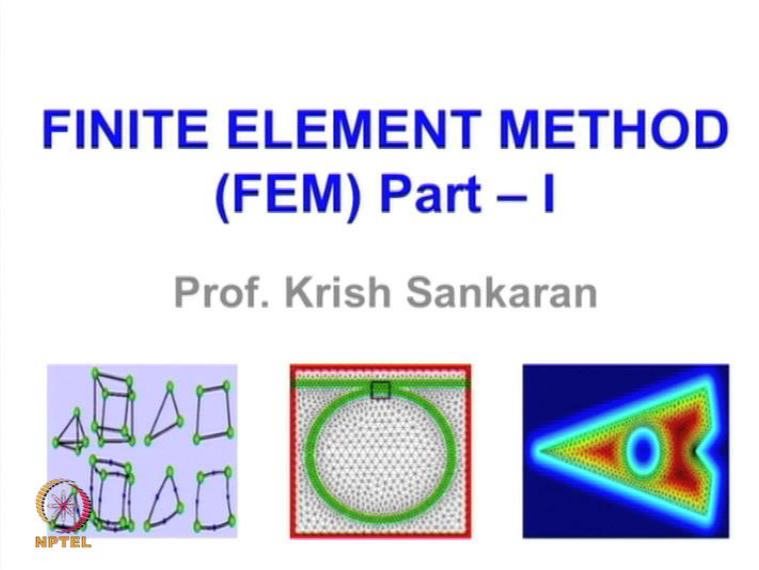
Computational Electromagnetics and Applications
Professor Krish Sankaran
Indian Institute of Technology Bombay
Lecture No 15
Variational Methods

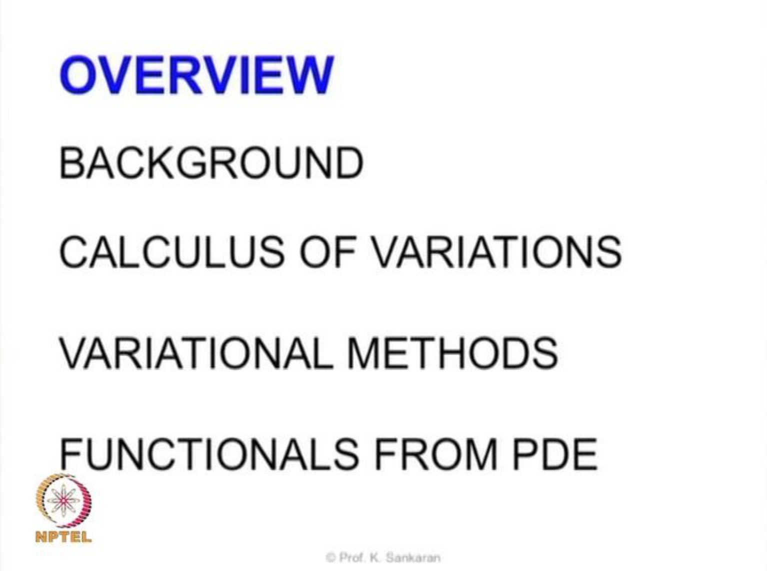
Hello welcome back so we have been looking into finite difference method in the earlier modules which was very basic approach to looking to any partial differential equation and defending it with certain stencils which we learned finite difference method has its own advantages and disadvantages we spoke about it so now we are going to go into a different class of methods which we call as variational methods the motivation for going into finite difference method has its own advantages and disadvantages we spoke about it so now we are going to go into a different class of methods which we call as variational methods the motivation for going into variational method a variational method is multi Force for example this method is going to be the basis for a larger class of problems later stage we will look into.

(Refer Slide Time: 00: 55)



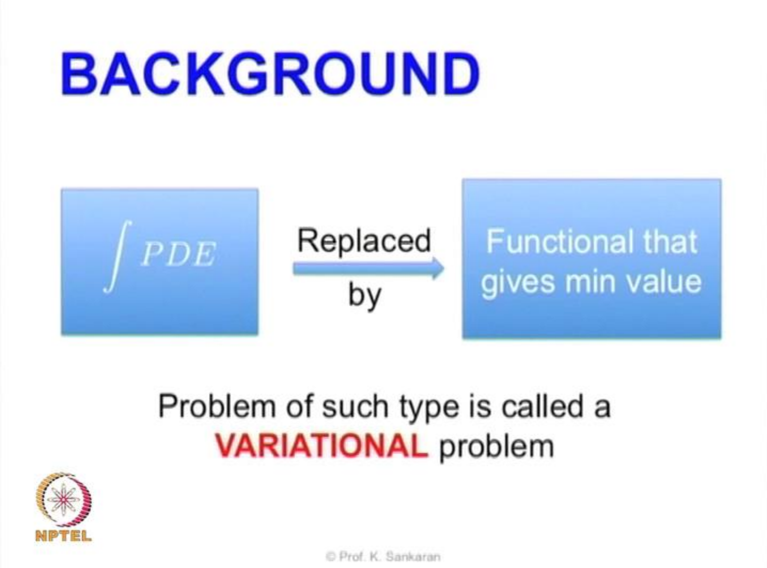
Which is called as finite elements method which is widely used in computational electromagnetics community not only that it is also used in other areas of Physics like mechanics aerodynamics excreta so there is a lot of motivation to look into variational methods as a starting point and then we will go into looking into other methods such as finite element method of moments so on and so forth so with this as a starting point let's look into today's overview.

