Computational Electromagnetics and Applications
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Lecture No 5
Finite Difference Methods-II

Welcome back! So where are we now, we discussed about Finite Differencing Method whether forward differencing or backward differencing or Central Differencing. We talked a lot about accuracy and dispersion and thus it bring us to a very important point. Imagine that you are trying to build a house and you wanted the house to stay stable for a long time to come. It is not that you get into the house and you want you are inhabiting it and suddenly it all crashes on your head, this is not what you want.

Likewise in the case of any numerical method accuracy is very important but not enough, what we need is a stable scheme. What I mean by stability is something very particular to numerical methods and also very interesting aspect of computational electromagnetic.

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So let us say I am interested in making sure i have a error that is starting at one point and that error should not propagate into a big Fiasco and create an explosion in the sense of numerical error. So at time step n equal to 1 should not propagate from n equal to 1 to n equal to n to n equal to a very large number. So the most important thing about numerical stability is it is important that we understand the methods long time behaviour. In that sense stability is not only a sufficient condition or necessary condition but it is both a sufficient and necessary condition for any problem to converge.

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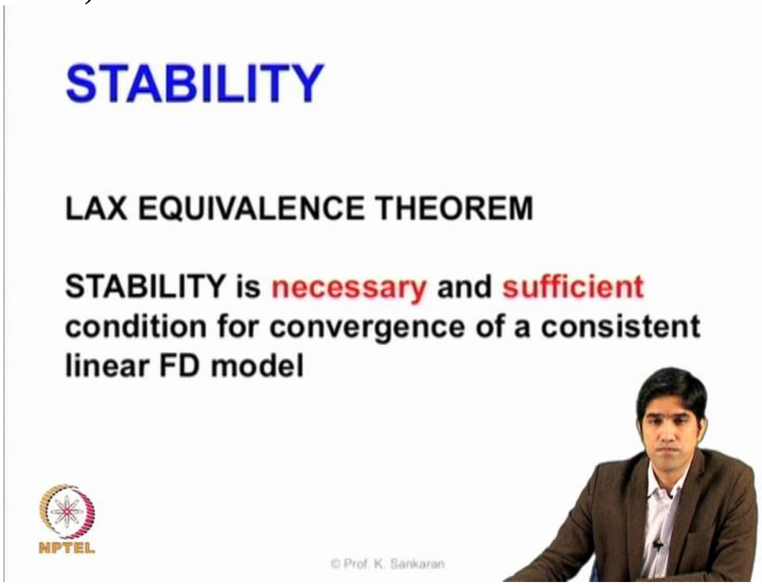

And that is what is put in a very very simple way in the Lax Equivalence theorem named after a famous mathematician Lax. So the Lax Equivalence theorem says stability is necessary and sufficient condition for the convergence of a consistent linear Finite Difference Model. In other words when we talk about a Finite Difference Model which is a linear one it is enough to know that it will be stable so that we can say that this method will sufficiently and necessarily converge over a period of time and spatial discretization. So if I can reduce the time step and the spatial step, if the method is stable it will converge and that is the main point.

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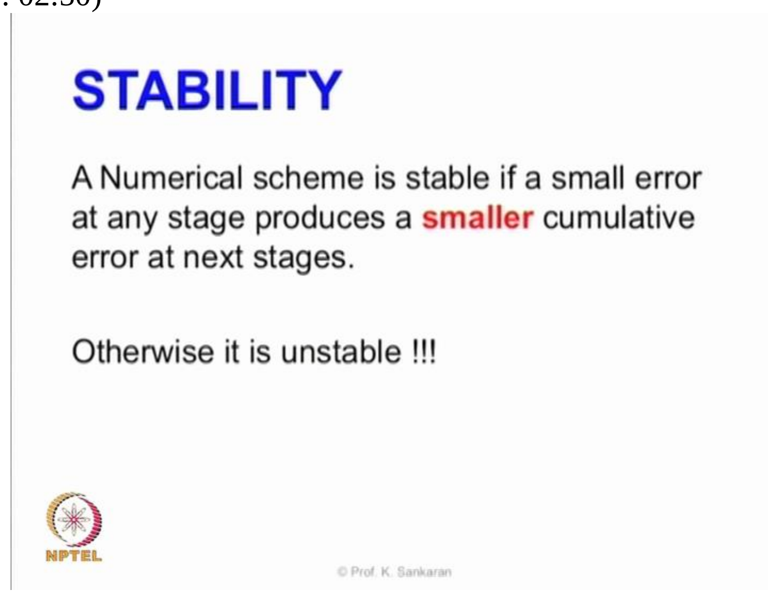

With this being said we can also say that numerical scheme is stable if a small error at any stage produces a smaller cumulative error at the next stages. So this is what we saw in the case of the example video where you saw the dominoes falling one after the other. So you have to make sure that a small error at any stage does not lead to a explosive numerical error at the later stages. So we have to make sure that the error is smaller over a period of time. Otherwise the scheme is said to unstable.

Let us say we wanted to know how one can compute the stability or at least how one can get a physical or mathematical sense of knowing what method is stable or what method is not stable. So for that let us do a simple mathematical analysis and we will see how this mathematical analysis helps us to know the stability of the method.

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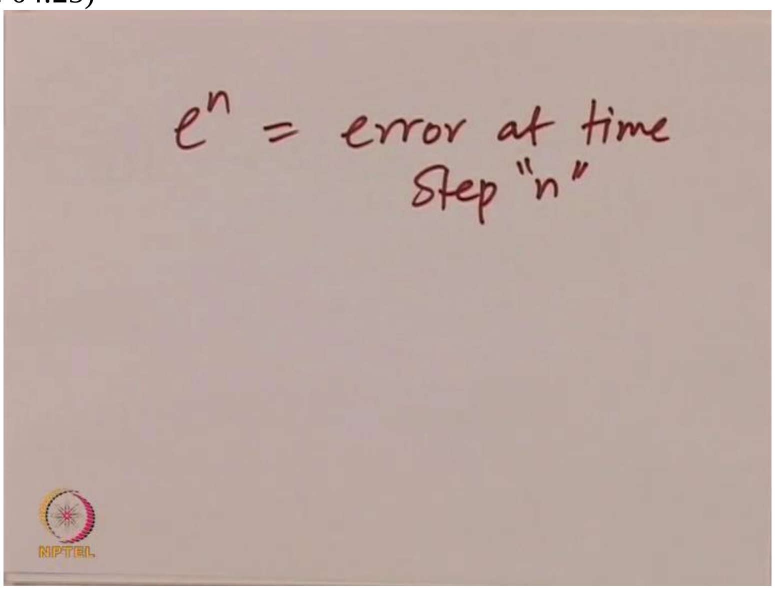

So let us say we define an error at time n as e n, so we say that the error at the time step n will be given by e of n (error at time step n).

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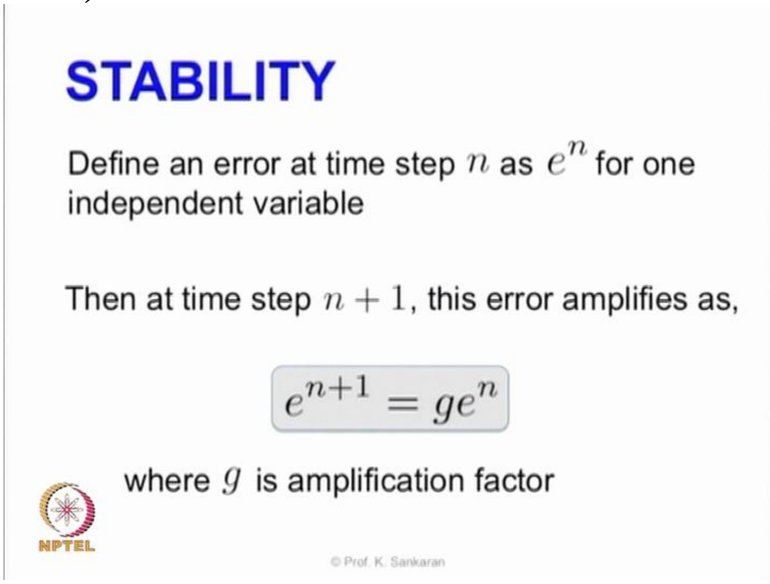

So this will lead us to ask a question, what will be the error at time n plus 1. So when we go from n to n plus 1 this error is going to get amplified, but how? So we say the error at time e n plus 1 is going to get multiplied using a factor g which is going to be called as the amplification factor.

