Control Engineering

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Lecture - 37

We have been looking at the root locus method for getting some idea of the transient response of the feedback control system or even open loop control system and we saw that some of the ideas associated with the root locus can be used to determine the value of the gain beyond which the system may become unstable. Now we look at another approach to the same problem which is based on what is called frequency response.

Historically, the root locus method was invented in the 1950's whereas the frequency response approach to the feedback control system analysis and design was developed in the 1930's and 40's. As I told you earlier, people who have working with telephone network, first came across this problem of stabilizing of amplifier and Harold Black invented the idea of or rather applied the idea of feedback to stabilize amplifiers and because of this people who were studying amplifiers till that time started looking at feedback in amplifiers and this is how the frequency response method was developed. In the case of an amplifier, whether it is useful speech which was the main application transmission of voice signal over long distances, over a large telephone network. This was the first application, today of course one can talk about other communication networks wired or wireless.

Now in these applications, frequency responses are very important concept a signal being sinusoidal is a very meaningful signal for example, a pure tone if one looks at it on the oscilloscope looks like almost a perfect sinusoidal signal and parts of speech also look like periodic waveforms and therefore, we know from Fourier series approach that it can be a thought of has consisting of number of sinusoidal signals because of this, it is quite natural to think of frequency response when studying amplifier performance and so, the frequency response techniques were developed in the 1930's and 40's for not only studying the performance of amplifiers but also studying the effect of feedback on the behavior of the performance of amplifiers.

So we will take a look at it now, although it seems that the root locus method which was invented later does a better job for this purpose than the frequency response base analysis for method. Now, we will start with the very simple result which is as follows you have all studied functions of a complex variable in your mathematics courses. So, suppose z is a complex variable that is z stands for any complex number, what so ever then you might have a function of z which is also a complex number that is its complex value of function of a complex variable and typically in the maths courses, the symbol z and q are use for the 2 complex numbers. So w equal to f z represents what is called a mapping or a transformation of the complex plane into the complex plane and in order to show this diagrammatically, one shows the 2 numbers z and w, start with the complex number z apply the function f to it, you get the value f of z that is another complex number w.

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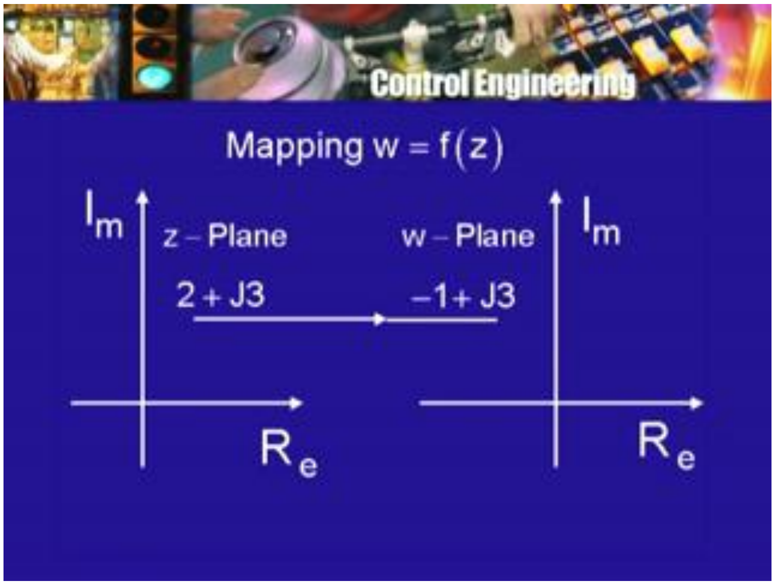

So these 2 numbers z and w are shown into 2 different planes that is you think of instead of 1 single complex plane, you think of 2 different complex planes. So one is call the z plane, other is call the w plane. So let us say here is the z plane, in other words in this plane or in this part of the diagram, I am going to represent the complex number z and here is the other plane. Let us say, we call it the w plane that is in this part of the diagram, I am going to represent the value of the function at a complex number z.

So for example if I choose a complex number z, here it may be some complex number like, let us say 2 plus j 3 g, right. It is a complex number in the first quadrant of course, I am not put the real and imaginary axes heading here. So if I do that this is the positive real axis, this is the positive imaginary axis for the z plane likewise, I will have the real and imaginary axis that is real and imaginary parts of w, in the w plane. So let us say there is some function f such that its value it 2 plus j 3 is let us say minus 1 plus j 3. So where is the point minus 1 plus j 3 all that will be a point 1 plus j 3 all that will be a point somewhere here, here is the point minus 1 plus j 3, the real part is minus1 the imaginary part is plus 3 and so, one says that this point 2 plus j 3 is mapped into the point minus 1 plus j 3 under the function or mapping f and this function or mapping f, we will do different things to different points in the complex plane, different points in the complex plane will be mapped in the complex plane, z plane will be mapped into or taken into different points in the w plane and one uses this terminology of mapping z is transformed into w under the action of f or f maps a point in the z plane into point in the w plane.

So this is the way in which one looks at such functions and what they do to complex numbers. Now, let us consider a very simple complex function namely w, which is f z is simply given by let us say Z minus 1. So, if I start with the complex number z then what this function f does it, it simply subtracts from it the number 1 and I get a new complex number and that is the value of the function, it is a function as simple as that.