(Refer Slide Time: 01: 27)



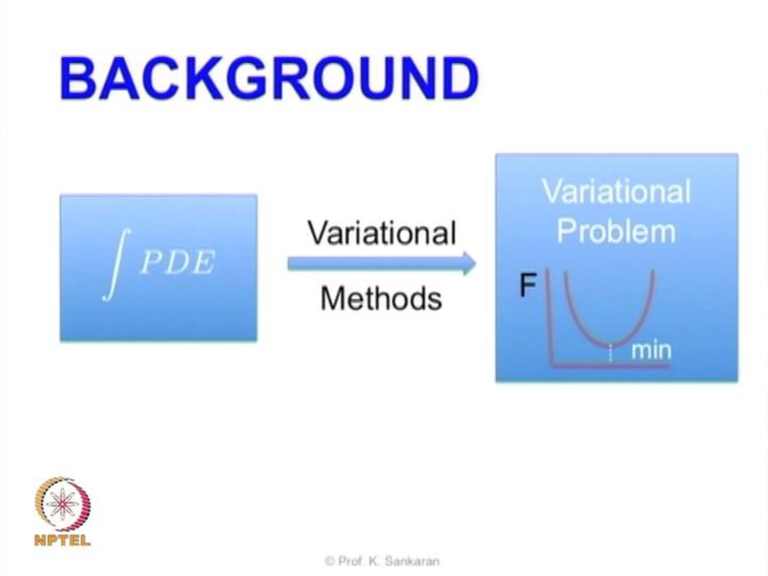
So we will have initial part we will look into the background of the variation methods and then we will discuss something what is calculus of variations I will come later on to explain why we call it as calculus of variations not just calculus simply and then there is some introduction into the variation method itself and then we will discuss how to derive a functional from a particular PD so with this at the starting step let's go into the background of variation methods.

(Refer Slide Time: 02: 02)



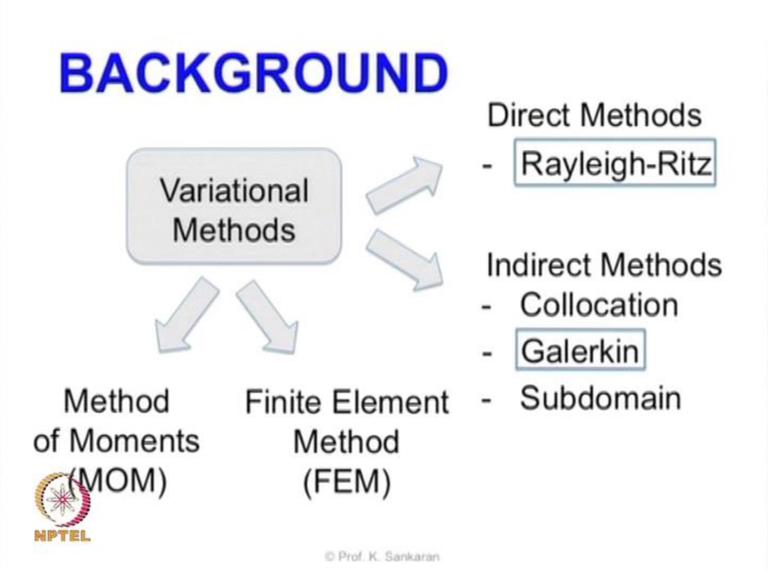
So in the case of the variation methods we are going to start with the basic partial differential equation which is given here and we are going to replace the partial differential equation by a function which is the mean value for a certain integral so the idea is in the calculus of variations which we will see we are interested in the necessary conditions for the functional to achieve a stationary value so the exceptional idea is we are going to look into a functional which is indirectly going to address the PDE later on. The problem of such type is called as a variational problem and we will discuss this much in detail in the next slide.

(Refer Slide Time: 02: 55)



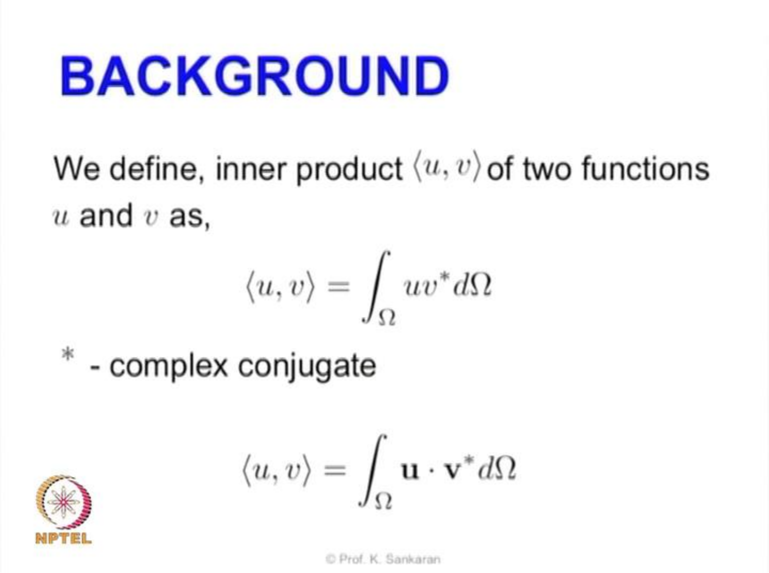
So basically what we said is for PD there is going to be functional that will have a minimum value so that is a pictorial depiction of what we said verbally.