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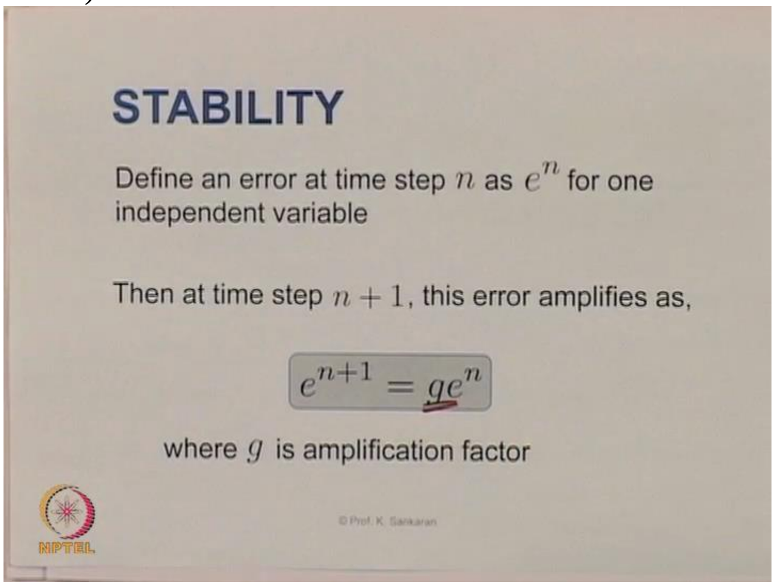

So in this case we write g as one number. But you can quickly realize when we go from a scalar equation to a vector equation what you will essentially have is e vector. And similarly you have g as a matrix. So in that case the g will be a matrix.

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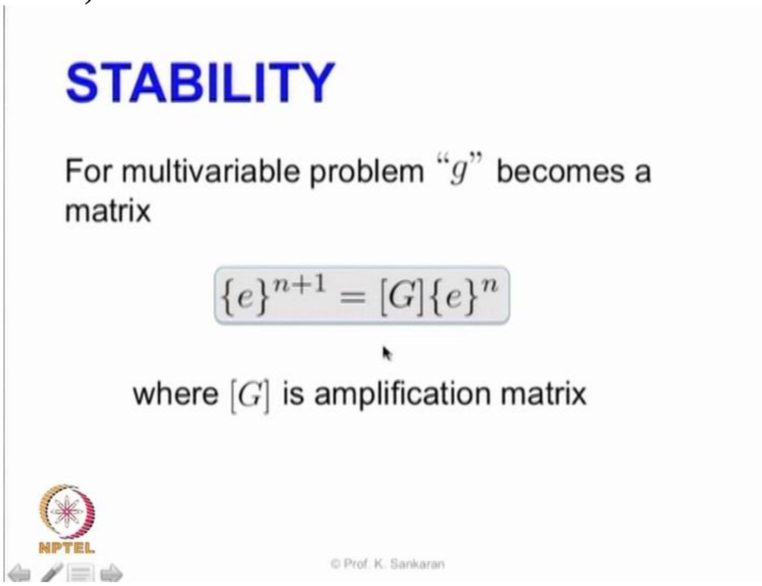

So when you have a multivariable problem where your independent variable is more than one variable. Let us say we have x, y and z g becomes a matrix here.

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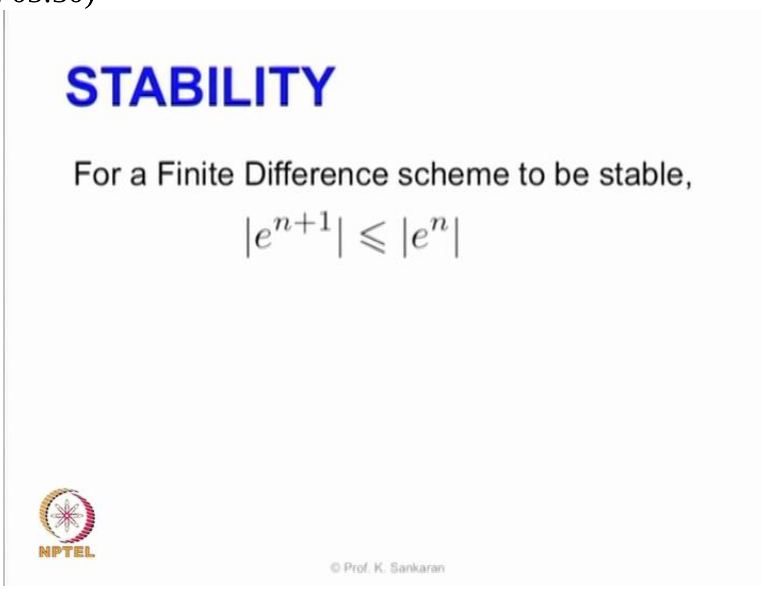

So for a Finite Difference Method to be stable we want certain condition to be satisfied. The condition is the net amplitude of the error at step n plus 1 should be less than or equal to the error at step n. In other words over a period of time I am going from n to n plus 1 so on and so forth or the amplitude of the net error should be less than what it was in the step before or else it is adding up.

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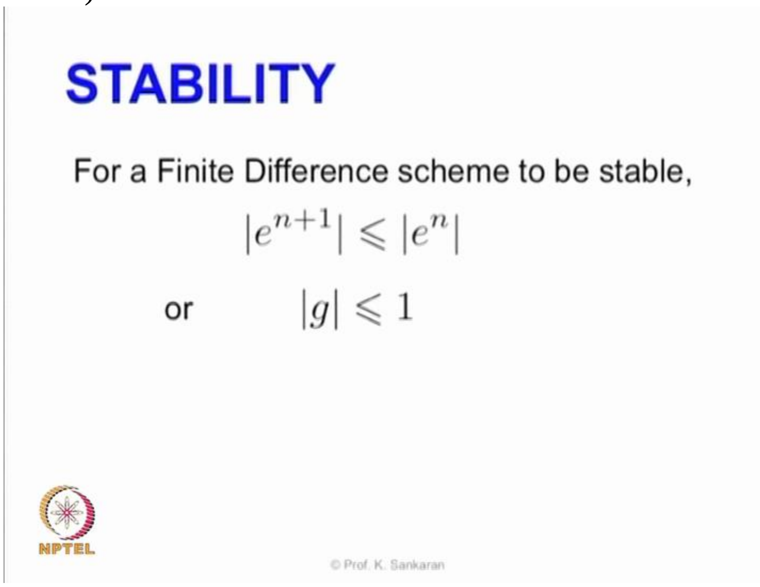

So in other words in terms of g it means the scalar value g should be less than or equal to 1.

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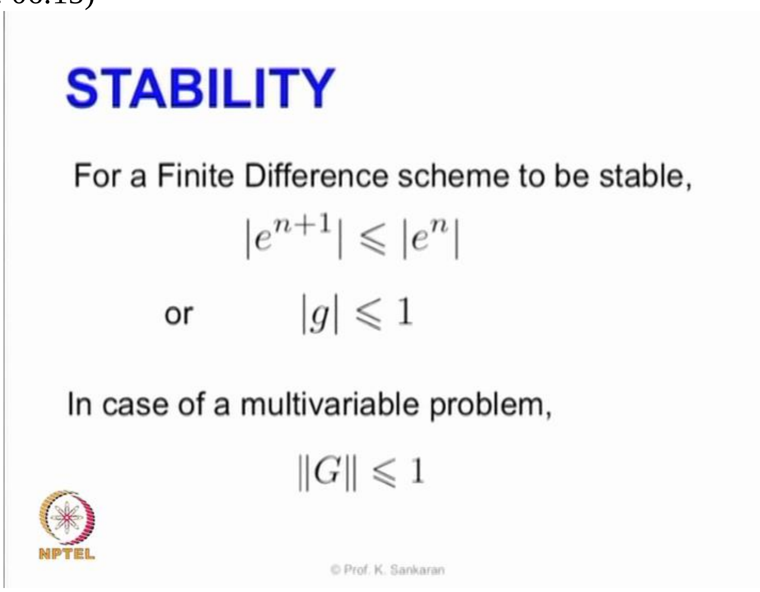

In case of the multivariable problem it will be the determinant of the matrix should be less than or equal to 1.

