Computational Electromagnetics and Applications
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Lecture No. 10
Boundary Conditions

So we saw in the earlier module how to arrive at one-dimensional absorbing body conditions I also mentioned in the case of the one dimensional problem the reflection is going to be actually ideal as in zero because we can pretty much arrange the value of delta t and Delta x in such a manner we can make reflection the the reflection go to zero but this is not going to be the case in high dimensional problems like in the case of two dimensional or or three dimensional problems we are going to see how we can extend this idea to a two dimensional case and that would be the focus of this module.

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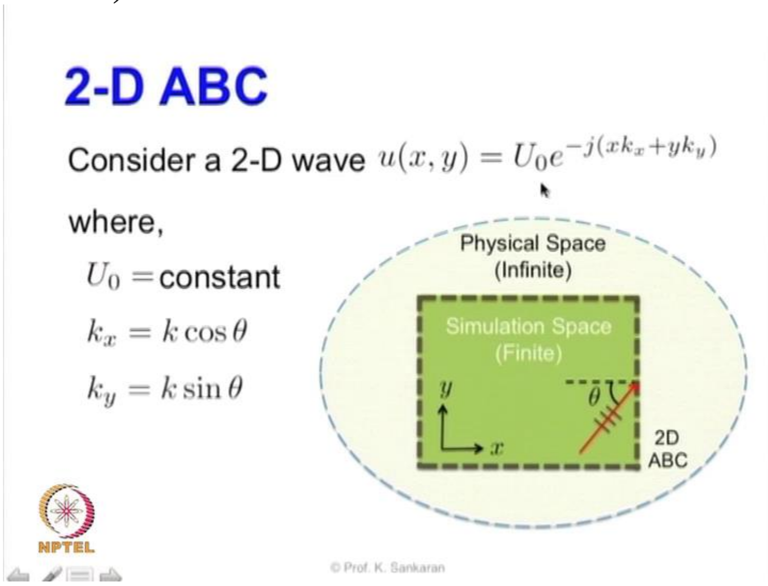

So let's start with a two dimensional problem where we have a 2D wave equation given by this u (x, y). So u is going to have X and Y dependents and U not is the magnitude of U and and e power minus J x k x + y k y is going to be the spatial dependents in the X and Y direction. And now from Elementary Mathematics we know that for a wave equation of this sort the k value can be split into k x n k y.

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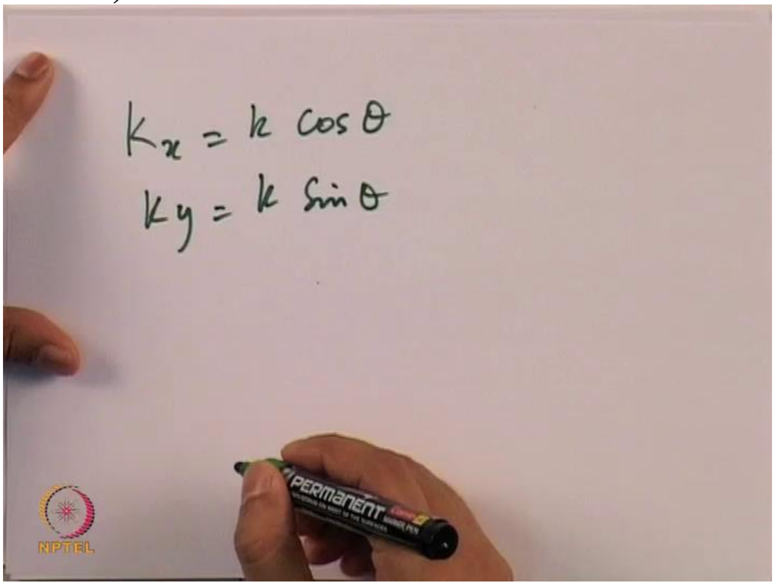

And they are given by the value k x is equal to k x cos theta k x cos theta K Y is equal to K multiplied by sin theta. So the value of the theta is going to be very important to k n o w what is going to be.

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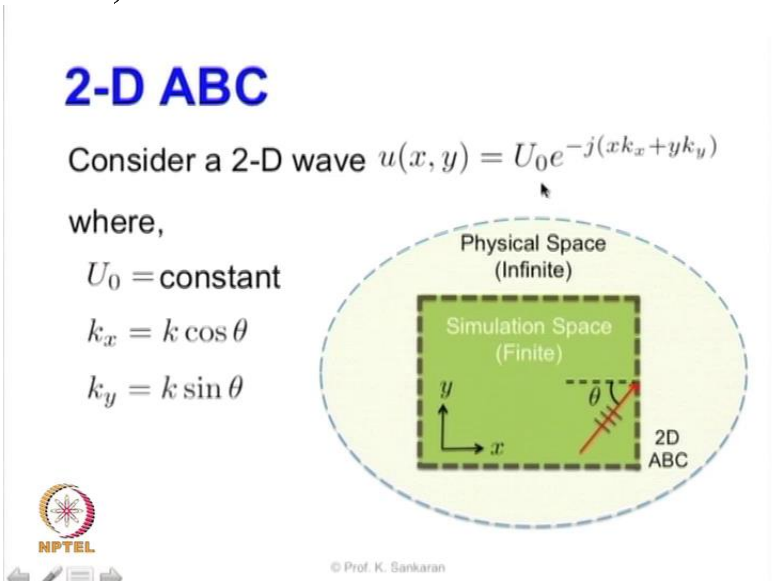

And we can see that from the illustration here theta is nothing but the angle at which at which the incident wave is going to impinge the boundary.

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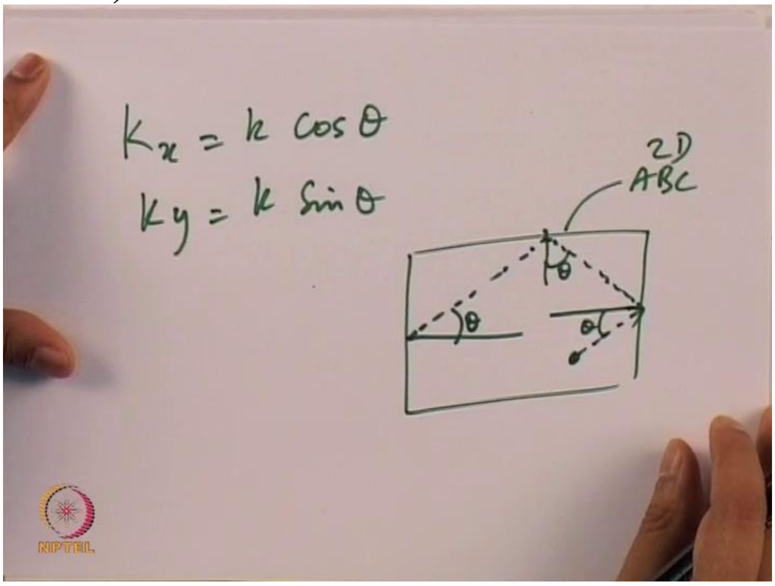

So in this case we have absorbing boundary conditions all around it's a 2D problem and there is going to be a source and then the wave is going to come and hit at this particular boundary and this is the theta we are talking about. Obviously when it comes and hits here it is going to get reflected and it will have a theta here and getting reflected and has a theta here. so we are not going to talk about all the different theta we are going to talk of first one thing and then we can get a generalized equation for other angle of incidence at different boundaries or absorbing boundaries that we are considering. Let's consider the first theta it is here.