(Refer Slide Time: 03: 10)



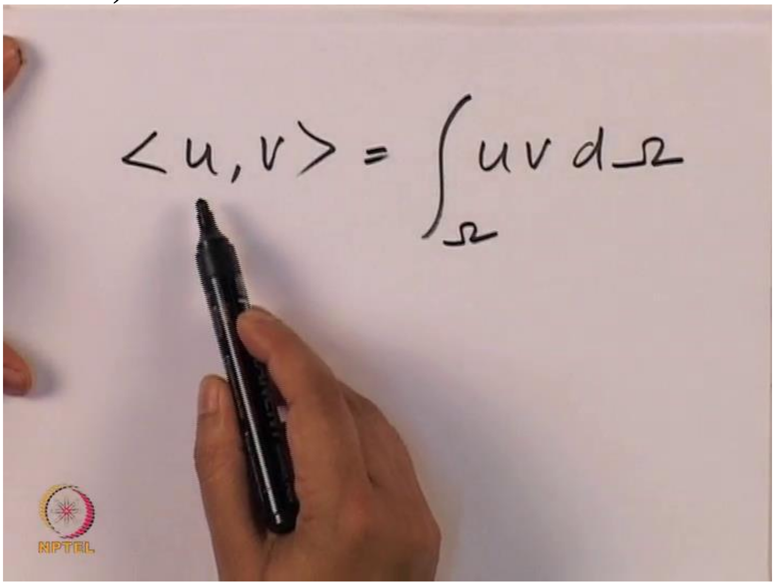
And the variational method as such gives rise to quite a lot of different methods as I told you the direct method would be the Rayleigh Ritz method which is due to both rates and relay and the method has been quite successfully applied to various class of problems not only in electromagnetics but also in mechanics and so on and so forth and there are other indirect methods namely the collocation method galerkin method subdomain method so on and so forth we will in this lecture focus mostly on rayleigh Ritz and the gallery can approach and these two approaches are the most common and most widely used methods and of course there are other two types of problems numerical schemes that are directly coming from variational methods namely the method of moments and the finite element method so we will look into the method of moments and finite element method in our next part when we start our discussions on method of moments. So right now the focus is around basic idea around the variational methods and how do we get the formulation for the rayleigh Ritz method and also we will see under what condition it will become a kind of a galerkin approach so with that let’s start into the basic background.

(Refer Slide Time: 04: 37)



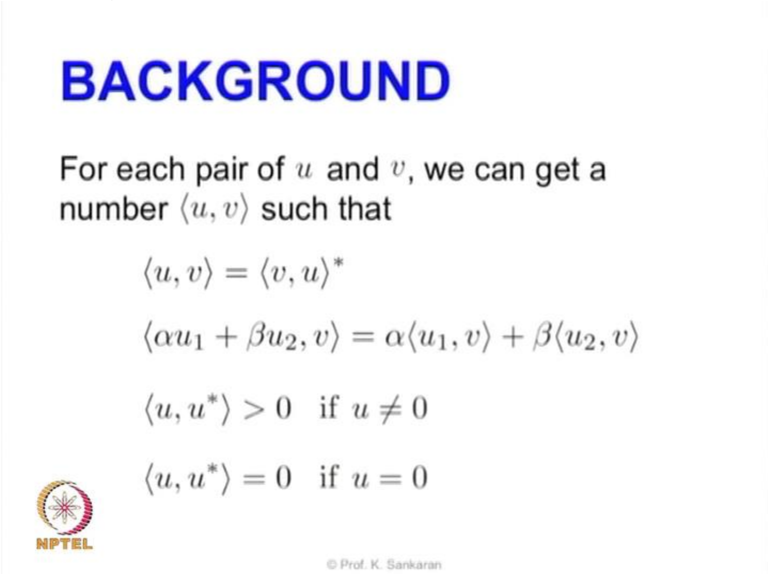
So let’s take two functions given by the letter u and v and they are scalar functions they are return with normal font and if they are vector functions they will be written with bold letters so in this case let’s assume that they are scalar functions and latest define the inner product as such as an integral who are certain domain. And we are defining it as uv star what is Noman so if the function is complex numbers then we are going to have the value here has written here with the star symbol and if they are not complex numbers and they are real numbers they are going to be just UV over this domain so as I said if they are going to be vectors we are going to talk about the dot products but in this case for simplicity we will assume that they are just real numbers.

(Refer Slide Time: 05: 42)



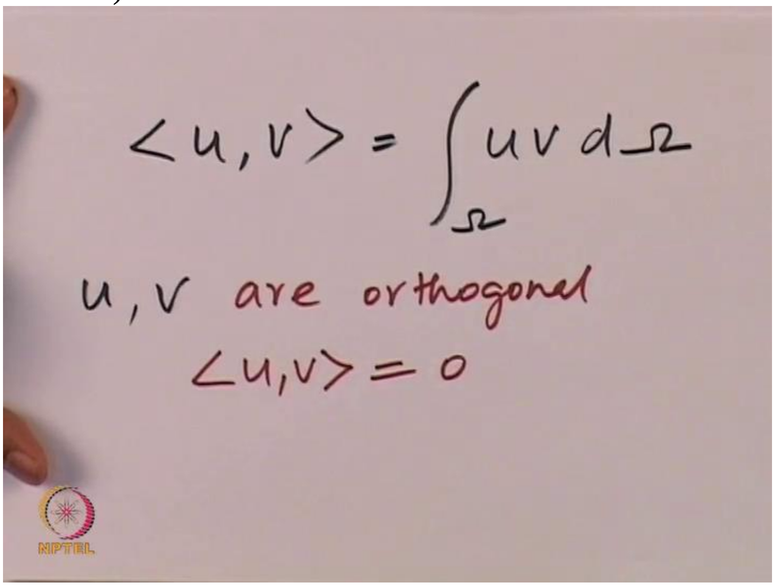
So for these two functions u and v we can get a number which is written with the angle brackets so let me explain this a little bit further. So we said we are going to have u , v and we are going to define the function UV over this domain and this is the definition of the inner product but what does this in a product mean physically. So basically what is happening here is here trying to see in a way a projection of u along the direction of v in other words what we are seeing is how this you and we are correlated.

