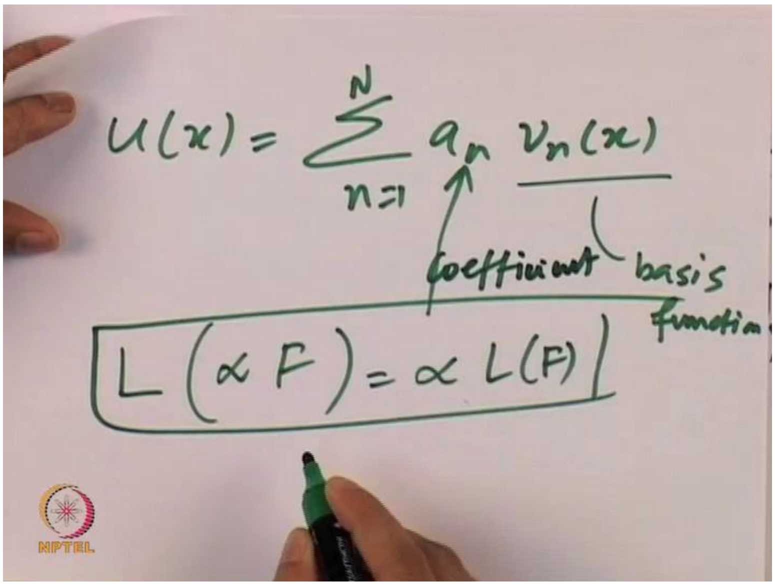

So when an operator who is operating on a let us say alpha of a function we can basically bring this alpha in the front and then we can say L(F). So this is one of the properties which we saw. So this is the coefficient. So let me write this in a separate page.

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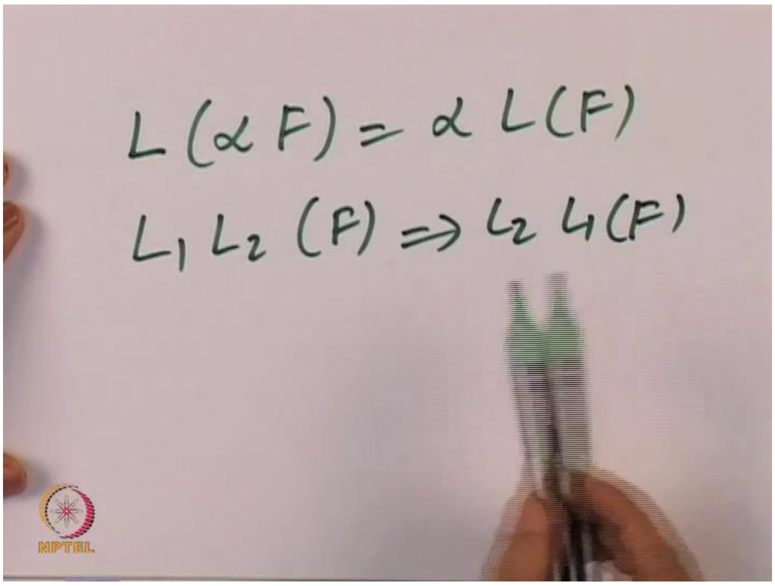

Remember that we said that the operator L operating on a scala multiplied on a function will be given by L(F). And then there are also some other properties L1 L2 operating on a function is equal to L2 L1 operating on a function. So basically the operator has certain properties.

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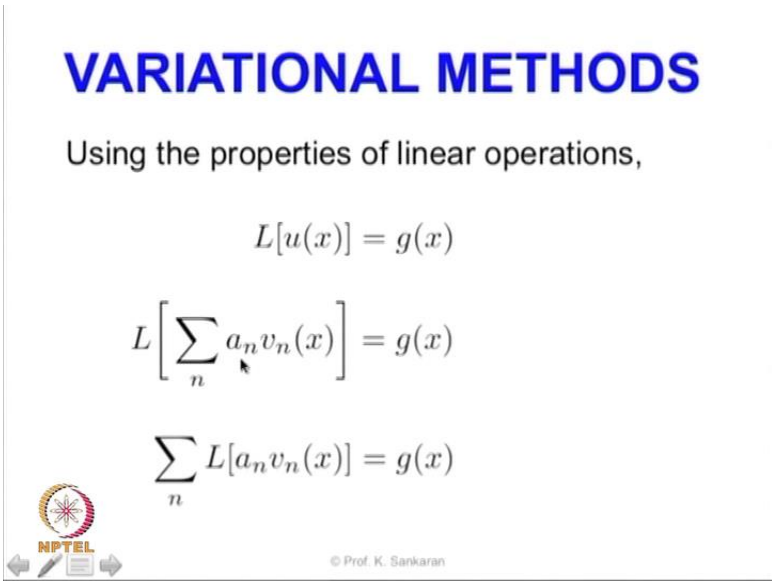

So what we are doing in this slide is we are bringing the operator L into the summation and taking the summation out. So that is what we will do here so have moved the order of the summation and the operator which is doable because we said the operator is going to have certain properties. So once we do that we have set the basis for doing various operations on this function. We will come to the point of continuing towards the Rayleigh Ritz method where we will approach the problem directly from the action principle itself or directly from the integral itself and see how we can arrive to the point of finding the solution.

And what we have done in this module is the basis for what we are going to do in the next modules. Whether it is going to be Rayleigh Ritz method or Galerkin method and at the next point we will look into the Rayleigh Ritz method

Thank you!