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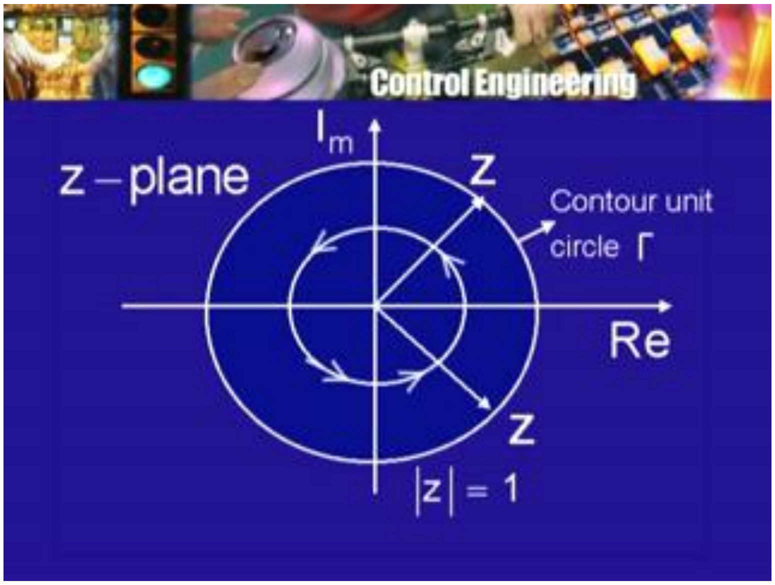

Now, let us look at the following. Here is the z plane, in the z plane I draw simple closed curve and a very special one, I will draw what is called the unit circle. So suppose this represents the unit circle in the z plane, what does this mean to say that this is the unit circle in the z plane, it means that it is a set of all points z such that modular sub z is equal to 1, it is in way is equation of the circle unit circle in the z plane, mod z equal to 1. So, here is a point which is on the unit circle so here is the complex number z and this is the corresponding vector drawn from the origin to the complex number z or this vector can also represent the complex number z.

Then, the modulus of this z which is the length of this vector must be 1 and so all the points which lie at distance one from the origin will satisfy this condition and conversely, any points satisfying this condition must lie at a distance of 1 from the origin. So this is a curve in the complex z plane, it is a simple closed curve and you should go back to your complex variables or complex functions course and find out, what is the simple closed curve. Now one goes a little further, one thinks of tracing the curve or moving along the curve in a particular direction. For example, I will put the arrow here this way which means now, I am going to trace the curve or move along the curve in the counter clockwise direction.

Now, such a curve with an arrow on it which indicates the direction in which you want to trace the curve is called a contour. So, I have chosen a contour in the complex plane and it is conventional to use the Greek letter capital gamma to denote any such contour. So in this particular case, I have a closed curve gamma in the z plane which is traveled along or travels in the counter clockwise direction. I have this curve but there is a direction given a or marked on it which is the direction in which I am going to travel along the curve as it were or travel direction of traverse in the counter clockwise direction. Also, it is clear from the figure that if, you have to sort of moves around this curve and at the origin you had to some mark or some pole or something. Then from the figure it is clear that as you move around the curve starting somewhere and you come back to the same position, you have gone around the origin through one revolution.

So, it is very simple to visualize that because you can think of this radius vector and as you move around the curve, the radius vector will go on turning and if you move around the curve once, starting at a point move in this particular that we have chosen and keep going till you come back to that point. At the end of it, you would have completed one full revolution in the counter clockwise direction around the origin of the z plane. So one says that this curve gamma with its direction along which it is supposed to be traversed, encircles the origin in this case, once in the counter clockwise direction or one says that this curve gamma with this direction of the traveling around it, the number of encirclements, this is the terminology that is used the number of encirclements, the number of times the point, the origin is encircled, the number of encirclements in this case is exactly one.

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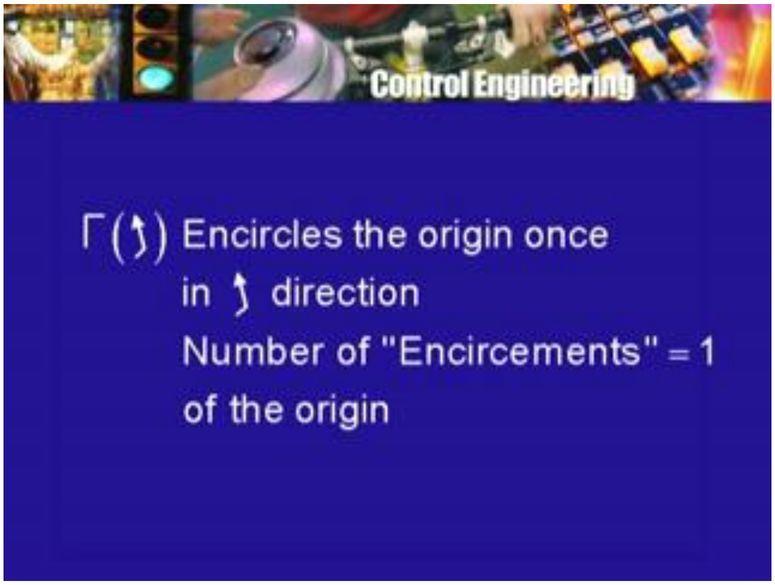

Now this is very simple for the circle but it is not much more difficult for more of different curve. So let me take another example let me draw another curve, here is my curve gamma. Now which is no long circle and here is the arrow which tells you, this arrow tells you that you are going around this curve in this direction. Now, what is the number of encirclements on the origin by this curve. The curve is no longer a circle but I can still talk about number of encirclements because I can think of this radius vector whose length is going to change as one moves around the curve but as one completes one traverse of the curve, the radius vector will have moved through 360 degrees in the counter clockwise direction and as you know, counter clockwise direction is taken as the reference direction in the complex plane field and therefore, the number of encirclements, we will take it as 1.

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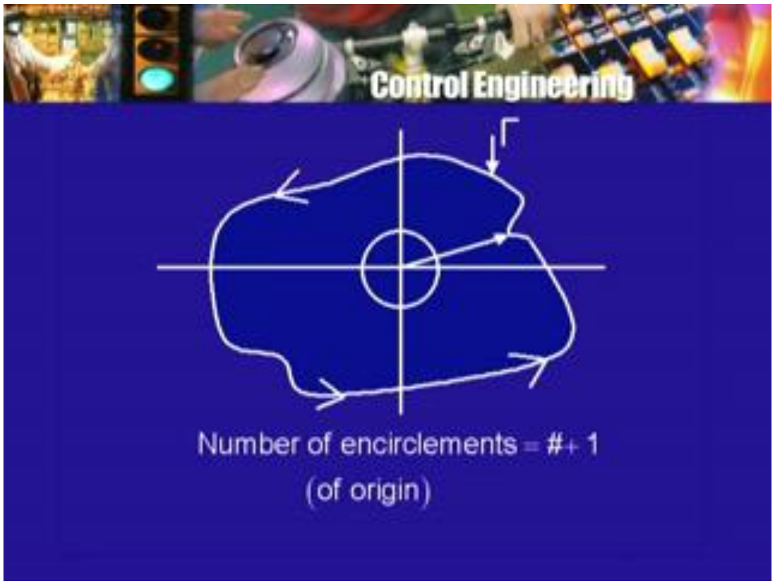

So in this case this curve gamma encircles the origin 1 and 1, when you say that you mean once in the counter clockwise direction as you traverse the curve gamma in the direction which is shown on the gamma. Let us take one more example, here is this curve gamma and I traverse it in this direction now, what is the number of encirclements by this contour or curve gamma of the origin, it goes around it once but in the clockwise direction and therefore, we will count the number of encirclements as minus 1 rather than plus 1.