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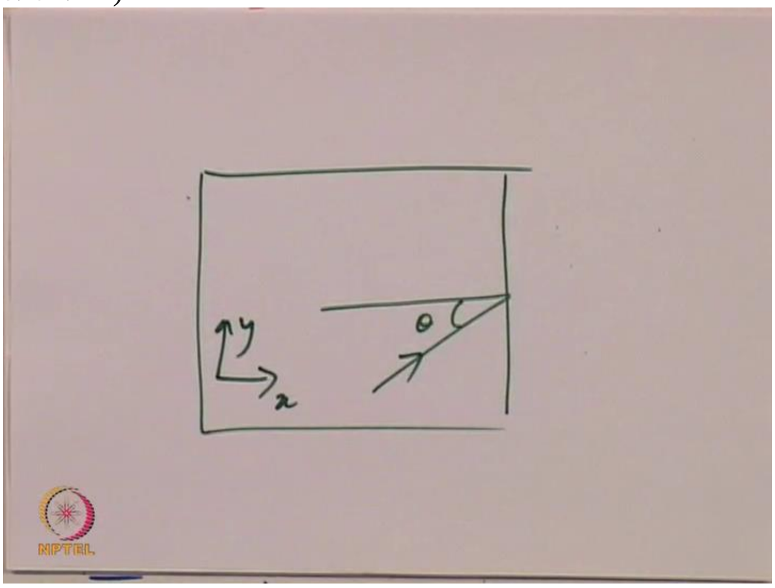

And we will write it as a separate sheet. So this is the normal and this is the angle of incidence. So what we are going to see is this is the x-axis and this is the y-axis we are simulating this infinite domain and finite space that is shown in this figure.

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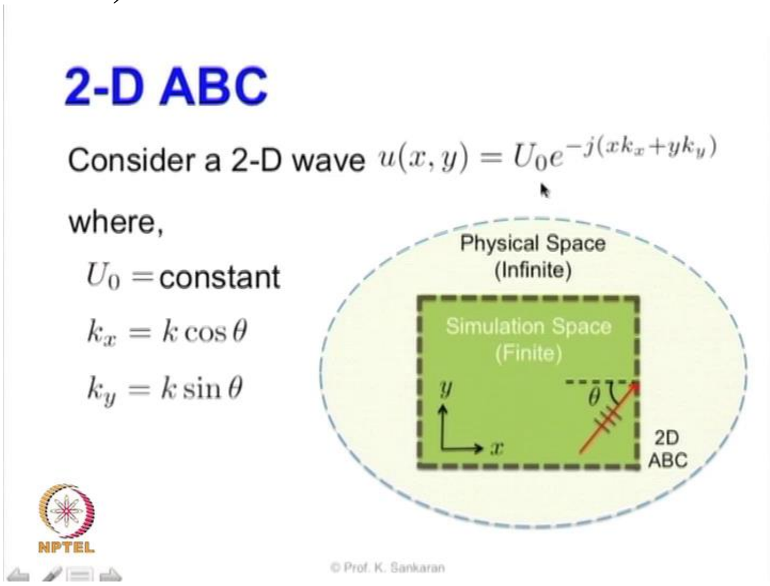

And now we can write the same way the partial differentiation of the solution U of x,y with respect to X and get the expression for that what we are going to do now.

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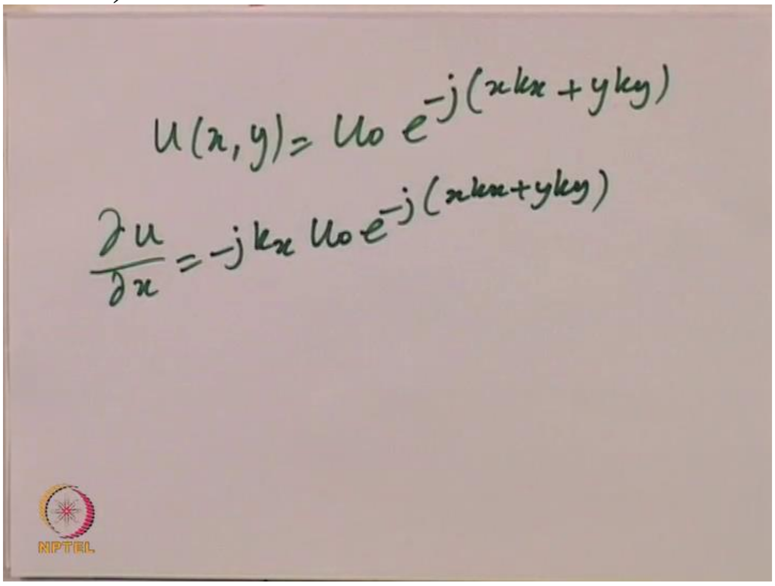

So we start with the solution of U of x , y which is equal to U 0 e power minus J .(X k X Plus Y k Y) . And now I am going to differentiate it with U with respect to X. What we have got is minus J kx and then U 0 e power minus J (XkX Plus YkY). Again this is nothing but u (x) itself. So we can write the equation as follows in a simple way.

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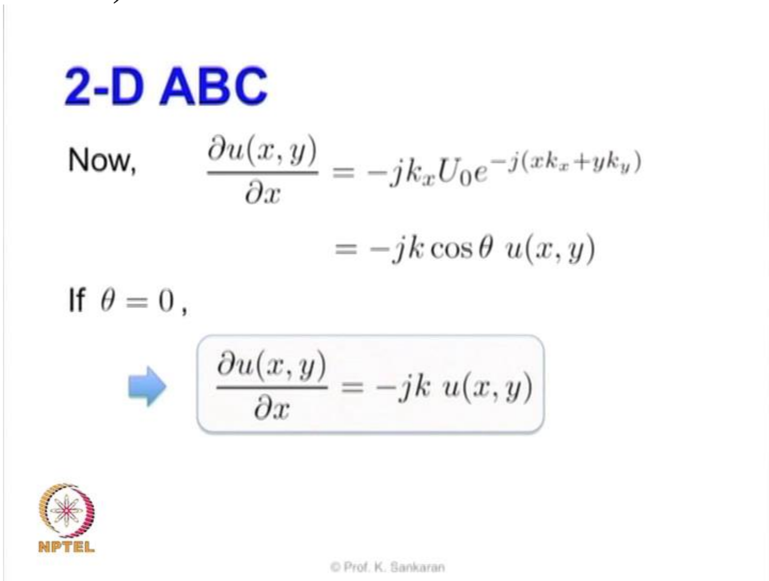

And here the K x value is going to be the value that we have written before. We are substituting the value KX equal to K cos theta and we are substituted the value for U(x , Y) for this entire U 0 e minus J (xkx Plus YkY) . And now if we put the value theta equal to zero we see that we get the standard equation that we got in the case of 1 one dimensional wave equation that we solved in the first part of this module for a 1D ABC.