(Refer Slide Time: 06: 34)



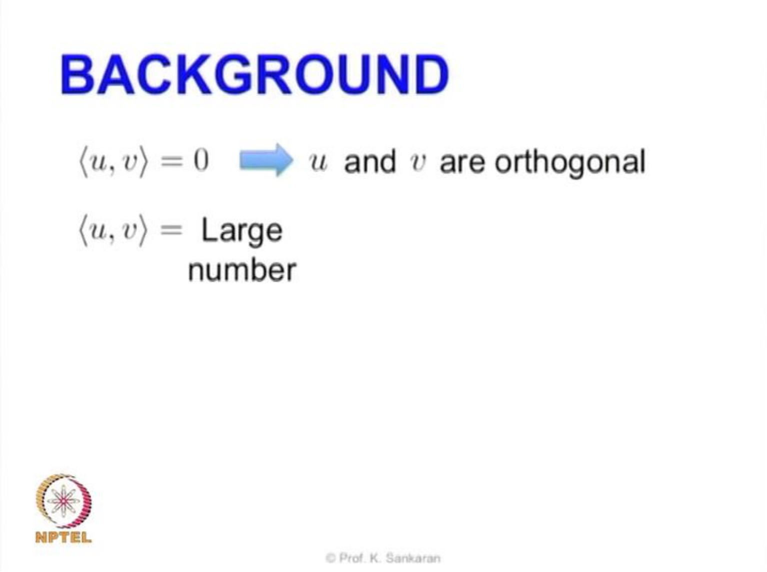
When we have you, we and We kind of swap the number U and v and this will be the complex conjugate and if they are real numbers it doesn’t matter so you, we will be equal to we, you with an angle bracket and the second thing is let’s say we have scalar constants Alpha for which we have Alpha you 1 + beta which is another scalar constant you too, v will be returned as Alpha angle bracket you one, we plus beta you too, v so basically the scalers go out so this is a second property of this particular inner product and the third property is when the inner product on the same number it will always be greater than zero if you is not equal to zero and if this value is zero then we can say that you is going to be zero so with that being said so we are going to see what are the physical meaning of this in a product so the mathematics is clear but what is the real physical meaning of it which will allow you to understand why we always take the inner product and what is the meaning of this in a product in this calculus of variations.

(Refer Slide Time: 08: 05)



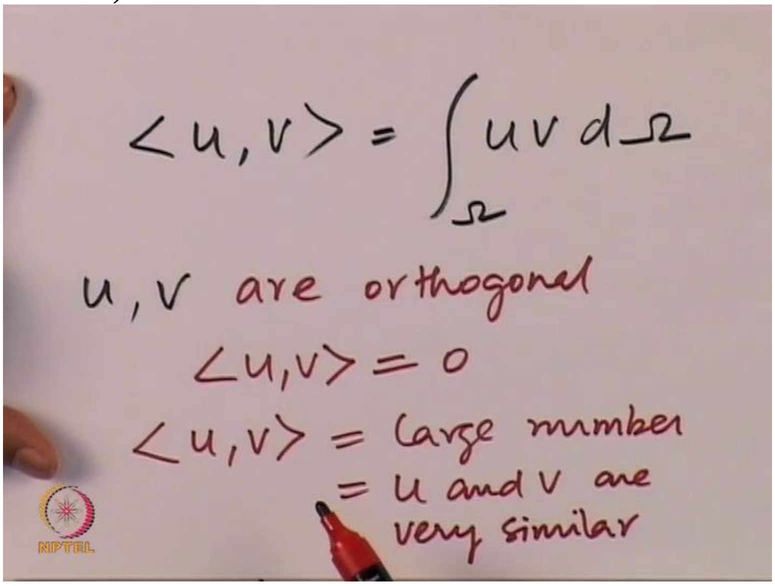
So as I told you what is happening here is we are trying to see how you and B are correlated to each other so let’s assume that you and we are completely orthogonal to each other so that means the projection of you on we will be equal to zero so in that sense when u and V are orthogonal the u, v will be equal to zero so that is the first property.

(Refer Slide Time: 08: 45)



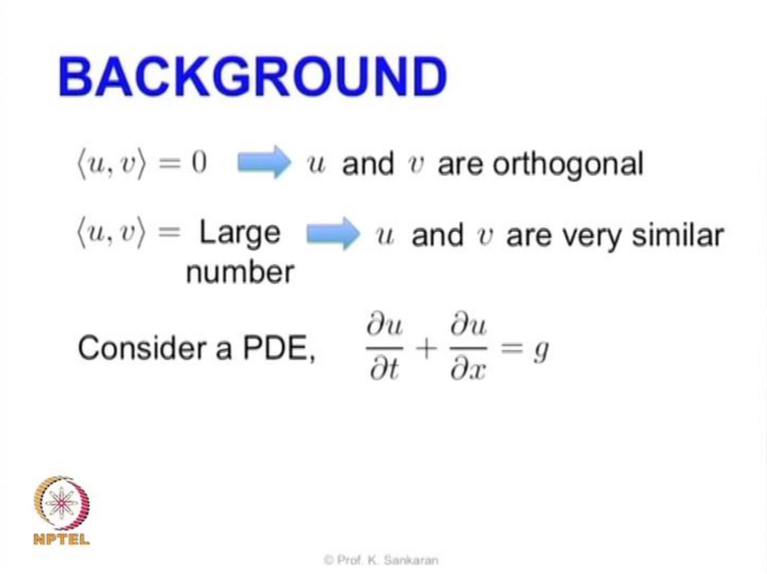
The second property is if u, v is equal to a large number.