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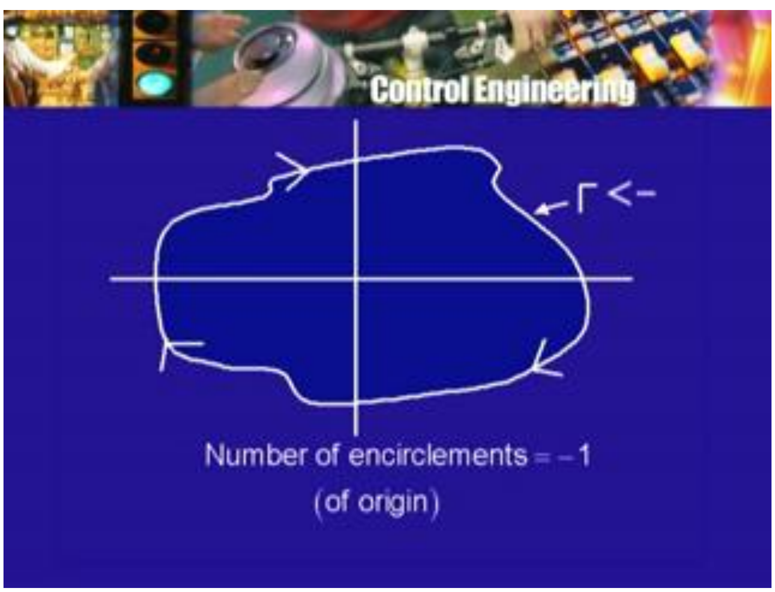

So in this case that number of encirclements was plus 1 and in this case, now the number of encirclements is minus 1. So the number of encirclements will not just be a number but it will also have a sign and essentially what matters is the direction in which the curve gamma is traversed. For example, this same curve gamma if I traverse in the opposite sense then the number of encirclements will be plus 1 whereas, if I traverse it in this sense then the number of encirclements will be minus 1. So the number of encirclements depends on this direction of traverse or movement around the simple closed curve. Let us take one more example, here is the close curve and I traverse it in let us say in this direction now, how many times does it encircle the origin.

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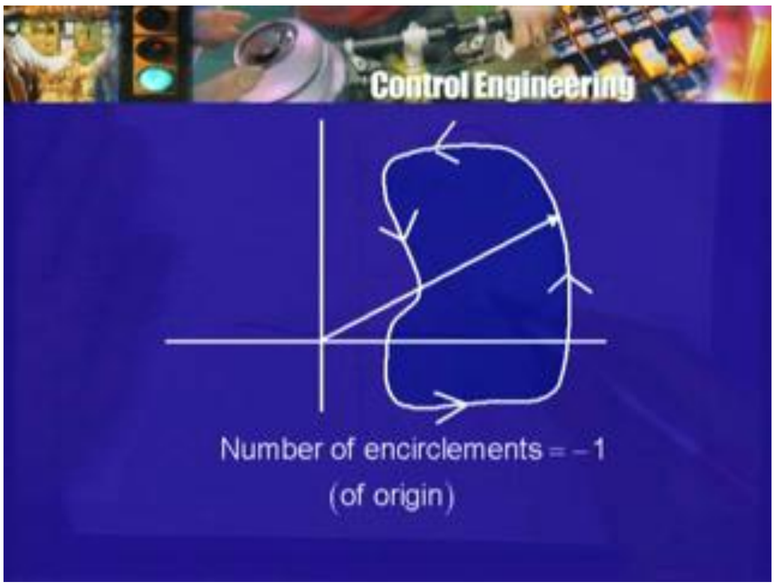

So what do I have to do think of a typical point on this curve, think of the radius vector and think of the point moving or think of yourself as traversing around the curve in the chosen direction. See what happens to the radius vector and in this case, you can see that the radius vector is going to move this way but then it is going to move back, then it is going to move this way, then it is going to move back. So what will be the total change in angle or the total rotation of the radius vector. The answer as you can see is going to be 0 degrees, the radius vector is going to move in the counter clockwise direction but then it is going to start moving in the clockwise direction and then, it will again move in the counter clockwise direction, when it gets back to the original position.

So the rotation of the radius vector total rotation from the beginning to the end is 0 degrees, is through an angle of 0 degree. So the number of encirclements is 0 and of course visually it is very clear, the origin does not lie inside the closed curve and therefore, you can explain that the number of encirclements by this closed curve of the origin is 0. So the number of encirclements can be 0, the number of encirclements can be a positive number or the number of encirclements can be a negative number. So given any simple close curve in the complex plane, we can immediately talk about number of encirclements by the closed curve of the origin. Now what is true of the origin one can apply for some other point. For example, here is the simple close curve and I can talk about the encirclements by this closed curve of this point which let us say is minus 2 plus j 1, what do I have to imagine then first of all of course I need a sense or direction for moving along the curve, so I am putting that arrow there.