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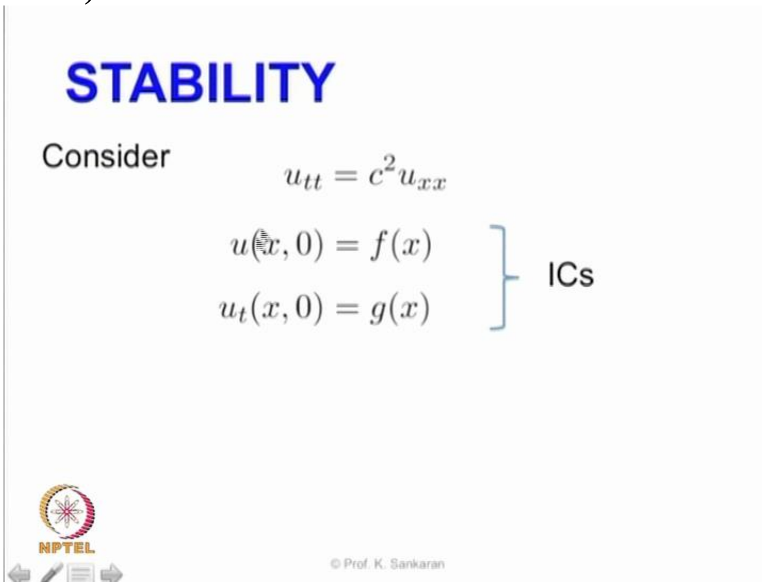

So let us consider the same old problem which we started in our initial analysis so this problem could be a very good problem to start working on stability analysis. So what we are going to do now is we have a problem given by this partial differential equation and obviously it is a second order equation so we need two initial conditions.

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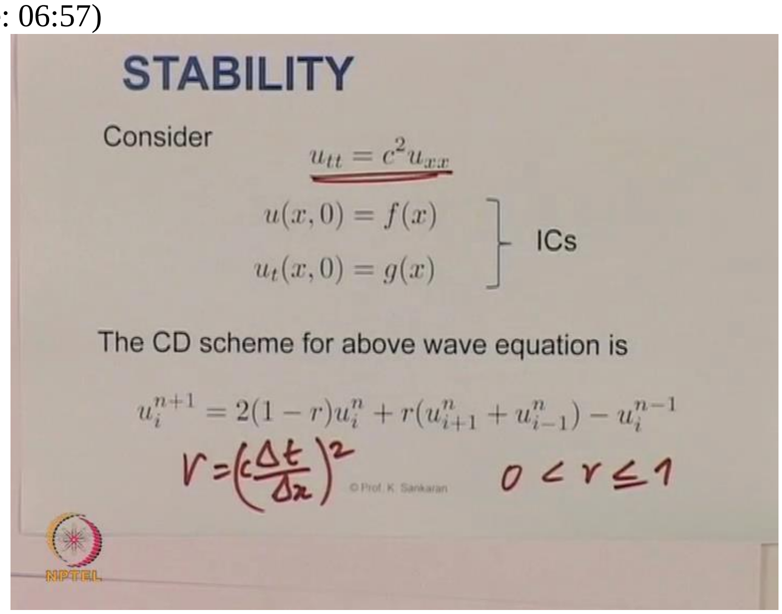

So the condition 1 is about the field itself and the condition 2 is about the time variation or the partial differentiation with this variable with respect to t. So these are the two conditions that are given. And we know that for this particular partial differential equation we can write the value of the Finite differencing scheme or we can write the entire partial differential equation in the finite differencing scheme. And we can take the u n plus1 of i on the left hand side and the other terms on the right hand side. And as you can see the value of r as we saw before is (delta t by delta x multiplied by c) the whole square and this is the same expression what we have. And we also saw in the previous case when r is equal to 1 this term will disappear we will have only the term on the second and third term. So for now we will take the general expression where r can be anything that is between 0 to 1. It is quickly more than 0 and it could be 1 or less than 1.

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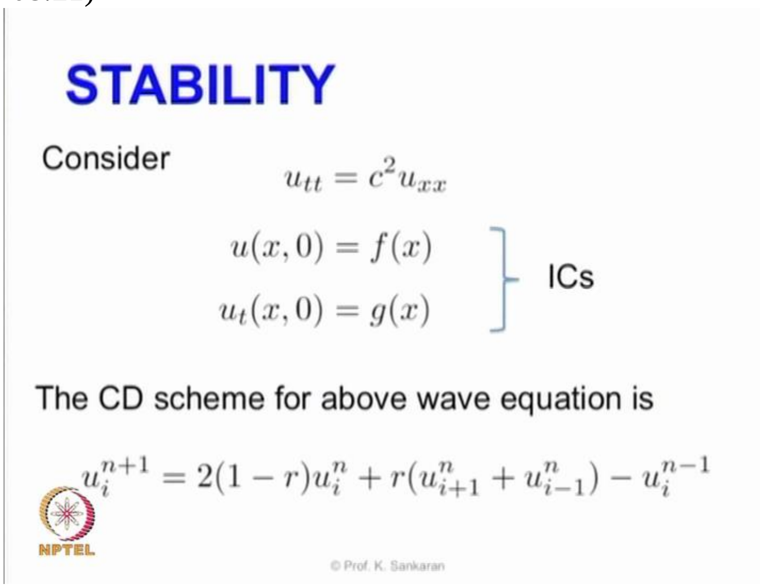

So for this particular scheme when you write the value of stability what we have is?

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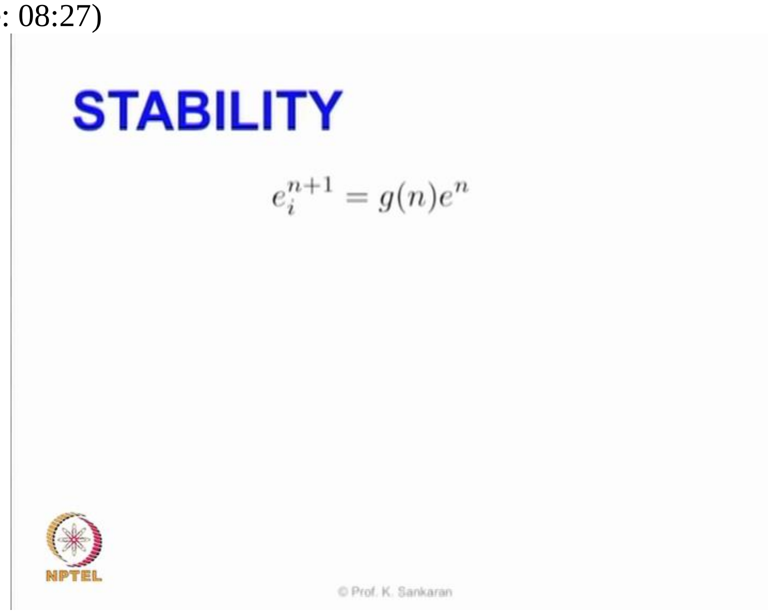

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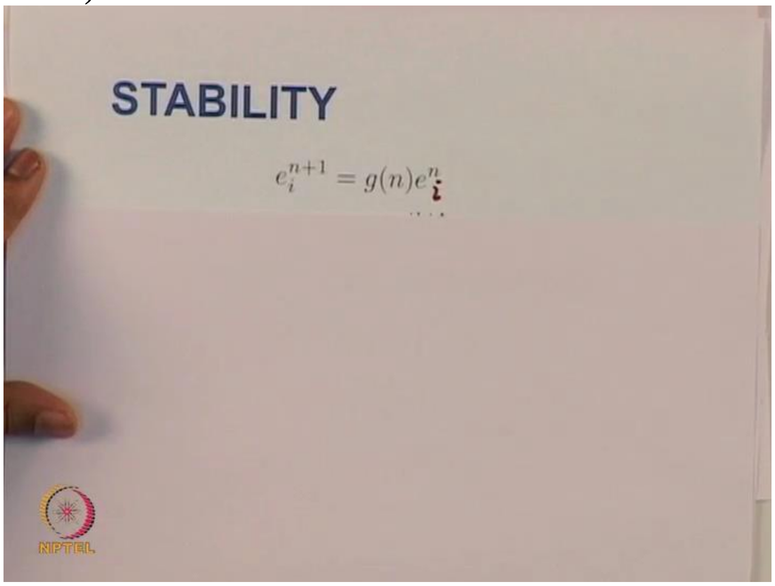

We have the error that is coming at the i th node at the n plus 1 th term will be a function of g (n) e n i. So here what we need to say is we have the value that is given by e n plus 1 is equal to g (n) e n i. So we are talking about the error that is coming from the i th node at n plus 1 th time step is equal to g (n) e n i.