(Refer Slide Time: 08: 54)



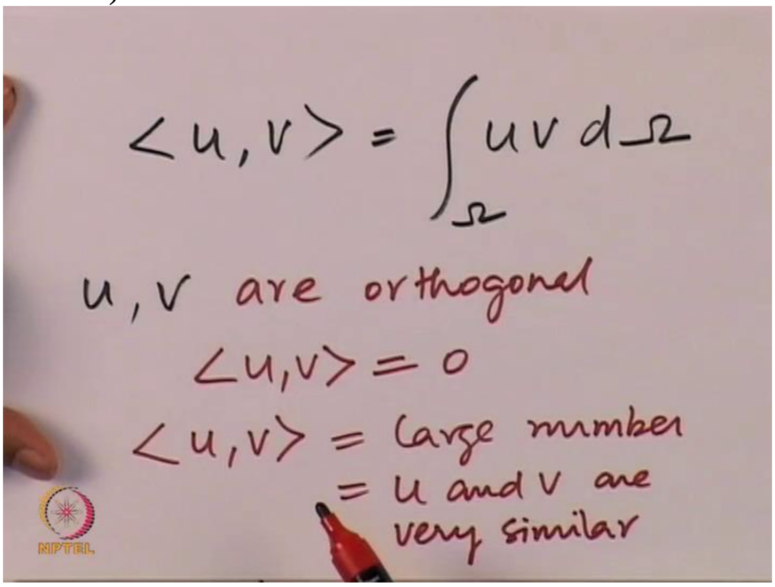
So in other words you, we is equal to a large number that means U and v are very similar so the more similar they are larger the number is going to be for this case.

(Refer Slide Time: 09: 24)



And the last point is when u and V is equal to a small number we know that they are not a similar but they are not orthogonal so anything in between them is going to be a number between zero and a very large number.

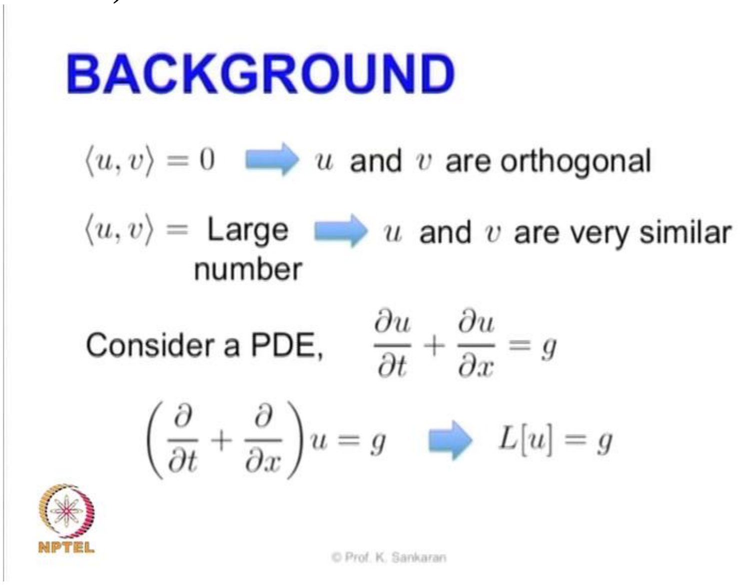
(Refer Slide Time: 09: 40)



So in other words what we have seen here is the inner product is nothing but a way to correlate between the function you and function we in most of the cases what happened is one of the function will be our test function about which we know a lot and one of it will be an unknown function so we are going to test the unknown function using a function which we know very well.

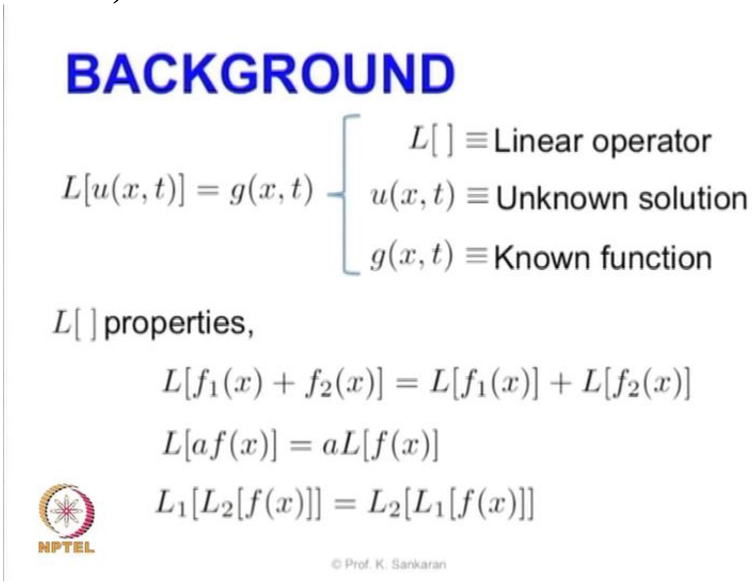
So that is the basic idea behind it. So the reason why we do the inner product is to get an understanding of how the unknown function is with respect to a known s function. So with that being said let us look into a simple partial differential equation.