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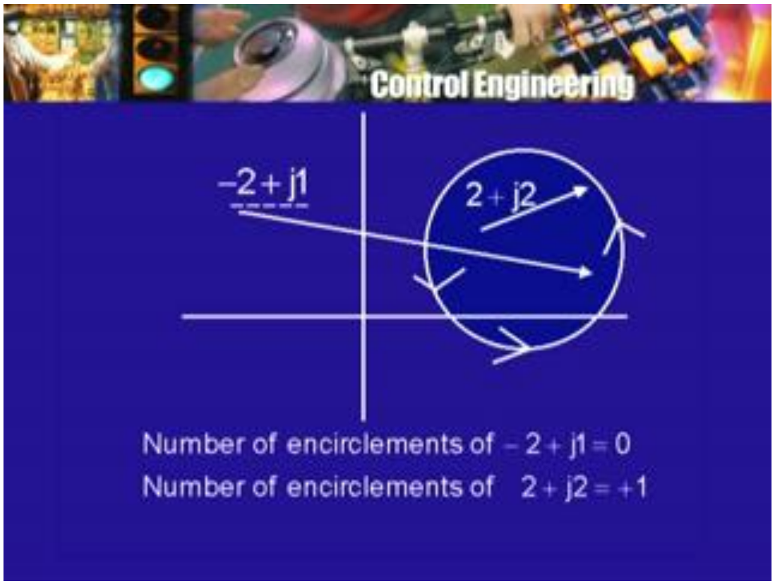

Now think of a radius vector going from there to there from this point, around which you are trying to find out the number of encirclements to a point a on the curve and imagine, what happens as the point moves along the curve. Now this radius vector is again going to swing back and forth and the total rotation of the radius vector, when this point has moved completely around this curve will be 0 degrees and therefore, the number of encirclements of this point by this simple close curve will be 0. Once again, it is obvious that this point is not inside the curve whereas, if I had a point it here let us say which is inside the curve and now, I wanted to calculate the number of encirclements of this point by this closed curve in the same direction of the traverse. Then, you will see the answer will be 360 degrees or number of encirclements will be plus 1 and of course, we see that this point is inside the simple closed curve.

So all this is very simple in fact that is why the curve is called a simple close curve but we can make situation more difficult and this is because such a thing will happen, when you deal with a transfer function or when you deal with a system which has poles and 0s, may be more than one of each. For example, let me draw another curve which is not a simple closed curve, here are the axis for the z plane and here is a curve that I am going to draw. Of course, I have traced it before you so if you remember that I start from here then, I go here, here, then here, here, then here, then here, then here and then, I come back because there are these intersection points, so it is not very clear which way I am going but I am telling you how I have gone from say this starting point A and I have come back starting from A, go this way and now not come here but go this way go here, now go not go this way but go this way, go around here then, come here then not go this way but come here then, not go this way but come here and then come back to A.

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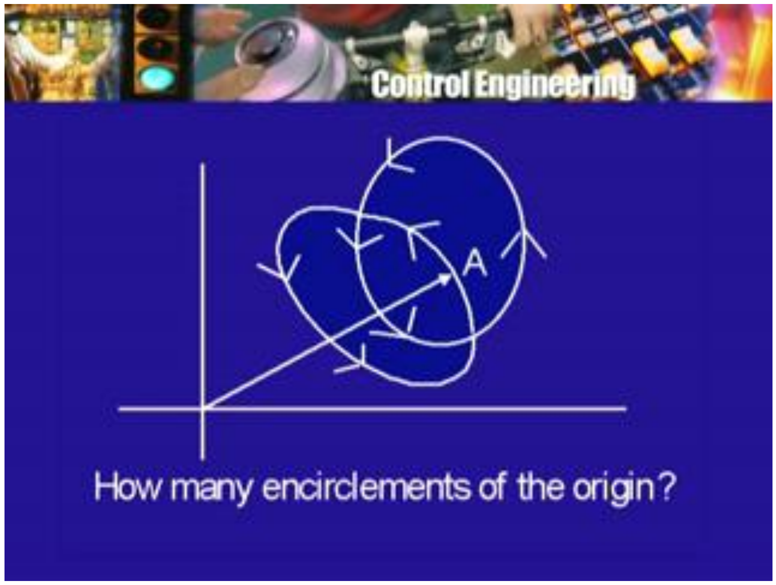

Now this is not a simple closed curve, it is closed curve because I have come back to the starting point and I have moved continuously, there is no break in the curve but it is not simple any longer because as we can see, the curve the sort of intersects itself in a number of places is 1, 2 and 3 points of intersection. So this is not a simple closed curve but if I say now, how many times this simple closed curve encircles the origin, what is the answer? Well, what do I do, think of typical point here draw the radius vector from the origin they are interested in encirclements of the origin. So draw the radius vector from the origin to a point on the curve and starts moving along the curve, see what happens to the radius vector.

Now you can follow and see that the radius vector is moving this way counter clockwise first then, it will move clockwise, again then it will move counter clockwise, again then clockwise then counter clockwise, till it finally comes to the old position A and the total rotation of the radius vector will be through 0 degrees. The total angle, the net angle, meaning positive and negative are to be counted as separate and to be added up. So the net rotation, position the total change in the angle is 0 degrees. So the number of encirclements by this curve of this point, the origin is 0 and again, we see that this point the origin lies outside this simple close curve.

Now, what is inside and what is outside is not really very clear and that is why one has to sometimes go through this exercise of imagining a point on the curve, this radius vector and then trace the point on the curve, see what happens to the radius vector, how does it swing back and forth and whether, it swings totally through 0 degrees or through an angle which is different from 0 degrees, this is what you have to see. Let me draw one more figure, I will draw similar looking curve so it will start with that and erase the curve and being traced in some particular direction starting of the point A and now, suppose I have a point P here. Now I ask what is the number of encirclements by this closed curve say gamma of this point P, now is the point P inside the curve or outside curve or what, it is not so easy to answer because the curve itself is not simple any longer, it is intersecting itself in 3 places. So what is going to be the number of encirclement.

Well, we start the same way draw this radius vector from this point P, we are interested in finding out how many times the curve encircles the point P.