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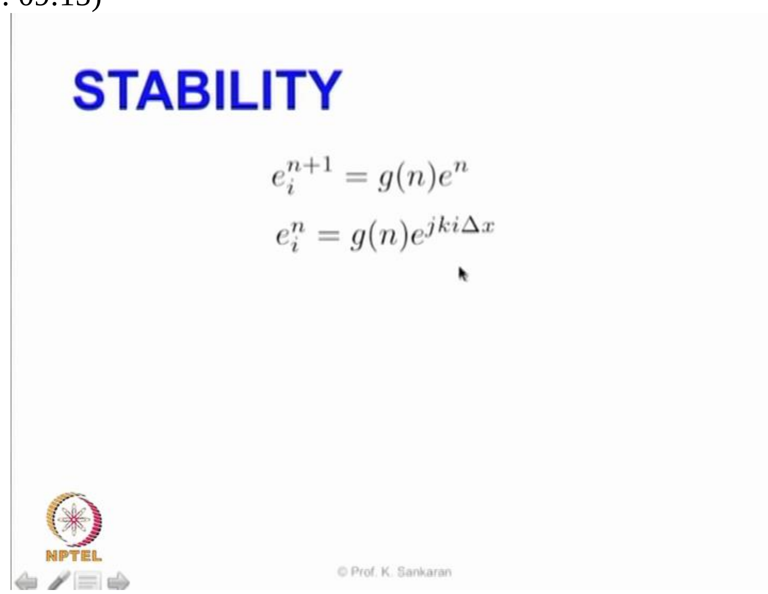

And obviously this is something that we can write in terms of delta x and delta t. So what we have is we write e of i n is equal to g (n) e j k i delta x. And what we have done here is we have substituted the value of the error as a function of the solution itself.

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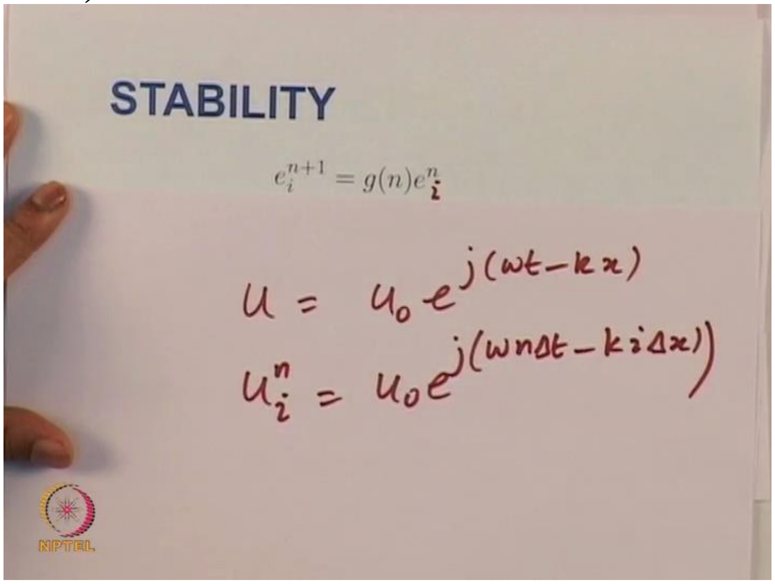

So it is not very important to know how we are coming here but obviously we say that the wave has the behaviour u is equal to u 0 e j (omega t minus k x); and the finite difference solution will be u i n is equal to u 0 e j (omega n delta t minus k i delta x). So we say in the case of the error that is coming at the step n plus 1 we can write this as a function of delta x.

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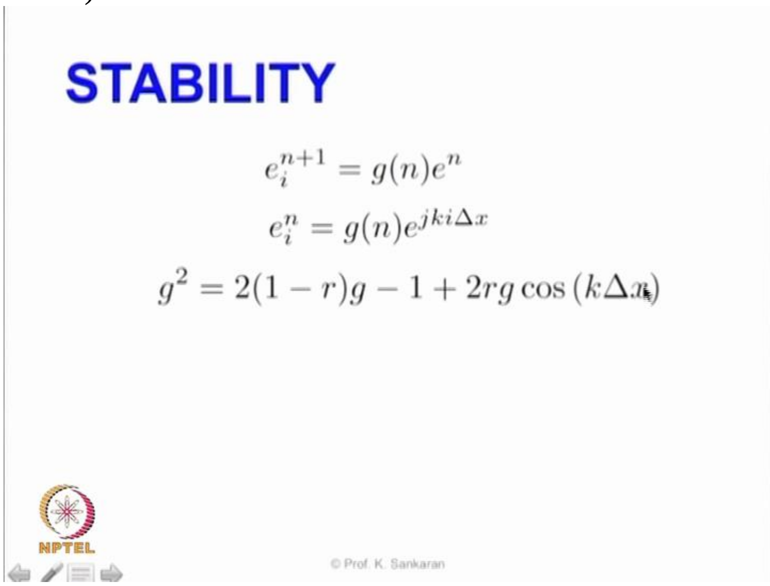

And that is what we have done in the point here so what we have written here is e i n is equal to g (n) e j k i delta x. And here we are only interested in the value of error that is coming from the delta x and of course the value of delta t is hidden within this n, so what we have is we can write the entire thing as the function of delta x and r as given in this particular expression. What we have is the amplification factors square is equal to 2 times (1 minus r) g minus 1 plus 2 rgcos (kdelta x).

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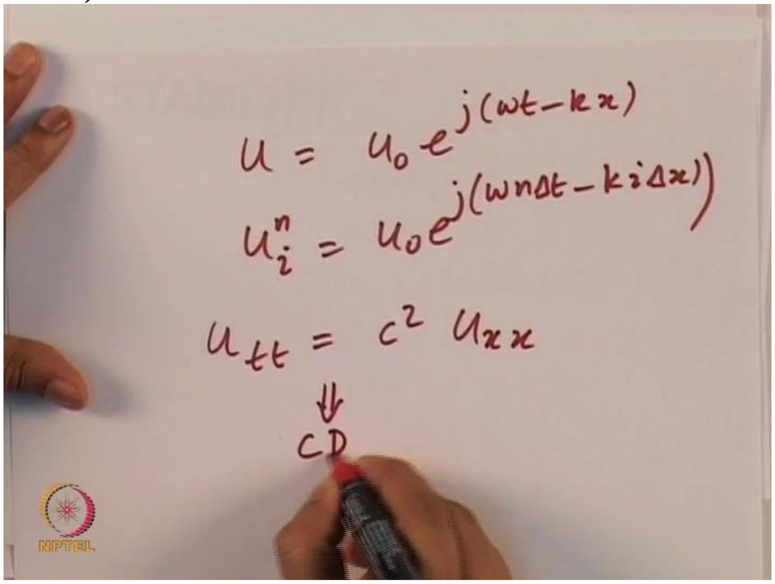

Here this particular expression is only valid for the case where our problem is defined by this partial differential equation and we are discretizing this by central differencing method. Finite differencing it using some other method this expression will change accordingly.