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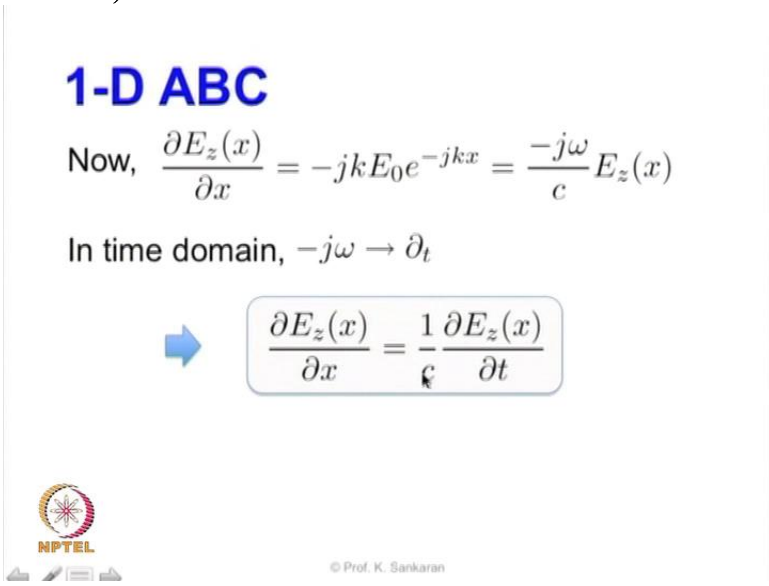

So this is the same as what we had seen in the case of 1 Dimensional ABC which is here.

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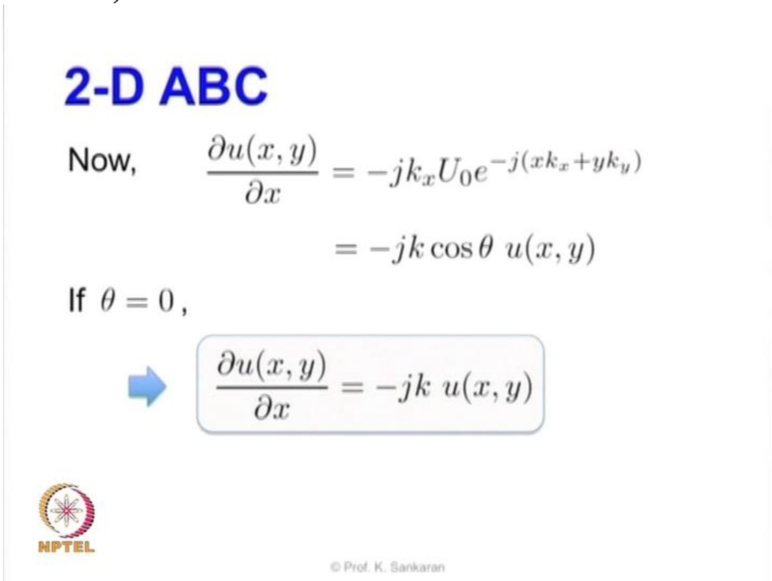

And what we have done is only substituted the value for J and K.

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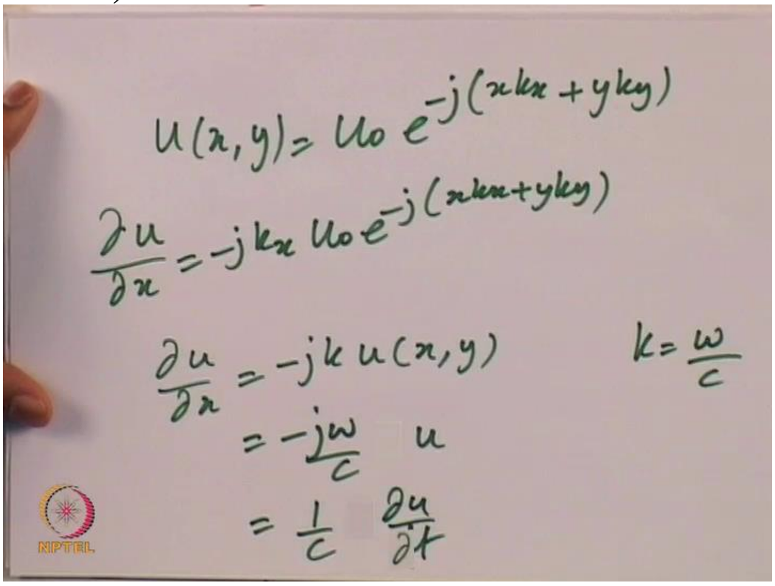

For example what we have got here is du of dx equal to minus J k u of x , y and I can substitute the value K equal to Omega by c so this will become minus J Omega by c u and J omega is nothing but minus dt. So this will be written as one by c du by dt .

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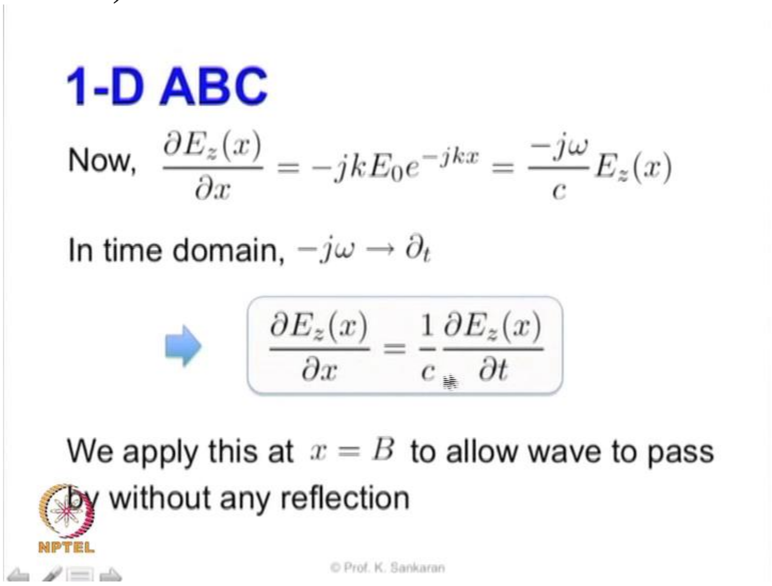

So this is nothing but the initial equation that we had also in the case of one dimensional ABC. So this is the exact equation.

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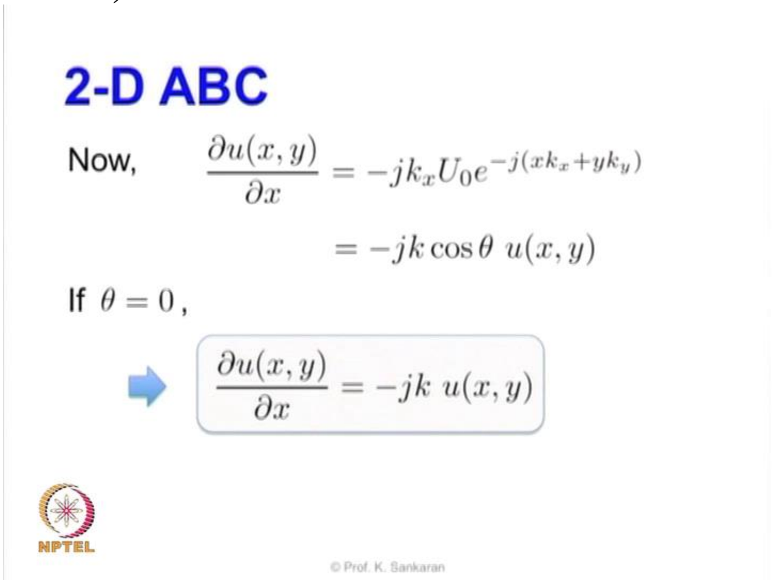

So this is making us feel confident that the formulation that have arrived at is the right equation for any angle theta.