(Refer Slide Time: 10: 27)



So what we have here is we have a partial differential equation which is with respect to t and with respect to x and it has a right hand side which is g. So what we do now is we take out u outside and then we have an equation which is written of this form so the reason for doing this is we want to have a form which is L[u] which is g. So basically what we are trying to achieve is so we are kind of creating an operator which is basically a differential equation operator. And we are trying to separate the operator from the function on which it operates and we call that operator as L. So in this case it happened to be a differential operator. So essentially what we are trying to achieve is of this form L[u] is equal to g.

(Refer Slide Time: 11: 28)

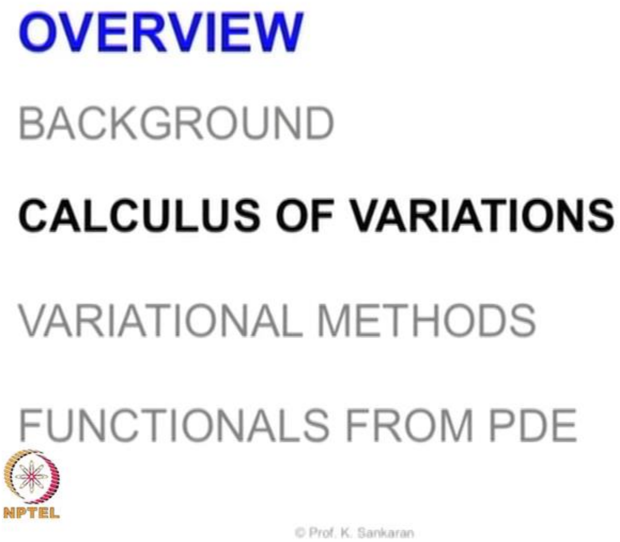


So we can write the old partial differential equation of the form L[u(x,t)] is equal to g(x,t). So we say that the operator is operating on an unknown function and then we have the right hand side given by the forcing function. So of this L is a linear operator u(x,t) is an unknown solution that we need to find and g(x,t) is the known function. And like in the case of the inner product we also have certain properties for the operator itself.

When the operator is operating on two functions, the same operator is operating on two functions we can basically split them into L[f 1(x)] plus L[f 2(x)]. The second one is when the operator is operating on a function which is scaled by the scalar variable we can basically take out the scalar variable and we can operate it on a function. And also the last property is yu can basically change the order of the operator if L 1[L 2[f (x)] you can basically swap it into L 2[L 1[f (x)].

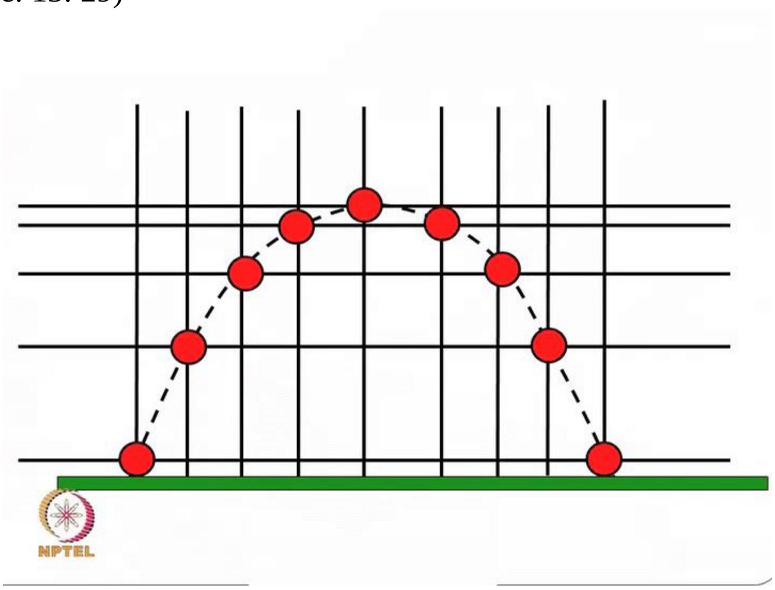
So these things are quite good to know because these are going to help you to manipulate the mathematical equations because at the end of the day if you wanted to get into a form which is easy for you to compute on a computer. So you wanted to have them on an algebraic form so you basically do a lot of mathematical manipulation and the idea of the mathematical manipulation is possible under the conditions that operator has certain properties like we discuss before.

(Refer Slide Time: 13: 24)



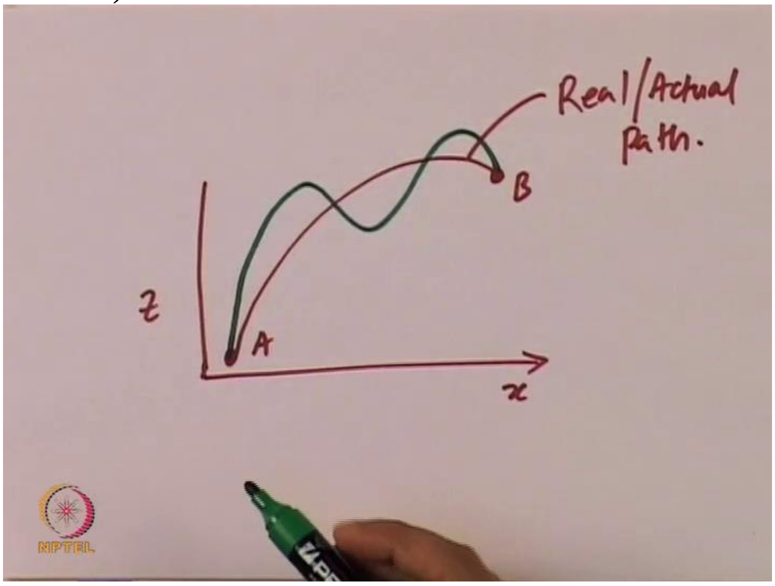
So now what we will do is we will look into a calculus of variation itself. So I said we will discuss about the calculus of variation.