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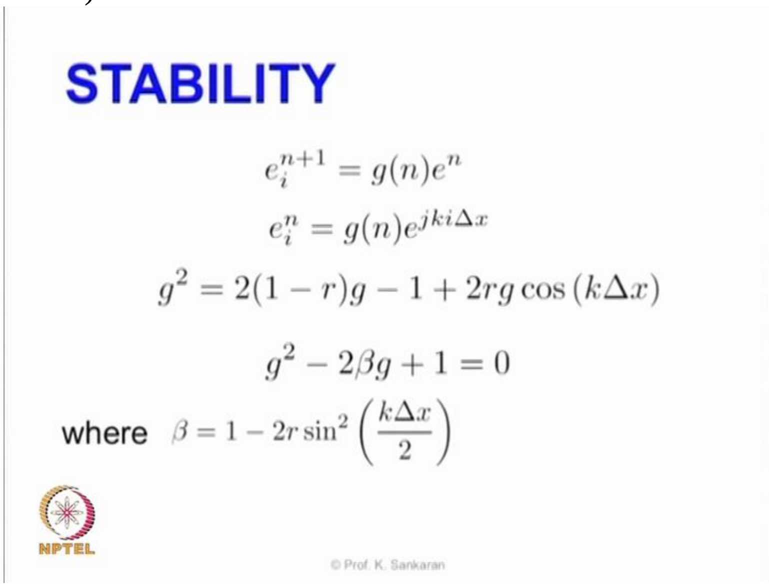

So what we have here is particular case where we are using central differencing method and accordingly you are getting an expression as a function of delta x r, and we are able to now bring the value of g as a value that we need to compute.

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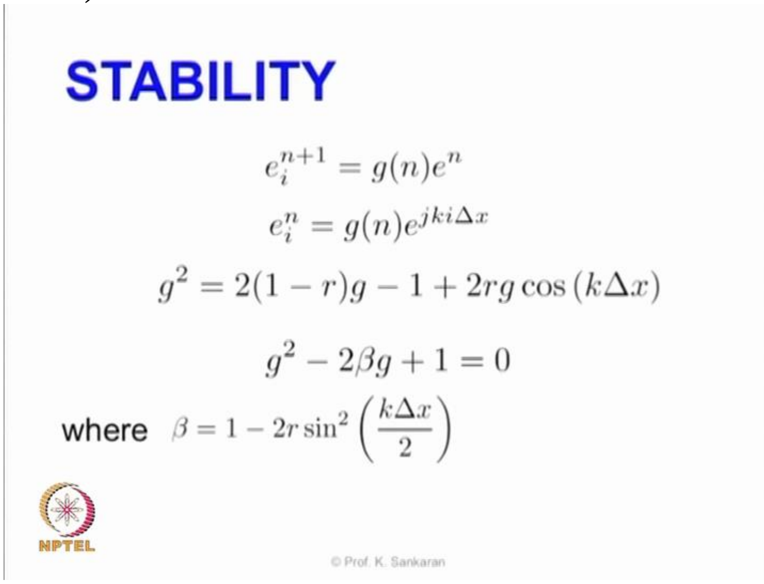

So what we can write is we can write this expression as a second order equation in terms of g as g square minus 2 beta g plus 1 equal to 0 where beta is given by this particular expression so the most important thing here is not to be worried about this entire set of equations. Sometimes it could be intimidating. But it is important to know that once you define a particular finite differencing scheme, this finite differencing scheme is going to create certain error and this error is going to go propagating in step one to step two to step three so on and so forth.

What you are interested is knowing whether this error is contained or is it going to grow and for that you can compute the value of g which is the amplification factor for that particular numerical method which we are using.

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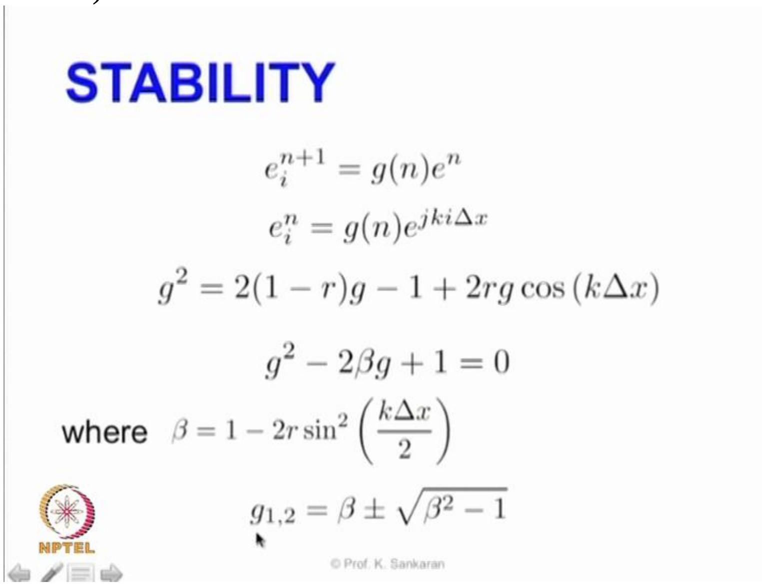

So in that way in this particular example where we have got a second order equation of the amplification factor and obviously you will get two values for g. And this value is nothing but beta plus or minus square root of beta square minus 1; where beta is given by this particular value. And Beta depends on the value of r which we choose and the value of delta x we choose. So once we fix the value of delta x and r or beta is going to be a value that will uniquely define what is going to be amplification factor oh boy! What we have got into? A lot of equations and expressions. So if you see this slide for the case of what we are using now what we get is the value of the amplification factor g has two values. Since it is quadratic so we have beta plus or minus square root of beta square minus 1.

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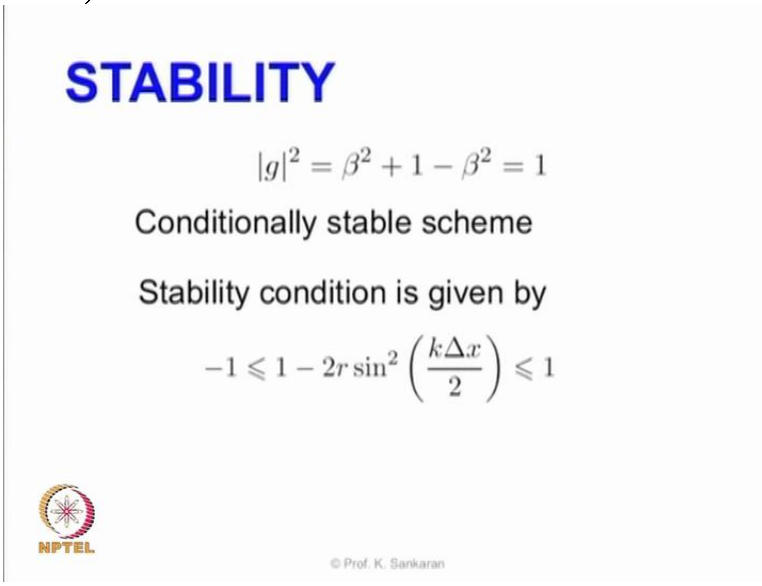

And for this to be equal to 1 scheme is said to be conditionally stable and we have got a stability condition given by the expression that the value of beta is given by 1 minus 2r sin square k delta x by 2 should be within the bound of minus 1 and plus 1.

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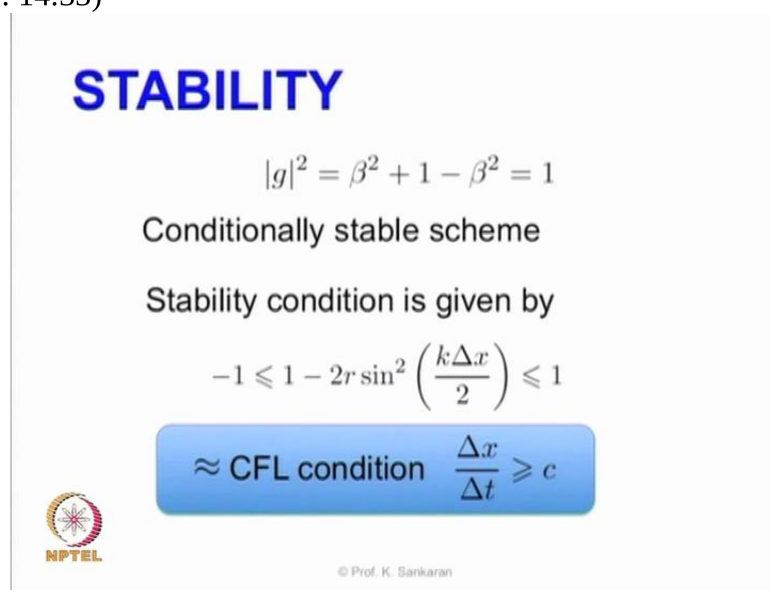

This is very similar to the CFL condition which is named after three well known mathematicians. German and American Courant, Friedrichs and Lewy. So Courant, Friedrichs and Lewy they have done quite extra ordinary work in the early days of numerical mathematics to bring out certain fundamental theorems related to numerical methods and one of the main results are related to the CFL condition which essentially says what should be the maximum time stepping for a given spatial discretization. So what you see here is a CFL like condition, that is why we say it is almost equivalent to a CFL like condition which is nothing but delta x by delta t should be greater than or equal to c.