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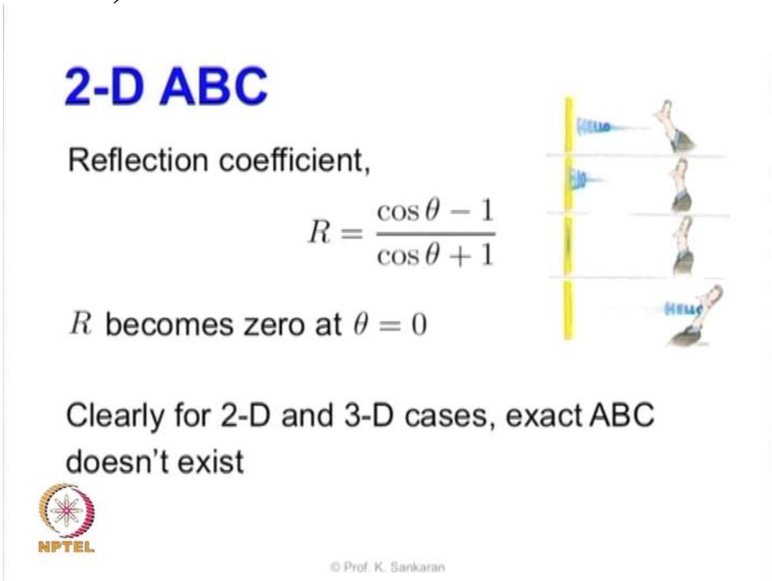

Now we can compute the value of the reflection Coefficient this illustration here is a funny way of representing what and ideal absorber should look like. You have something you are yelling at the wall and then it gets reflected that ideally nothing gets reflected but the really wall is not like something gets reflected and what is getting reflected is something that we can compute in the form of the reflection coefficient and that is what we are doing here the value of the reflection Coefficient is going to give us what is going to be the numeral reflection that we are going to get out of the particular boundary and it is going to be given by cos theta minus 1 divided by cos theta + 1 it becomes zero at theta is equal to zero like in the case of one dimensional ABC like we discussed earlier. Clearly as I said for two and three dimensional cases we cannot have this exact condition satisfied exact ABC is not possible. So we have to somehow come to terms with this when we are modelling with we see in two and three dimensional case.

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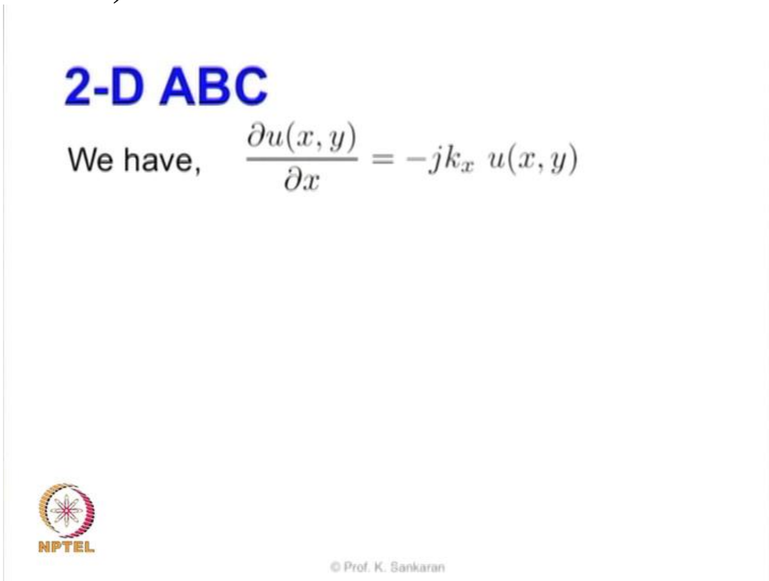

We have du by dx given by – Minus jk x u (x, y), which we derived before.

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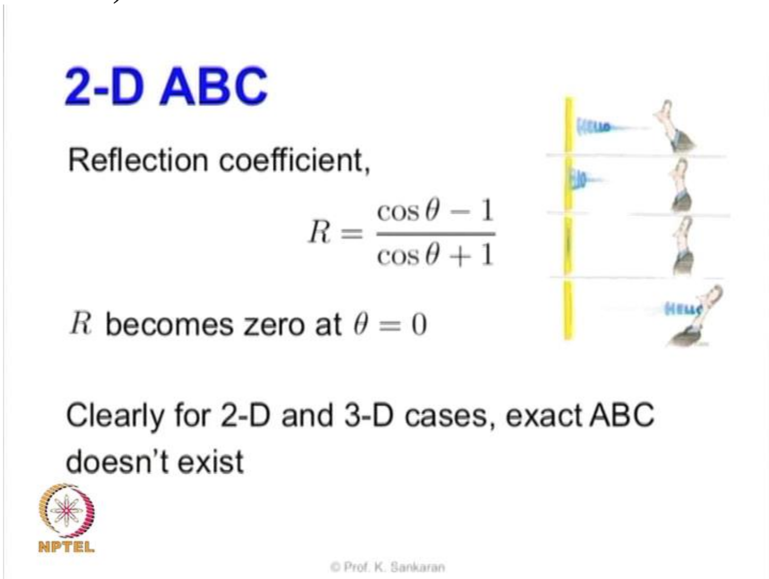

We can improve the order of accuracy of this particular problem by going higher in order because here the order of truncation is going to be very low. We can increase this accuracy by improving the value that we are going to compute for the for the boundary conditions.

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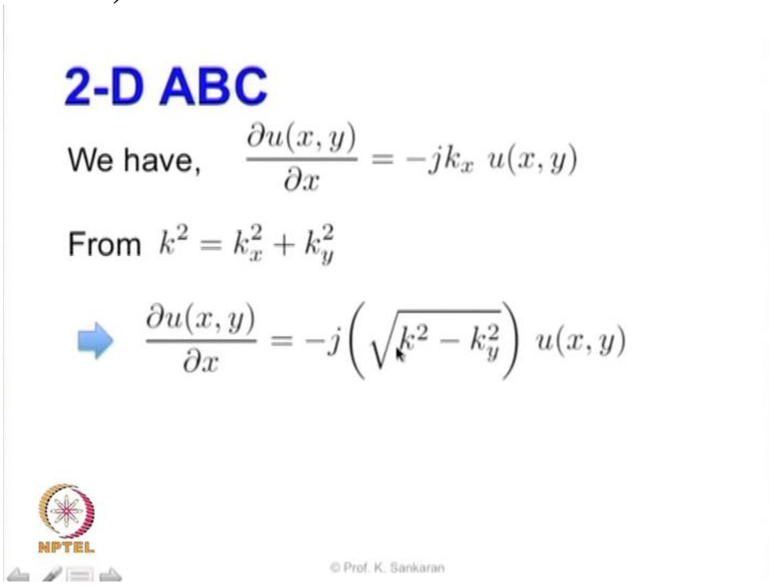

What I mean by that is we can we can improve this boundary condition itself if we can get a better approximation for k of x.