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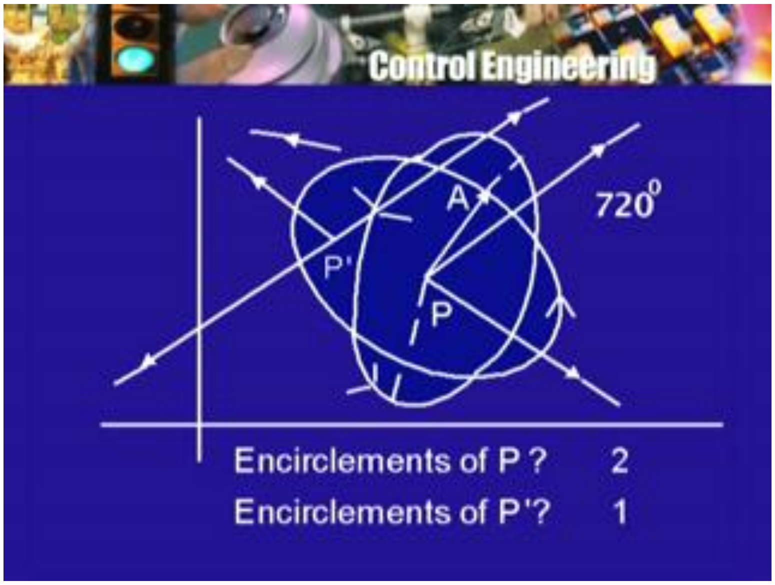

So start with radius vector and let A move around the curve, see what happens to the radius vector, see what is the total angle through which the radius vector turns. Now, if you do that carefully, what will be an answer as I go from this position to this position, the radius vector would have turn through 180 degrees in the counter clockwise direction. As I go further from here to up this point it would have gone through a further 180 degrees in the counter clockwise direction. Because, it will coincide with its initial position PA but the net angle will be 360 degrees in the counter clockwise direction then, it starts from here let us say and continues further. So counter clockwise all the way till this point is reached, when 180 degrees has been completed continuing further another 180 degrees in the counter clockwise direction.

So that total rotation of the radius vector will be 720 degrees in the counter clockwise direction and therefore, what is the number of encirclements by this curve gamma, traverse in that direction of this point P, the answer is 2, 720 degrees divided by 360 is 2, 360 degrees is 1 rotation that counts as one encirclement but it is 720, so that number of encirclements is 2. Now when the curve gets more complicated than this and sometimes it can become more complicated there are number of intersection then, how do you work it out for example, I put the point P here, so let us call it P dash.

Now what is the number of encirclements of the point P dash by the same curve gamma and I can do it the same way of course draw the radius vector from P dash to A, imagine going around that curve, see what happens to the radius vector and then, follow the change in the position of the radius vector and figure out, what is the total rotation of the radius vector, the total angle through which the radius vector has turned in the counter clockwise direction divided by 360 that will give you the number of encirclements. Now is this point P dash also inside the curve or is it outside the curve.

Now someone may say, it is still inside the curve but if you find out the number of encirclements of the point P dash by this curve then, your answer will be 1 and not 3 or 2 or 0. Now I will ask you to go through this yourself in fact, you can draw copy this figure put the point P dash actually take your ruler or as I have done it, use your finger or your ball pen or whatever and literally sort of trace the variation, trace the moment of the radius vector as you go around the curve and as certain for yourself that the net rotation by this curve or by the radius vector, the net rotation as the point traverses around the curve is through 360 degrees and therefore the number of encirclements is 1 plus 1 and not 2 or 0 or minus 1 or whatever else.

Now there is a way of findings this out which is sometimes useful and that is the following. From the point P dash or P of whichever point around which you want to calculate the number of encirclements, draw what is called a ray or a half ray that is draw from a, just a line which goes away towards infinity, such a line, if you remember your geometry course is called half ray and of course, I can put a point arrow on it to indicate that I am going from P dash out to infinity along this direction. So such a thing is called ray. Now when you have such a half ray then, this half ray can intersect the curve in 1 or 2 or 3 or 0 points. For example, if I have point here and if I draw ray this way then this ray is not going to intersect this curve at all whereas from this point P dash I have drawn this half ray, this half ray does intersect the curve in 1 point.

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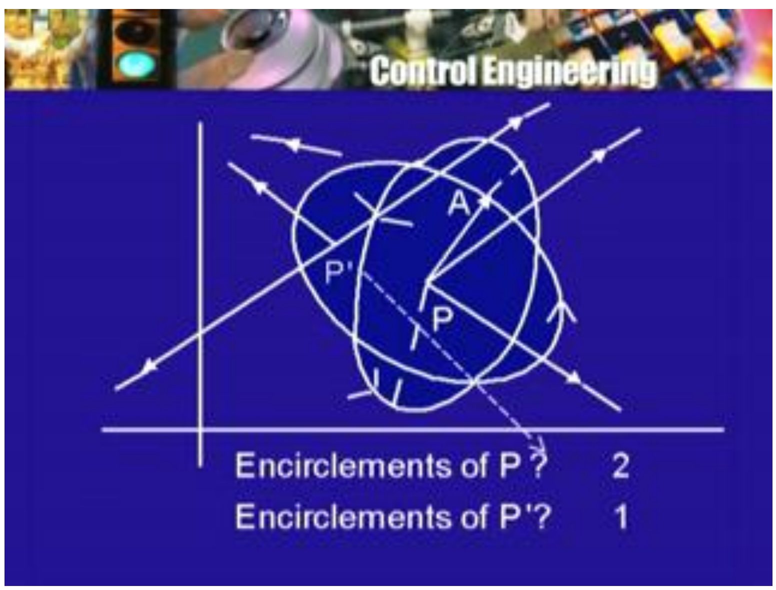

Now at the point of intersection, you have to figure out which way the moment of the point is taking place, point on the curve, the point on the curve is moving in the counter clockwise direction. So count that as plus 1 and this is the only intersection point and therefore the number of encirclements is plus 1, I can do it with another half ray. For example, I draw this half ray from P dash going in this direction okay. Now this half ray intersects this curve in how many points, there is 1 point here there is another point here, where actually it looks like 2 coincident points here okay. The way I have drawn the figure and drawn ray, why not.

Now for each intersection I have to find out which way the movement of the point on the curve is taking place around P dash. Now, when I take this point of intersection the curve is being traversed this way. So the movement around the point P dash is in the clockwise direction, so that is minus 1 whereas, if I look at this point there are 2 points actually, twice intersecting. So one of the intersections is let us say this one then, when I move along the curve this way around the point P dash I am going in the counter clockwise direction that is plus 1 then, I come back here and now, I move along this part of the curve, again it is counter clockwise, so plus1.