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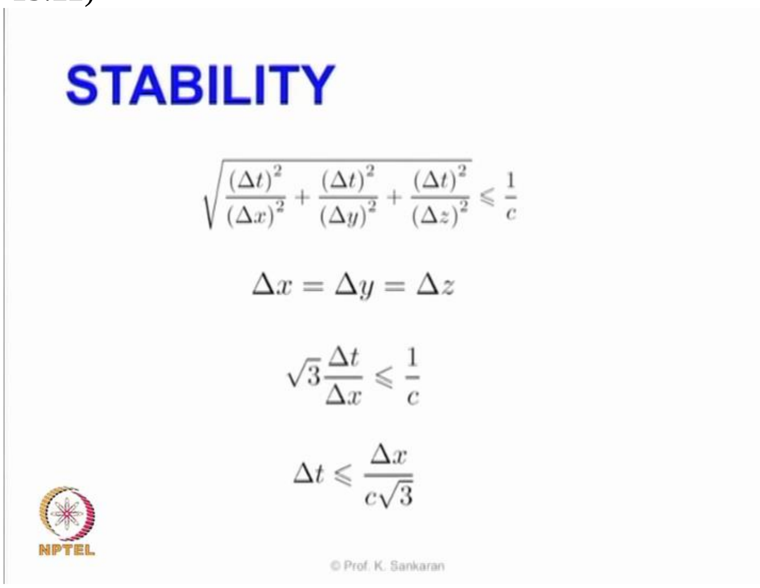

In the three dimensional case when we have more than one spatial discretization so you have also the component of delta y and delta z coming if we set delta x delta y and delta z equal, in other words we assume that the spatial discretization is going to be same in x , y and z direction. So in that condition or in that setting the CFL condition above turns into square root of 3 delta t divided by delta x which is less than or equal to 1 by c. So in other words what we get is our delta t should be less than or equal to delta x divided by c by root 3.

So I would like to give a more physical meaning of the CFL condition in a bit. But before doing that let us revamp on what we have just looked into.

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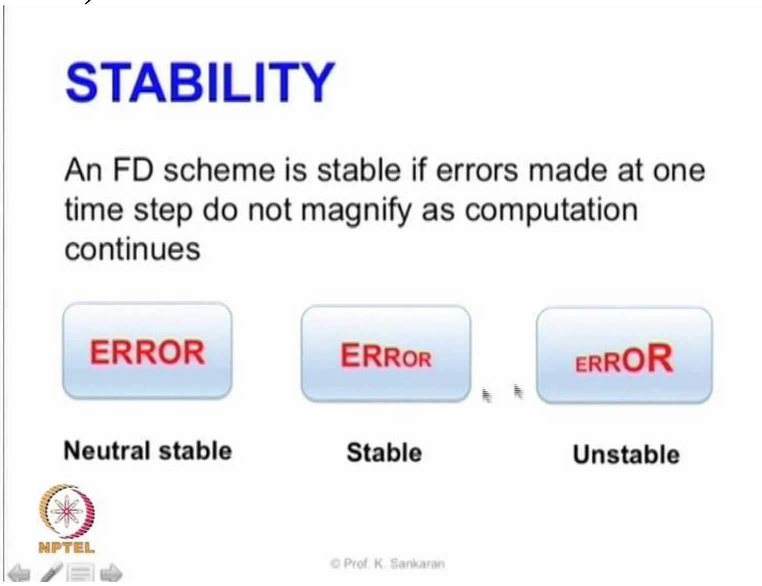

A Finite difference scheme is stable if the errors made at 1 time step do not magnify as computation continues so that means as I go in time if the error is within certain label or staying stable for a constant we are calling it as neutrally stable scheme. If the error is reducing over a period of time it is called as a stable scheme. And if the error is propagating or increasing, it is called as a unstable scheme.

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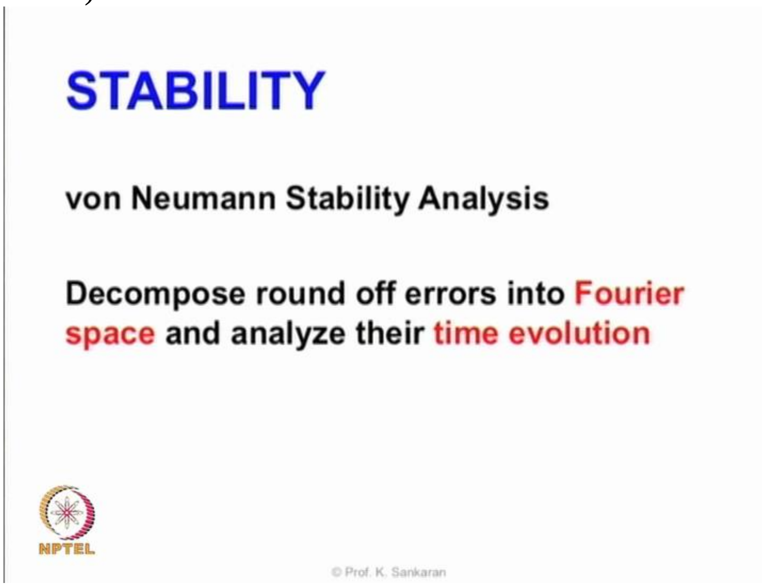

There are other methods of looking into stability as such. One thing is due to a very famous Hungarian and US American Mathematician Von Neumann was done excellent work in numerical methods and also in various other domains, he is truly a polymath in that sense. He has contributed to the stability analysis using particular technique called as Neumann analysis. So which is essentially decomposing the round of errors into Fourier space and analyzing their time evolution. This particular method is quite interesting if we go for spatial discretization and we know that it is easy to decompose the errors in the Fourier space and analyze them. We are not going to particularly look into the Neumann analysis in this lecture series. For people who are interested there is quite a lot of extensive material available in literature on this. But it is good to know or it is sufficient to know there is also another way of looking at it which is the Neumann analysis.

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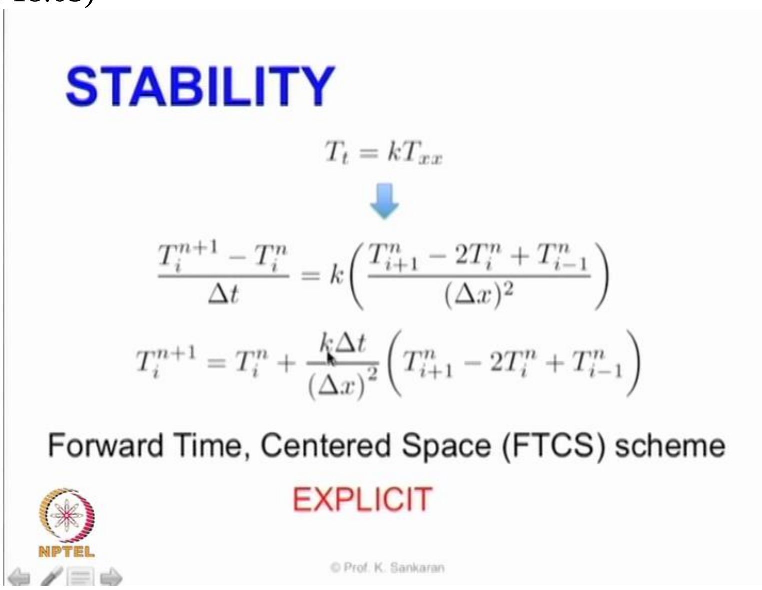

So let us take a very straight forward simple problem and apply our explicit algorithm and see where it leads us.