And that is what we are going to do now we are going to substitute the value of K of x as K square minus ky square square root. So this is directly from the equation K square is equal to K X square + k y square this is the wave number it has a x component and Y component and that is what we are going to substitute here. Once we do that, we know that we can expand this a little bit, so what I am going to do is I am going to change the situation a little bit by taking a out of the square root as follows

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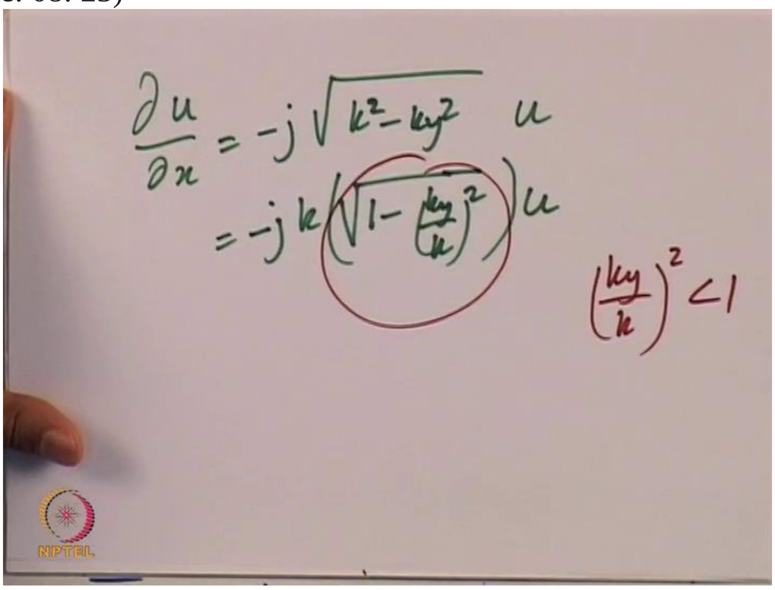

So what I am going to do du by dx is equal to minus J . I have got the square root K square minus K y square multiplied by u itself. Now I am going to now I am going to take the value of k outside the equation when I do that I have a square root again I will have 1 minus k y divided by k the whole square close the bracket multiply by U. And I know that I can expand this particular term using the Taylor's series because k y divided by K whole square equal to less than 1.

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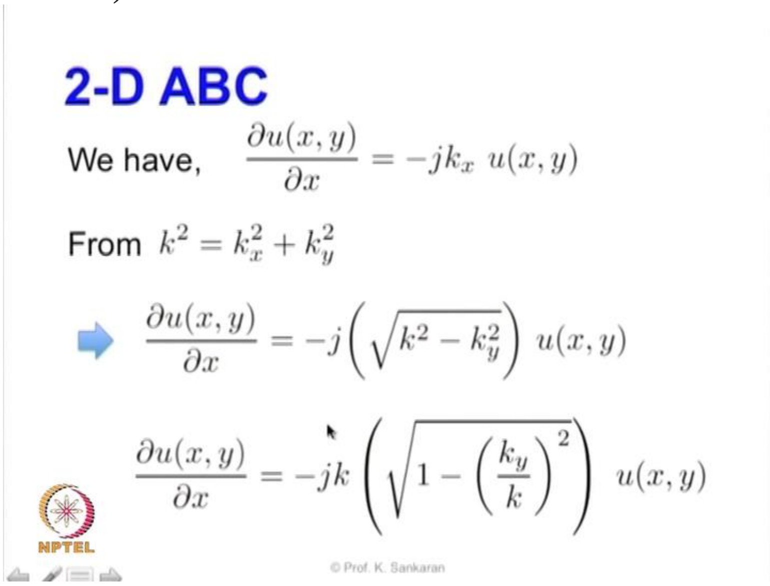

And because of this I can use the Taylor series expansion to compute the value for this particular term using the Taylor's series and this is something I have derived on paper you can see that step by step here. And now we are expanding value of this term using the Taylor's series as follows.

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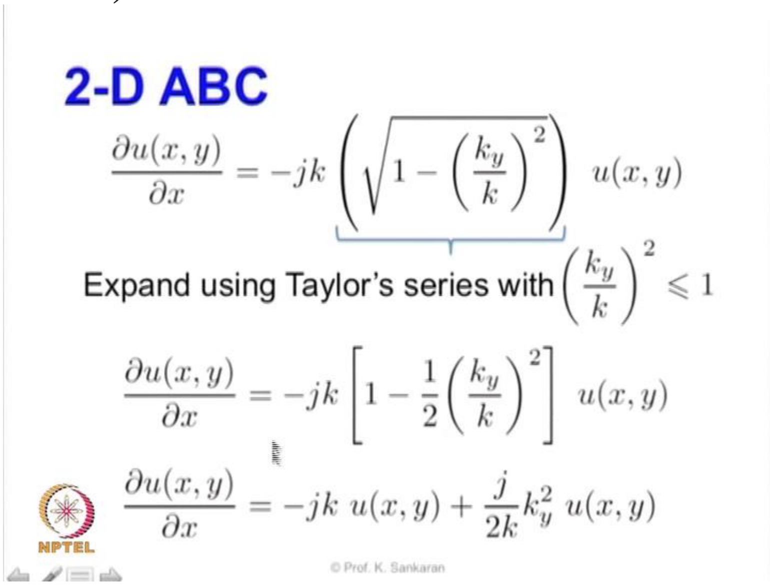

So here we are able to the value of e y square with not only the first term but also the second term. So the second term is going to improve the order of the accuracy of the boundary condition because only with this first term we are limited in the accuracy so we are bringing the second term also into the equation.

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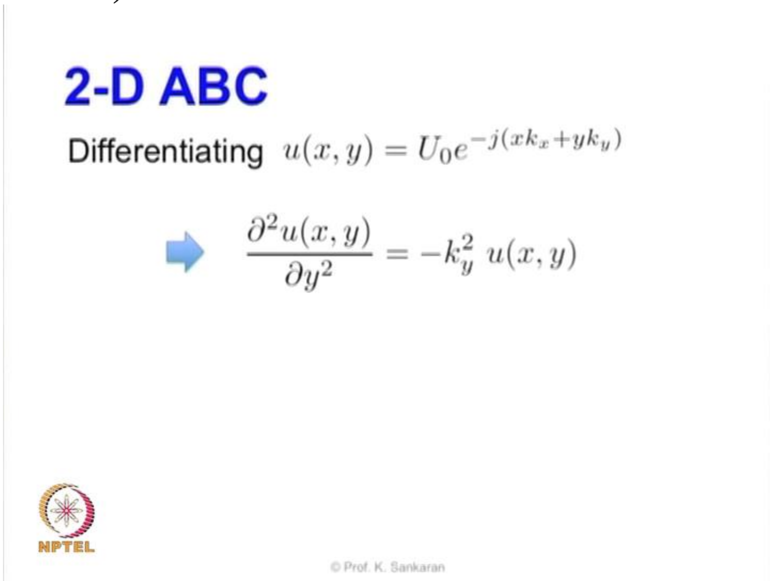

So this is again can be simplified by using the following approximation.

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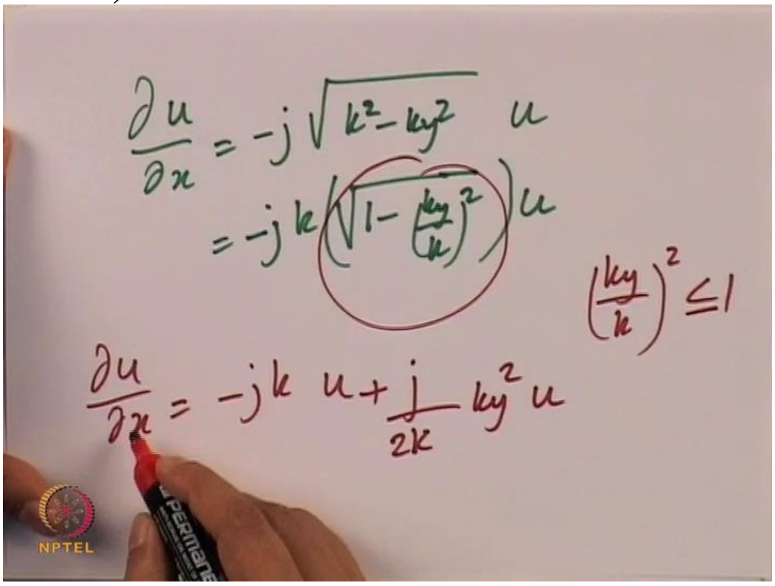

What I have got here is du by dx is equal to minus J k u Plus J divided by 2 K KY square and U. Now i know i can do the same thing for du by dy and I will get a set of equations .