So plus 1 and plus 1 that is plus 2 and this one is minus 1. So the total is plus 1 and that is the number of encirclements by the closed curve of the point P dash. This is a little more convenient or easier than drawing the radius vector and then, imagining the radius vector to turn around and then figure out what is happening to that radius vector, what is the net angle through which the vector turns, when it comes back to its original position. For example, with this point now what is the number of encirclements I can draw any ray.

So in particular I have drawn this ray well this ray does not intersect the curve at all and so the number of encirclements is 0 and I can draw a ray which does intersect the curve and I can go through the same rule, you will find out that the number of total number of intersections may be more than 0 may be 1, 2, 3 whatever it is but, if you use this rule that is counter clockwise plus 1 clockwise minus 1, for each one of these intersections and add up, you will get the number 0.
For example, here is a radius vector that I am going to draw like this starting from this point it has how many intersections, there is 1 intersection here, another intersection here, a third intersection here and the 4th intersection here.

So there are 4 intersections but the number of encirclement is not 4, I have to find what it is.
Now for these 2 intersections whereas, these 2 intersections this one and this one are both in the clockwise direction. So minus 1 and minus 1 that is minus 2, so the net is 2 plus minus 2 which is 0, so whether I draw the ray this way or the draw the ray this way, I will get the same answer.
In fact that is why this method will work and so you can draw ray which is convenient, so that the number of intersections as few as possible and then, from that you can figure out the number of encircle.

So although it looks like a very simple idea that you have a closed curve and you have point in a plane the closed curve is also in the same plane then, you imagine a point moving around that close curve and the radius vector joining the point on the curve to the fixed point therefore, moves what is the number of encirclements by this curve or the point that is moving on the curve of the other point. Think of yourself standing on some ground at some point and the closed curve may be something that is marked on the ground, the periphery of some region it may be a boundary of a field or whatever it is and think of somebody moving physically moving along that curve and as you stand here, your turn head, keeping your eyes on that person and I was looking at that person and then as the person moves around the curve, you have to turn your head and sometimes of course since you cannot turn your head by very large angle, you have turn your body.

So essentially you really can count the rotation that way and see, how many times that person has gone around you. This is the simple idea of encirclement of a simple of or not a simple closed curve by a simple or not simple closed curve of a point. Now this is a very useful idea and although it looks very simple, it is not really that simple and the examples that I have taken are such that actually occur in our calculations. So simple closed curves is not the only thing that you will come across when we use this, for our study of the feedback, effect of feedback on control system performance. Okay, let me get back to the problem then, here is the z plane, here is the w plane and I have a simple function w equal to f z, where f z is given by a simple formula z minus 1.

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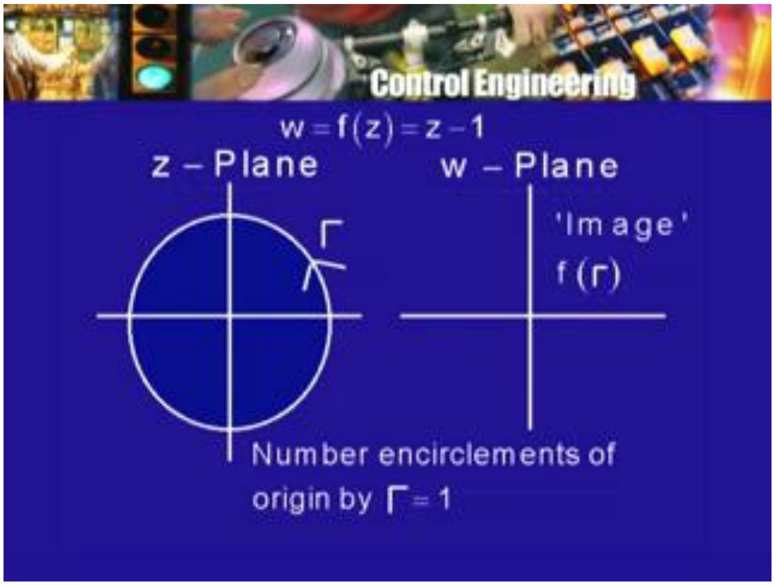

Now I choose to start with a simple closed curve in the z plane and let us go back and again choose the unit circle and trace it in a particular sense or direction. Let us say I have chosen the counter clockwise direction, so as the point moves around this curve this origin of the z plane is circled exactly one, the number of encirclements is plus 1. Now think of the corresponding point w that is for each point z here, I calculate equal to z minus 1 that gives me a point in the w plane, do that for each point on the curve I will get a curve in the w plane. This curve is called the image of the original curve gamma under the action of the function f and because this is gamma, points on gamma are being acted upon by f to produce points on this image curve, sometimes this image this curve is represented as f gamma. Actually, in your complex functions course probably you have come across something like but at that you time you may not have paid much attention to it.

Well, now is the time to either revise it or pay attention because this is going to be useful in our study of performance closed loop feedback control system. In particular investigations of stability of a feedback control system. So, let us look at the simple problem of a mapping of the z plane in to the w plane that is we have a function w given by f of z of a complex number or variable z, w also is a complex number and let us take it to be the simple function z minus 1. Now looking upon it as a mapping of the z plane into the w plane, we would like to see what the mapping does to a contour or simple closed curve, I have chosen this simple closed curve here which is the unit circle around the origin in the z plane, I have called it gamma and I have chosen the orientation as shown by this arrow the radius of the circle is 1.

So the points on this circle where do they go in the w plane. We can choose a few simple cases for example, take the point 1 j 0 or 1. So if z is 1 what is w, w is z minus 1 w is 0, so in other words the point by 1 j 0 is mapped to the origin of the w plane, this is now we talk about it.