So let us take a problem which is given by this expression, there is a variable T, so there is a partial differentiation of that particular variable with respect to time is equal to some constant times the second derivative with respect to x. Let us assume that it is a one dimensional problem in space and it has the time component as well.

So when we do the discretization in space and time like this what we are essentially doing here is we are doing forward differencing in time and we are doing a central differencing in space. How do I know that because I know I take into consideration only the component n plus 1 and n? So it is a forward differencing whereas here I am taking the components on the spatial side i plus 1, i minus 1 and i and i am dividing it by square of the spatial discretization. So this is a central differencing in space, whereas forward differencing in time.

So if we take the terms on the right hand side and keep only the left hand side term as n plus 1 what you get essentially is a forward in time centered in space scheme which is an explicit scheme.

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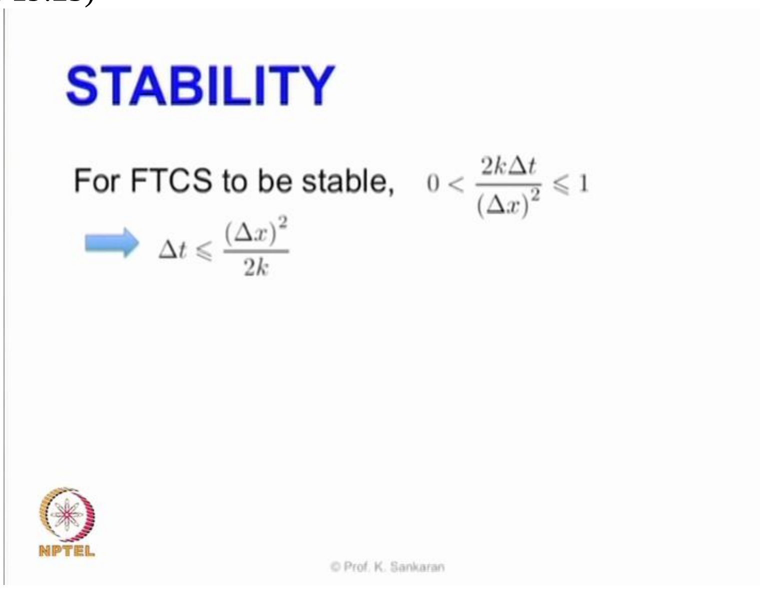

So for this particular scheme if we look at the stability condition what we are interested is we are interested to make that this scheme is stable under certain condition and this condition is going to be given by this 2k delta t by delta x square. The reason why we have 2k here is coming directly from central differencing algorithm here. So if we look at this central differencing algorithm the center point is going to be the one that is going to have the maximum information contribution. So it is weighted by 2 and we are interested in making sure that the central point the weightages is going to be the one we are taking to the CFL condition.

So if we rearrange this term and look only the delta t term what we get is delta t should be less than or equal to delta x square by 2x. And obviously this particular thing is only valid for the kind of problem what we are looking into

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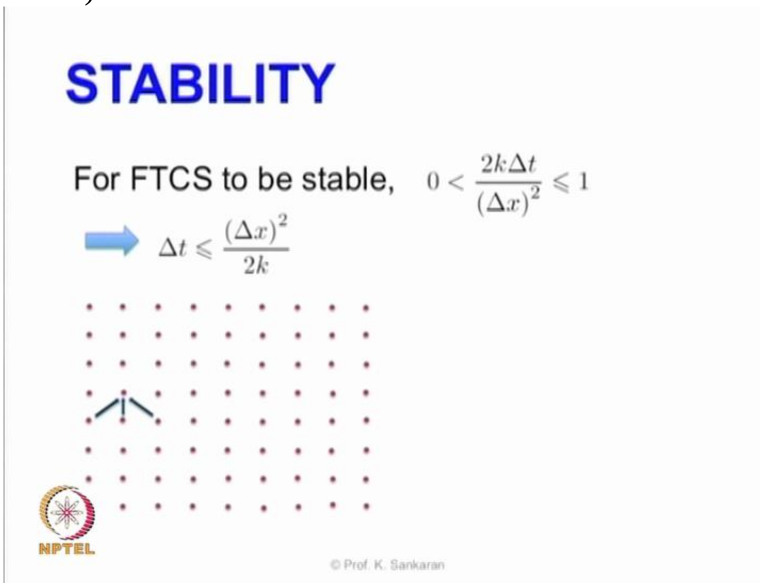

So let us look into this more from a physical point of view. So what is happening here is we have certain grid, and the grid has certain points. And then we are talking about certain influence of points at a particular point in time and space.

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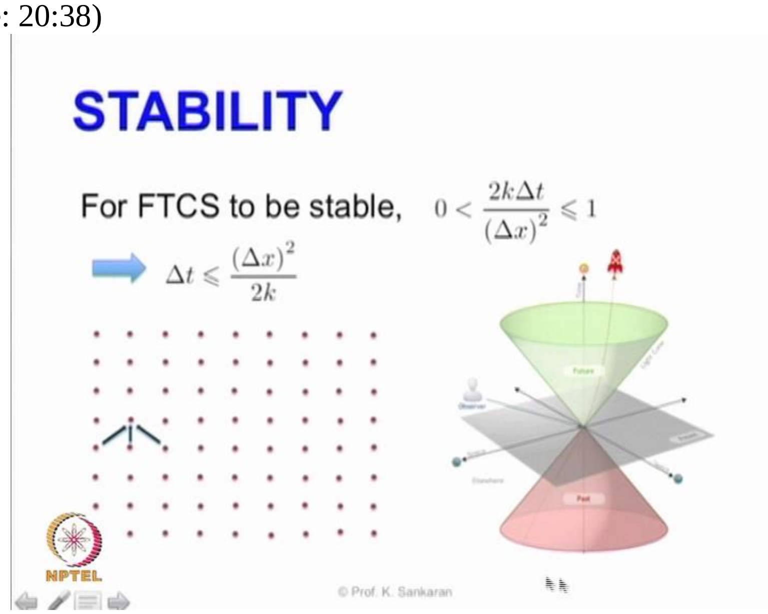

So I would like to draw the analogy to this particular setup to the famous space time diagram that is quite often seen in relativity subjects so when you talk about theory of relativity what you talk about is the past time zone and the future time zone and the time phone defines the way physically possible a range of solutions.

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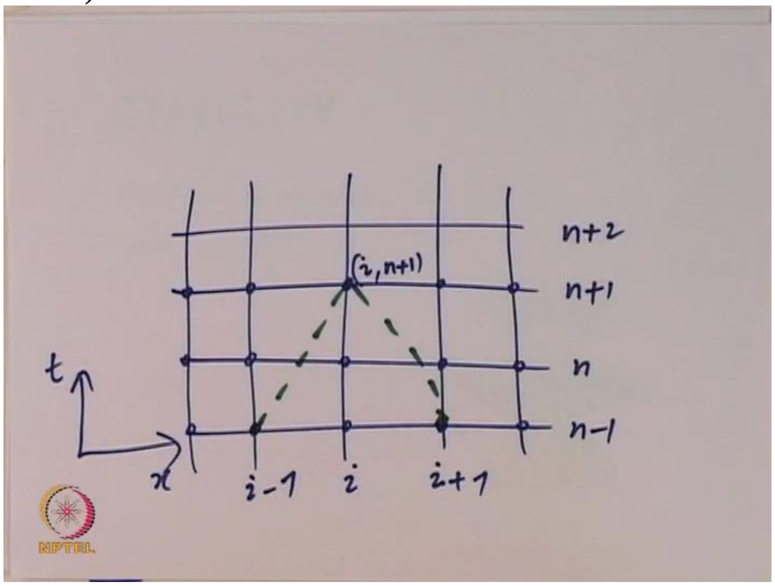

So let us look at this in a much more simple form, so what i wanted to say here is you have a grid that is going to be in the space and time possession like this. Let us say this is time n minus 1, time n, time n plus 1 and time n plus 2 so on and so forth. So this is the time axis this is and here I am talking about i, i plus 1, i minus 1. So what I am interested is I am interested in a solution at this point i at the time instant n plus 1. So i can pretty much say there are various points let us say in space. So i can consider this point this point this point this point this point any of the points in the space time grid. But how do I know what is the right number of points or what is the sufficient amount of points. So this is basically coming from the physics itself.