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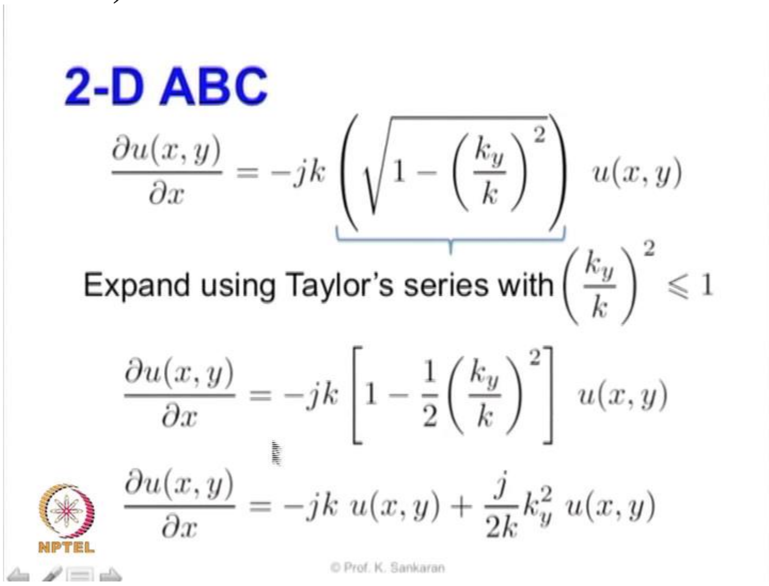

Let me do that for du divided by dy and see how i can club that equation into this equation.

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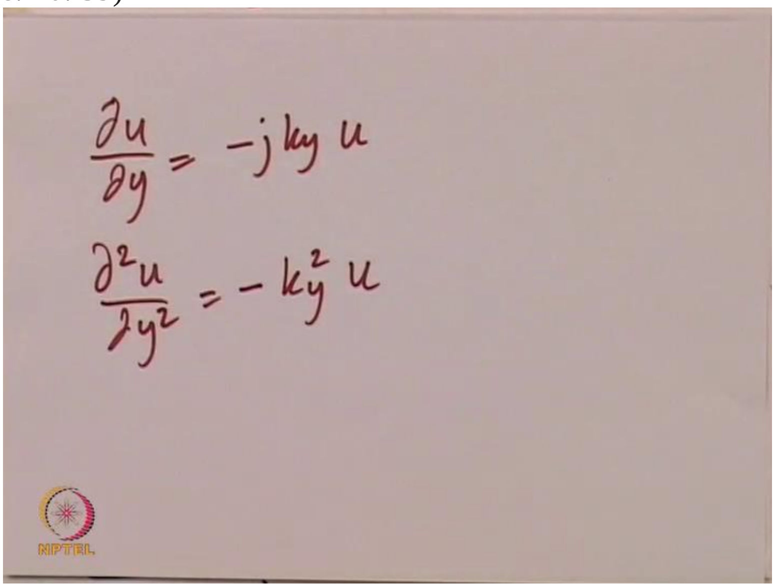

What I am going to do is I am going to differentiate U with respect to Y first time. What I will get is minus J ky U and now I am going to differentiate the second time d square U divided by dy square. This is going to be equal to minus J and I am going to get another minus J here so this minus J multiplied by minus J will become Plus J square and this will be equal to minus 1 and I will get ky square and U again.

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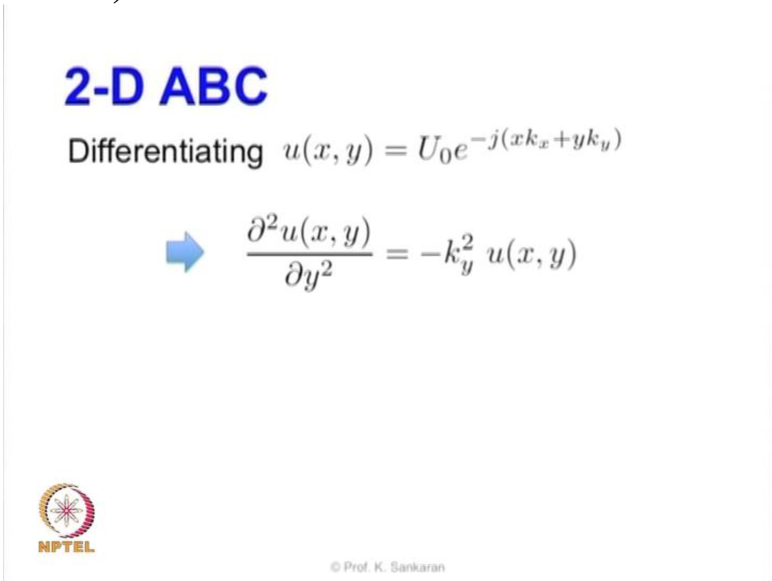

And that is what we are going to see in this particular expression here. You know how we got this equation by differentiating U with respect to Y twice. Once I know this is the value for d square U by dy square I can directly plug it in in the case of ky square.

Because remember in the case of the equation what we have here.

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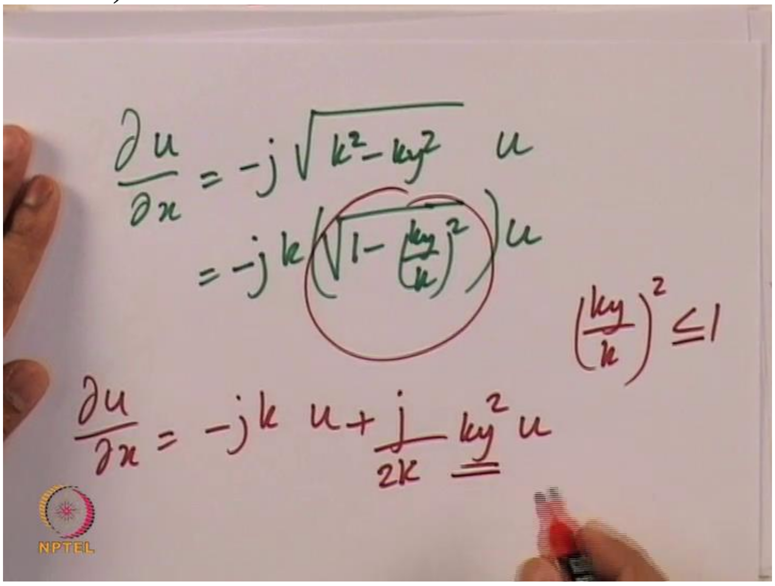

we have a term for k y square U

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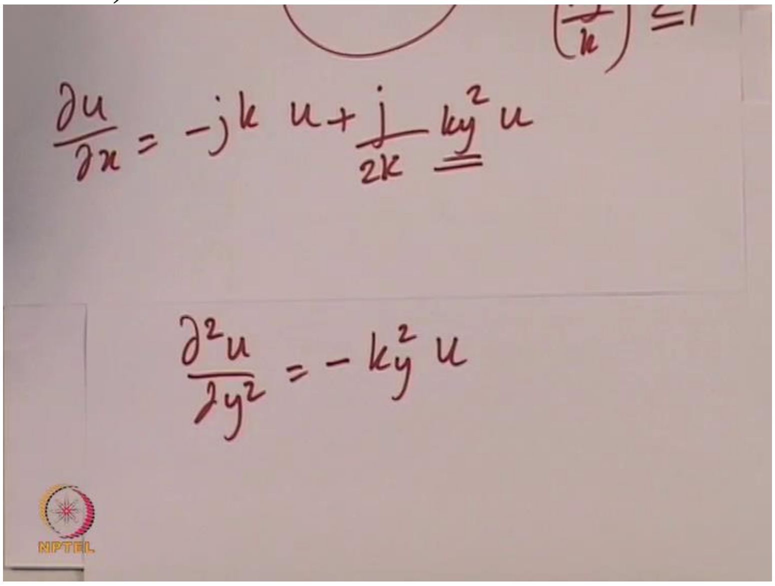

so I can substitute the value of k y square U as d square U divided by d y square. That’s what I am going to do here.