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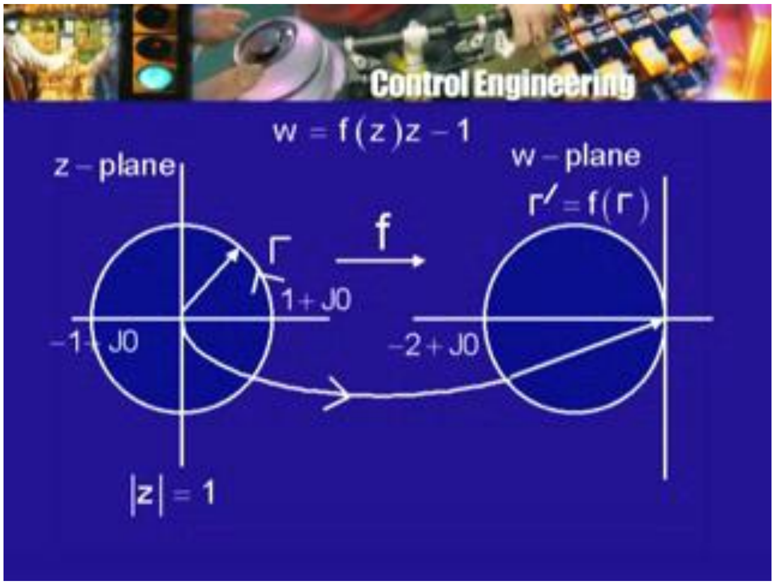

A point in the z plane namely 1 plus j 0 is mapped into a point in the w plane namely w equal to 0 by the function f z, in this case the function is z minus 1. Let us take another point the other extremity of the diameter through the origin minus 1 plus j 0. So z is minus 1, so what is w? W is z minus 1 and therefore it is minus 2 or minus 2 plus j 0. So this point minus 1 plus j 0 will be mapped to the point minus 2 plus j 0 and in fact, you can now see what is going to happen, the action of the function is to subtract the 1 from the number z.

Now subtracting 1 does what to the complex number. It does not affect the imaginary part because your are subtracting only the real number 1 but it affects the real part, what happens to the real part, since you are subtracting 1 the real part is reduced by 1. Therefore, you can see now that this circle gamma will be mapped into a circle, gamma prime or as I said we can call it f of gamma. It is called the image of the gamma under the function f as this curve gamma is traversed in this counter clockwise direction, the curve gamma prime or f gamma will also be traversed in the counter clockwise direction and the circle image mapped into a circle that is happing because this transformation or mapping is a very simple one, namely it is really a mapping which in mechanics you would call a displacement the circle is displaced by one unit to the left or in the complex plane there is a shift by minus 1.

So the circle goes into a circle, now the contour gamma encircles the origin once as we saw earlier and of course one can see it immediately, what about the contour gamma prime, does it encircle the origin of the w plane. Now this is where one can run into a difficulty because if you imagine that radius vector that I talked about going from the origin to a point on the contour and then, let the point move along the contour and see what happens to the radius vector, you will see that as one leaves the origin in this direction, the radius vector is pointing in the upward of the w plane whereas as we approach the origin at the end of your traverse. The radius vector is pointing in the downward direction of the complex plane and therefore, there is going to be a discontinuity in the orientation of the radius vector. Of course, a radius vector is never perfectly vertical that is the tangent but the tangent is not the same as the radius vector.

So if a contour passes through a point then, it is not so simple to talk about number of encirclements of that point on the control by the contour itself and so, we will exclude this case. In other words, when talking about encirclements we will make sure that if you are talking about the encirclements by a contour of a certain point that point does not lie on the contour. In this case then because the function w is z minus 1 and if we are looking at then in encirclements of the origin in the w plane then, we must exclude the point z equal to 1 because z equal to 1 makes w equal to 0. So you have to choose a contour which will not pass through the point z equal to 1. So let me choose a contour which does not pass through the point z equal to 1 and that can be done by simply may be shifting our entire circle already by 1 unit to the right.

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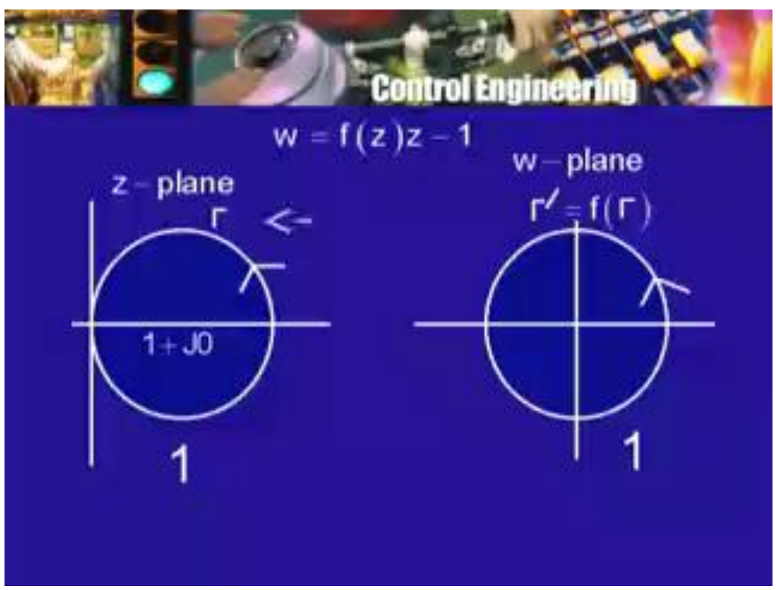

So this is the circle now, this is gamma here is the point 1 plus j 0, this point was mapped into the point 0. So now we are avoided it by not putting it on the contour gamma. In the w plane what will be the image contour, this circle is shifted by 1, so the image contour will be simply the unit circle in the w plane, this is gamma prime. Now look at the following the contour, the image contour, gamma prime or f gamma, how many times does it encircle the origin in the w plane. We will look at the encirclements of the origin, how many times does gamma prime encircle the origin into the w plane, the answer is 1.

Now look at the contour gamma of course, the contour gamma actually passes through the origin so we will not talk about encirclements by gamma of the point origin 0. But look at this point 1 plus j 0 which will be avoid it how many times does gamma encircle 1 plus j 0, the answer is 1. So in this case the number of encirclements by the contour gamma of the point 1 plus j 0 is 1 which is the same as the number of encirclements by the image contour gamma prime or f gamma of the origin of the w plane. Now is this a coincidence that there is 1 encirclement here and there is 1 encirclement here.