(Refer Slide Time: 13: 29)



Let us say we have a problem throwing a stone up in the air. So this is a problem that my school teacher has taught me while I was in school but later in my life I found out that we actually got this idea from a very famous book called Richard Feynmanns lectures on Physics. So any way does not matter where the idea came from but I am going to tell you the same story what my teacher used to tell me. So he said that take a stone and you are trying to throw the stone up in the air.

(Refer Slide Time: 14: 07)

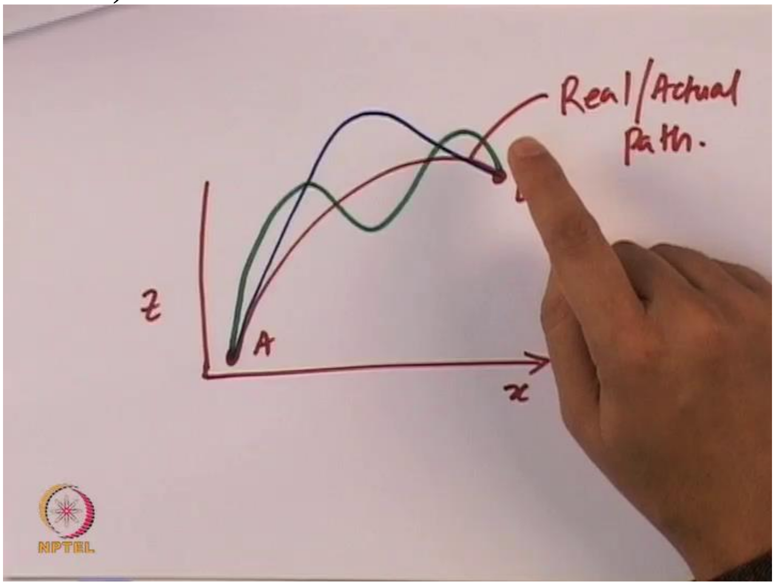


So let us assume that I have a stone and I am throwing the stone from this point I am throwing and the stone is going to reach the point b. So I assume that the stone is going only up and down and there is no motion in the x y plane. So only the motion is in the z axis.

So this is z axis and this let us say x axis so basically the stone is going up and then down. So if I know the point a where I am throwing from so let us say this is the starting point and then I know the point b where it is going to fall. He asked me a question that can you tell me what will be the path of the stone if you go from a to b.

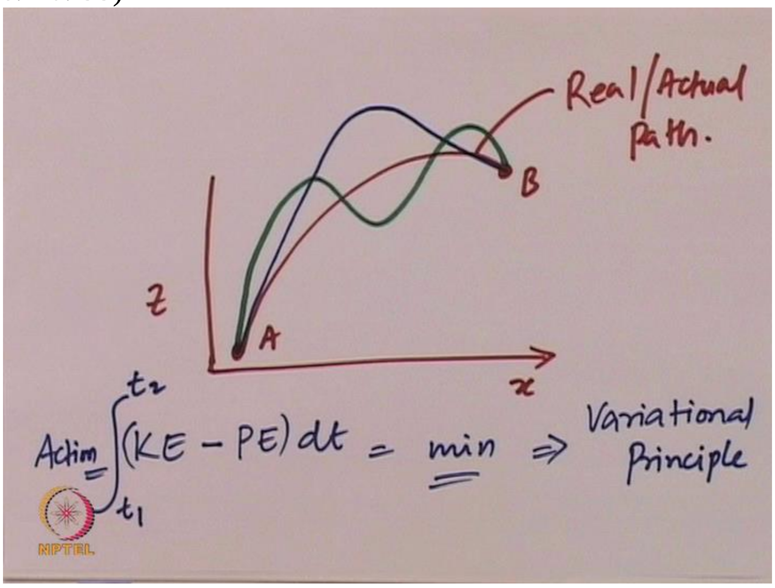
So in other words let us say I am throwing the stone in this manner and then let us say I compute the this is the real path. Let us say we call this one as the real or actual path. But let us assume that I do not know the actual path but I am trying to speculate what is going to be the path when I say the stone is actually going from here to here which is quite unphysical but let us assume that the starting point and the ending points are fixed. But I am trying to say this is going to be the speculated path.

(Refer Slide Time: 16: 00)



But how do I know this is right or wrong ok physics says that the stone cannot go in this particular direction but what if it is going like this. So there might be so many different paths, so there are n different number of paths that one can think of, provided the starting point and the ending points are the same and the time for the stone to reach the end point is also the same. Only one path that is going to be the actual path the other paths are going to be fictitious are let us say are not relevant paths. So what he said was very remarkable for me is he said the true path has something unique about it.