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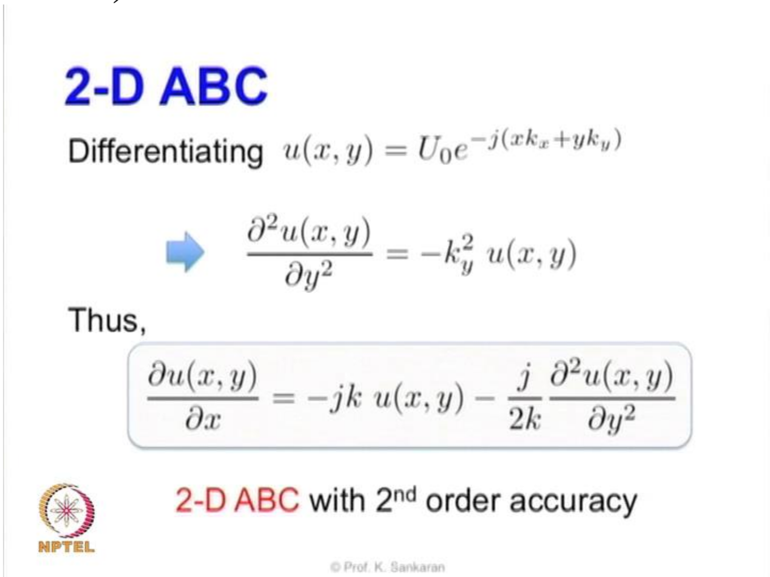

And I get now second order accurate ABC for the 2 dimensional problem. And this is boundary condition called as Engquist Majda boundary condition named after named after two eminent mathematicians who introduced this type of boundary condition. And the reflection Coefficient like in the case before going to be calculated for this particular boundary condition as well because we have a second order accurate that means our reflection Coefficient is going to get improved.

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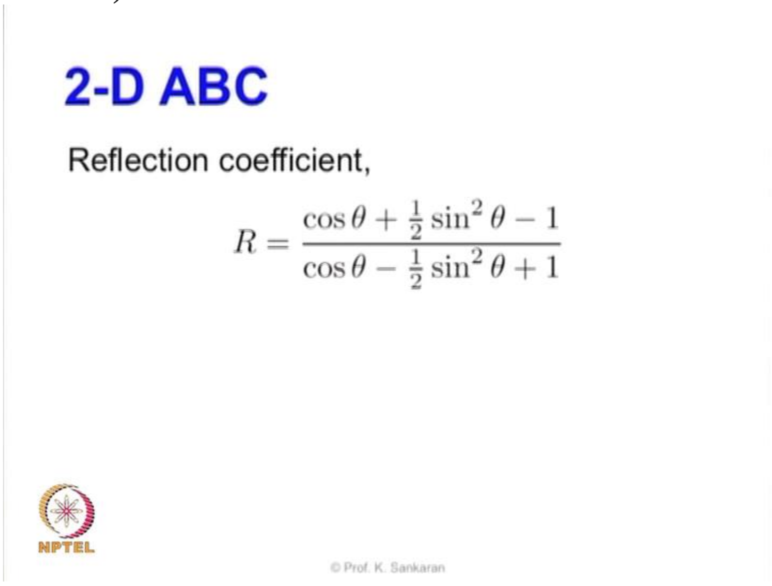

So what we get as a reflection Coefficient for this particular problem is given by this equation cos theta plus 1 divided by 2 sin square theta minus 1 divided by cos theta minus 1 divided by 2 sin square theta plus 1.

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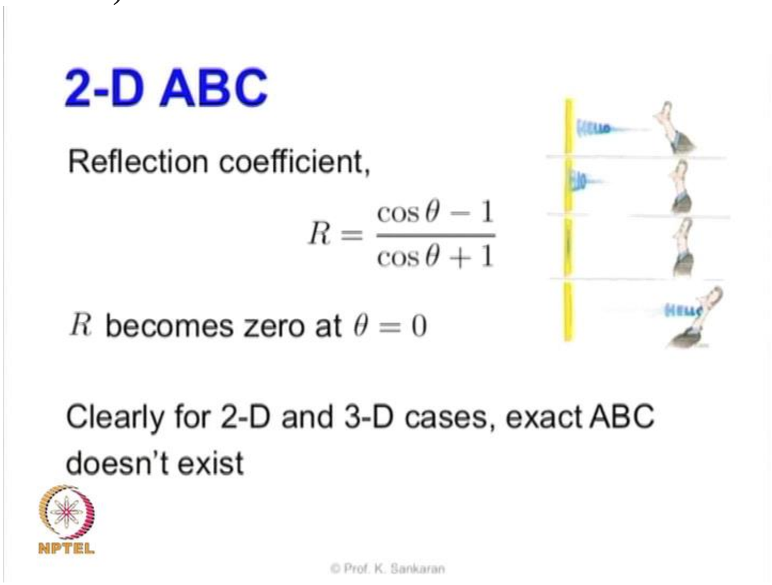

Remember in the previous case what we had as the reflection coefficient is cos theta minus 1 divided by cos theta plus 1.

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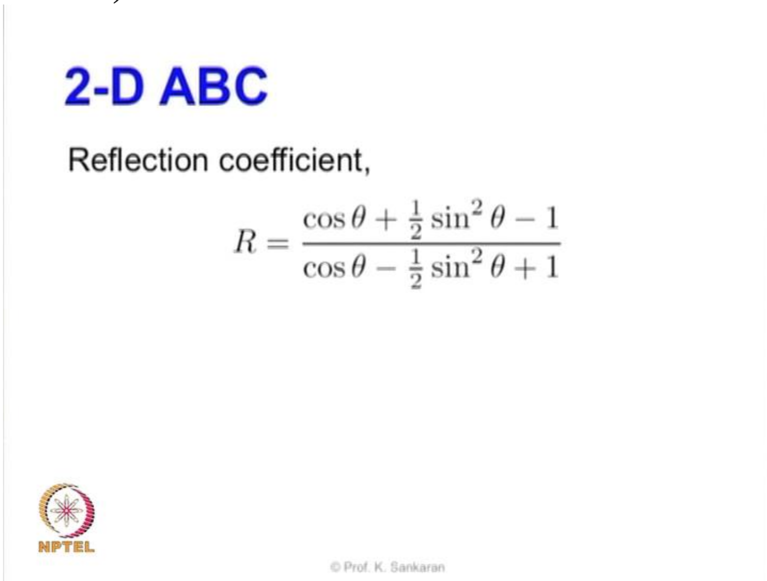

In this case what we are getting is cos theta plus 1 also there but you also have a term that is coming additionally so this is a second order accurate so you have additional term on the numerator and denominator and you can interpret this equation by a way that I have shown that you have an additional term coming in to play because we are getting higher order accuracy for the preparation for K of Y.