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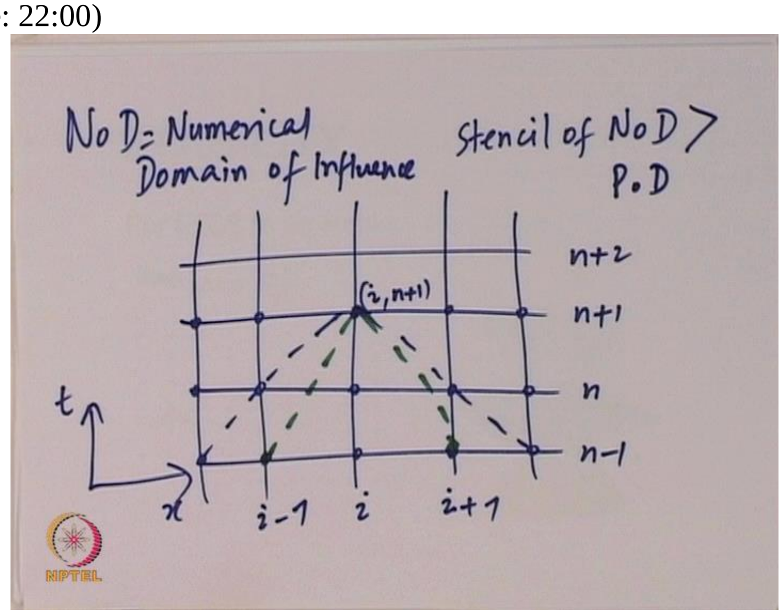

So let us assume that there is physical domain of influence, in other words the solution is going to propagate in a particular direction it has certain physical domain of influence let us say that physical domain of influence is given by this green dashed curve. So it is going to start from this point and whatever is going to come here has a physical domain of influence given by this arrows.

So the stable condition essentially says the physical domain should be contained within the numerical domain of influence. So the numerical domain of influence should always be larger than the physical domain of influence, So in other words for me to properly replicate the physics of this particular problem whose physical domain of influence is going to be in this way, to rightfully capture this problem in the numerical method my numerical domain of influence should be larger than the physical domain. In other words the stencil of numerical domain of influence (NOD). Let us say the numerical domain of influence should be larger than the physical domain of influence. So basically if this is satisfied my numerical method will be properly or rightfully capturing the kind of thing that we are trying to numerically model.

In other words the question comes how do I know what is the physical domain of influence.

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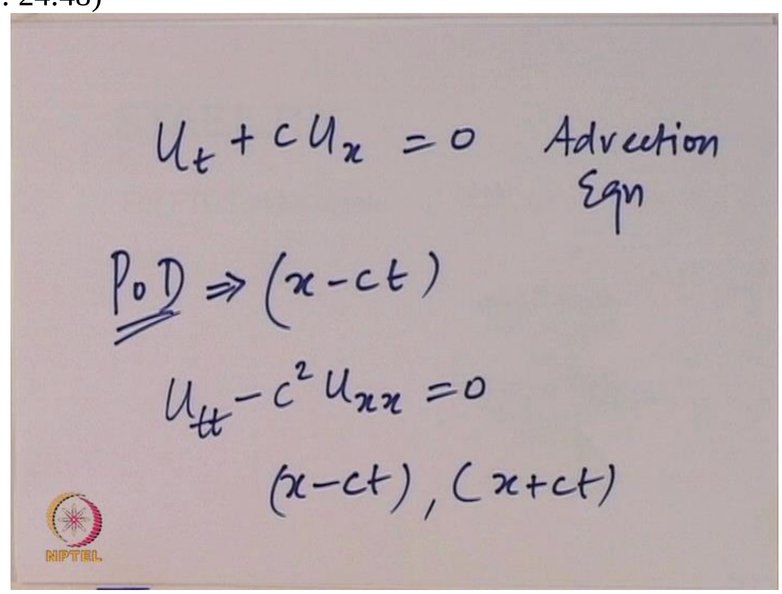

For example you might say I am talking about a one dimensional problem u of t plus let us say c U x is equal to 0. This is a simple advection equation. And I Know the physical domain of influence (POD) is given by the value x minus ct. So x minus ct will be the way the field is basically moving, the solution is propagating for a given x a given t it will propagate with a velocity c and this is what it is going to be.

Whereas if i have a second degree equation which is the equation what we have now. I know the solution will be the physical domain will have two solution one is x minus ct the other one will be x plus ct. obviously high dimensions we will have more components or more directions possible. But let us say we are talking about one dimensional problem there are two solution propagation direction. So the physical domain of influence is known through this particular way. And once we know that we can say that our numerical domain should always be more than the physical domain.

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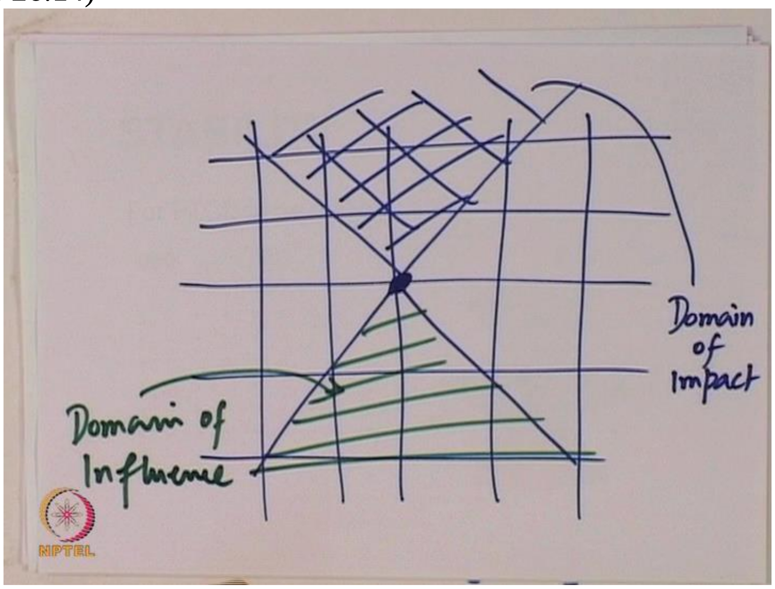

In other words if I have a numerical grid, for a particular point in space and time, there is always going to be a domain on which it is going to so let us say this is the the green one is going to be the domain of influence. So this one is the domain of impact and this is the domain of influence and this is what we saw in the example of the slide here

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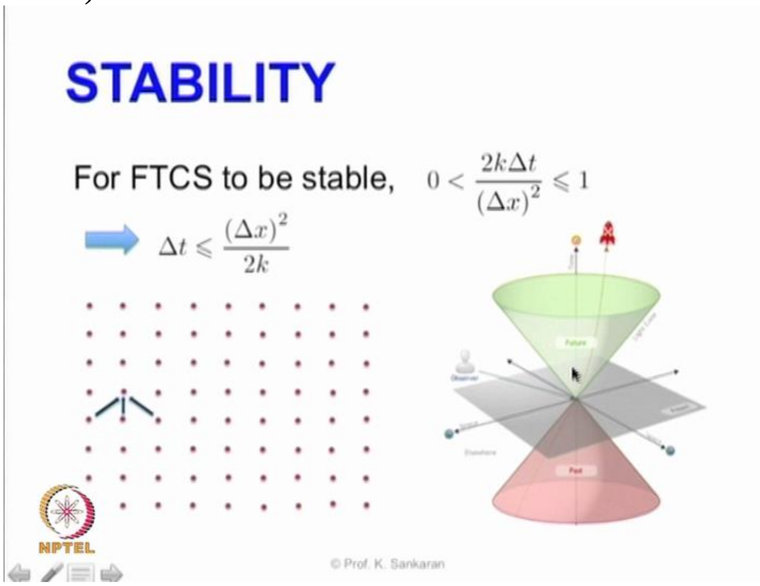

This is the cone which is the past cone and the future like cone. So the past cone will be the domain of influence and the future like cone will define the domain of impact.

So with that being said we could pretty much understand why the CFL condition is so fundamental for a numerical method and how we can understand for proper modelling of a physical phenomenon.

Thank you!