(Refer Slide Time: 16: 53)



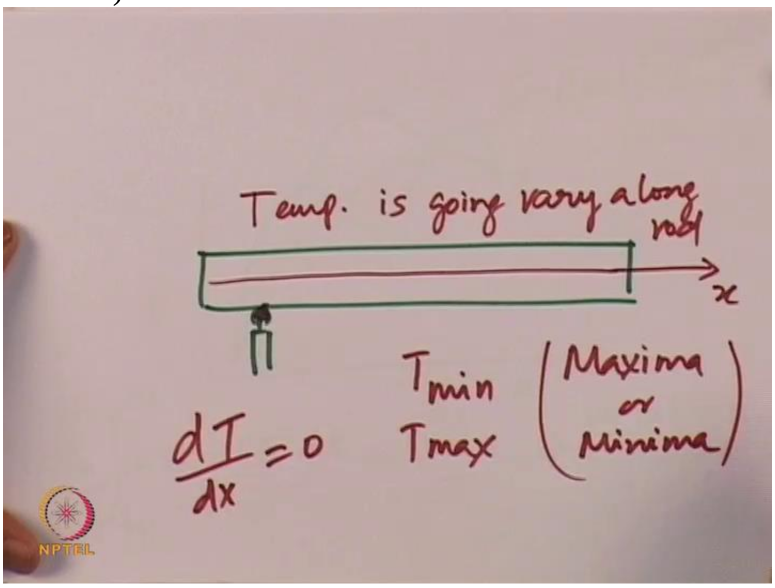
So he said let us say you take at each of the these moments in the journey, you take the potential energy and you take the kinetic energy so you take the difference between the kinetic energy and the potential energy and you integrate it from the point let us say you are talking about time t1 to t2 and I am integrating to dt and I am integrating on each of these instances. He said the true path will have this particular integral to be the minimum which was really fascinating for me because some how the physics has something unique about it.

What is happening here is regardless of whatever we come up we plan or we kind of predict somehow the reality in this case of this example has something unique about minimizing certain integral. And this integral is going to be the integral that defines the true path. So remember when we started with this entire lecture on variational methods we said the problem of integrating a PDE is going to be replaced by a function that is going to have the minimum at something.

We said it might have not clear to you at that point but you might be understanding or appreciating where we are leading to. So what is going to be the case here is the true path is going to be something that is going to have a minimum. And this particular minimum is gong to be something that we are talking about as the action so this integral is called as the action integral. So in other words it is also called as the variational principle. So the variational principle in this case is nothing but the integral from t 1 to t 2 at each of this instance where I take out the potential energy from the kinetic energy and I am integrating over the time.

So let us assume that there is no potential energy it will go the straight line because there is not going to be any pull so it is going to go in the straight line. But in the reality you have certain potential energy that is going to pull it in some ways. So we have a path in space in which there is going to be the minimum of this integral. So the reason why we call this as calculus of variation not just calculus is something that needs to be explained.

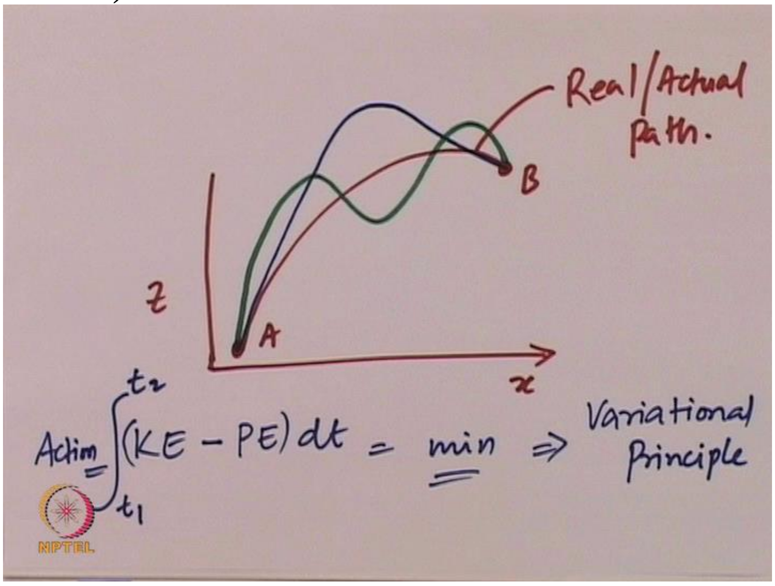
(Refer Slide Time: 20: 00)



So let us assume that I am interested in finding out the temperature maximum or minimum. So let us say this is a rod and I am heating this rod at a point I am heating it here, Let me make it little red so it is quite to be heated and the temperature is going to vary along the rod but there is going to be a point where the temperature is minimum and there is going to be a point where the temperature is going to be maximum. So finding the minimum or maximum for this particular case assuming that the rod is thin. So it is going to be a point in line where it is going to be minimum or maximum so this is very simple problem of finding the maxima or minima where you have to find the dt along the dx should be equal to 0.

So we are finding the temperature gradient for which the let us say this is the x axis you are differentiating along the x axis for which you find the maxima or minima you will have the value 0 so this is the calculus of finding the maxima or minima.

(Refer Slide Time: 21: 39)



But this particular problem is completely a different kind of an animal. Reason is we are not having 1 path like in the case of the temperature we are having multiple paths and the multiple path is the function. So it is a function of functions what we are talking about and we are trying to find which of the path is going to be the minimum. The reason why we call it as calculus of variations is because we are interested in finding out which of the path is going to have the minimum. And that is why it is called as calculus of variations.

So we have now looked into the basic background behind why we call it as calculus of variations and I also gave you an example using a simple throwing the stone in the air as a kind of a problem to illustrate what is mean by the function that we are looking into and also gave you a little bit understanding about the action principle. So in the next module we will take a simple problem and then we will try to compute the value of the action principle or variational principle and take it from there on, Thank you!