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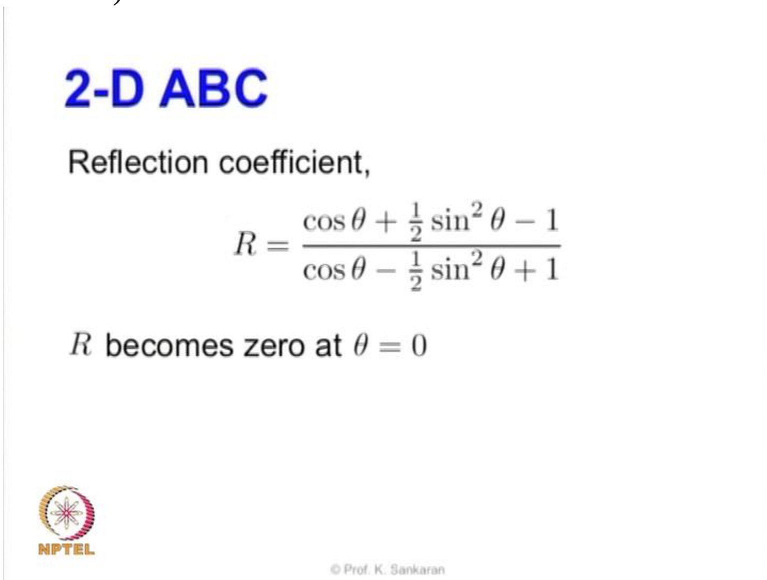

So now when we plug theta is equal to zero in this equation we should ideally get the reflection to be zero.

And that's what we are seeing here when you put theta is equal to zero this entire equation becomes zero because cos theta will become 1 and this will become 0 and then cos theta minus 1 and then cos theta plus 1 is at the bottom so this entire equation will become 0.

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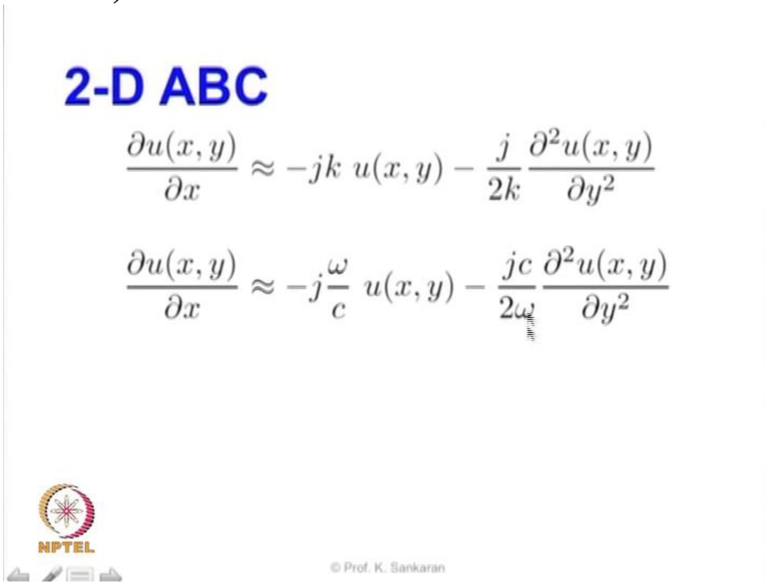

So this particular way we can expand the boundary conditions obviously we have to substitute the value for the first term as well we are going to do is we are going to substitute the value in Omega divided by C and we are substituting the value for K as Omega divided by C as well.

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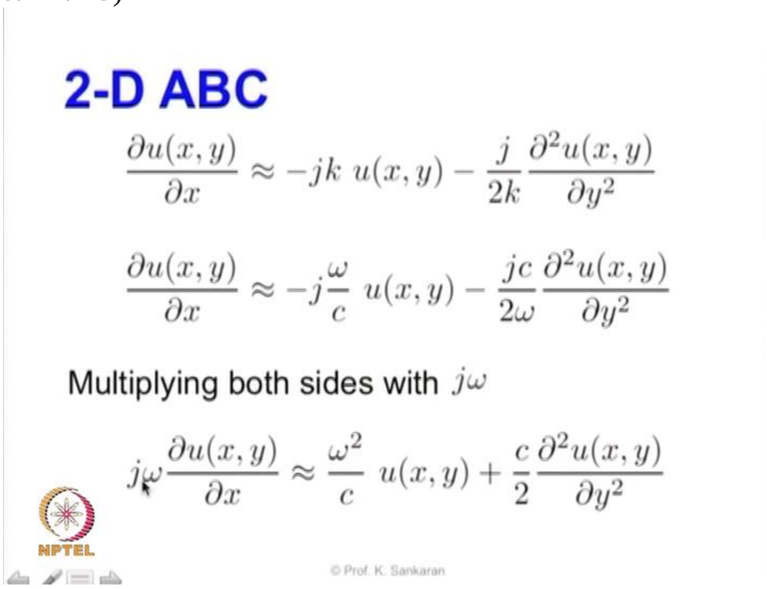

And we can multiply both sides by j Omega and we will get a formulation of this sort. So when you multiply by j Omega what you will get is the term will become omega square divided by C and this term will become C divided by 2 and the other term of J Omega is going to be on the left hand side.

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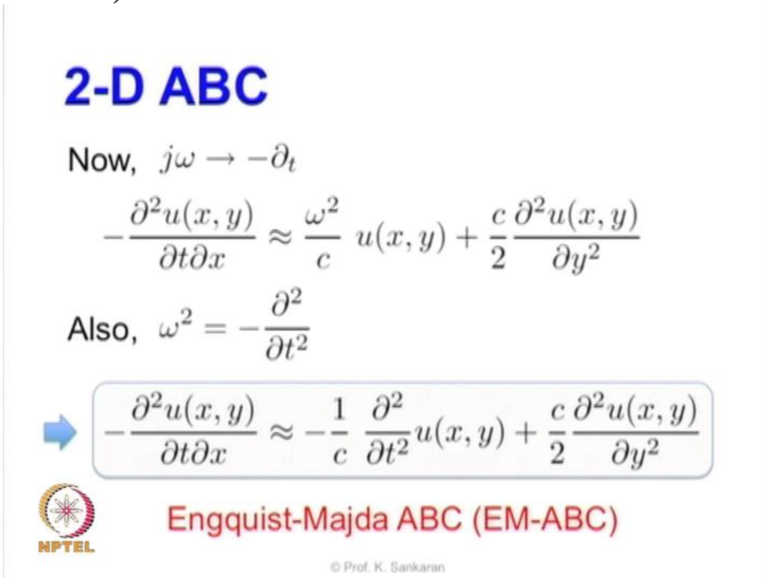

And now we can substitute J Omega is equal to minus dt so you will get a partial differentiation with respect to time and space on the left hand side. And you have a second order differentiation with respect to y axis on the right hand side. Again you can substitute the value omega square is equal to minus Doe square by Doe t square and and we get the final value for the Engquist -Majda absorbing body conditions what you get is the time derivative and spatial derivative here on the left hand side second order time derivative and second order spatial derivative on the right hand side.

This particular boundary condition is easy to implement in case of a final difference method where you can substitute for simple Centre difference scheme on the right hand side to get the value for the particular term and you can do both the forward in time and forward in space to get the left hand side so we have come to the end of the absorbing body conditions itself will stop here at this point will come back and look into the most important class of boundary conditions which is actually the boundary layer which is called as the perfectly matched layer and we will see that how we can implement for the final difference method and also give hints on on how we can extend it for for other methods thank you