Now this is not a coincidence and we can check it for our very special function z minus 1 by taking one more example. For example, I can take the contour gamma which is different from this one but I will choose it in such a way that it encircles the point 1 plus j 0. I can choose for example, square, so if I choose square in the z plane, let us say here is the z plane square whose sides are all 1, 1, 1 each and this is the contour gamma then, what is going to be the image of the contour. As we have seen we are going to shift 1 unit to the side of the it will have to be 2 units and not 1 because I want the point 1 plus j 0 to be inside. So it will be not be a square with unit side but it will be a square with side 2. So when we shift this curve by minus 1, we will get the square here of side 2, this will be gamma prime. The number of encirclements by gamma prime of the origin is 1 of course this is not circle now, the contour need not be a circle it simply has to be a closed curve, a simple close curve to start with.

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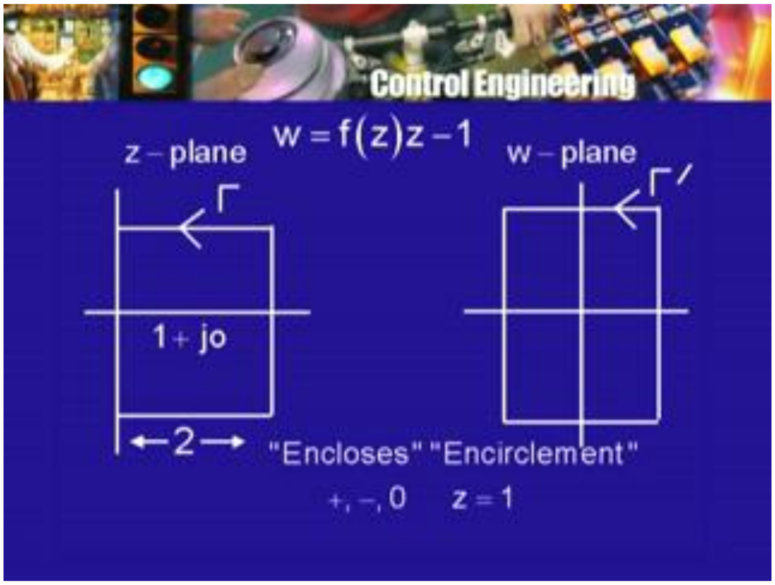

So the square is a simple closed curve and therefore gamma prime encircles the origin once whereas, if you look at the z plane, this is the z plane, this is the w plane. In the z plane gamma encircles 1 plus j 0 once actually what is important is not how many times gamma encircles the point 1 plus j 0 but because the point gamma does encircle 1 plus j 0,1 plus j 0 is inside the closed curve gamma. The function was z minus 1, w equal to f z equal to z minus 1 where does this function become 0 at how many or what to at what points this function becomes 0 only at the point z equal to 1.

So this function has only one 0 and the 0 is 1 plus j 0, the location of the 0 is 1 plus j 0, the curve gamma encloses. Now this is the difference between the encirclement and enclosure. In encirclement we talk about sense because the curve may be traced in 1 direction or it may be traced in the opposite direction and the number of times, the radius vector goes may be in the clockwise direction or may be in the counter clockwise direction. Therefore, an encirclement is a number which may be plus or minus or it could be 0 also, whereas encloses there is nothing like 2 times, 3 times this point 1 plus j 0 is enclosed by this contour gamma. Of course, gamma does encircle if traverse this way encircle it once if I traverse it in the opposite direction it will encircle it minus 1, so enclosure is what matters here.

So the result here is that I have a curve gamma there is the point which is 0 of the function f which is enclosed by the curve gamma or the image contour gamma prime. This image contour gamma prime encircles the origin, the same number of time as there are number of 0s enclosed by the curve gamma. Now this of course we are just verified by actual drawing by a for the simple function z minus 1, it holds more general functions, not too general but sufficiently general to be of use to us, in our work with feedback control system. Before, I say that general result let us take a look at one more example and I will ask you to go back and revise your complex variables, conformable mapping and things like that to get the answer. Now the function will be w equal to f z equal to 1 divided by z minus 1 not z minus 1 but 1 divided by z minus 1,notice that this what is called a rational function because its ratio of 2 polynomials, the numerator polynomial is just the constant 1, the denominator polynomial is of degree 1.

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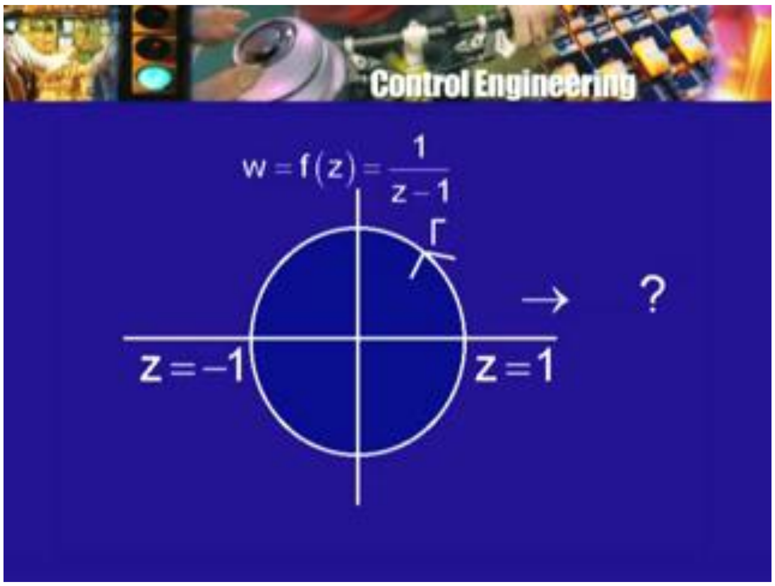

So this rational function has1 pole and the pole is at the point 1 earlier the function z minus 1 had one 0 and the 0 was at 1. Now there is a pole there is not 0 now, I will choose this same contour gamma as before in the beginning namely the units circle, what do you think will be the mapping of this contour gamma, the unit circle in the z plane into the w plane, will it be a circle, the answer is, no, the answer is that it will be a straight line not only that, here it is simple closed curve, the image contour or the image curve is not a closed curve at all, but it is an open curve and in fact, the open curve goes to infinity in both directions, what is happened is this curve gamma has passed through a point where the function has a limit or diverges to infinity as you approach that point or the function has a pole and as a result what happened is, the image contour of the image curve is no longer a closed curve since, it is not closed curve I cannot talk about encirclements and things like that.

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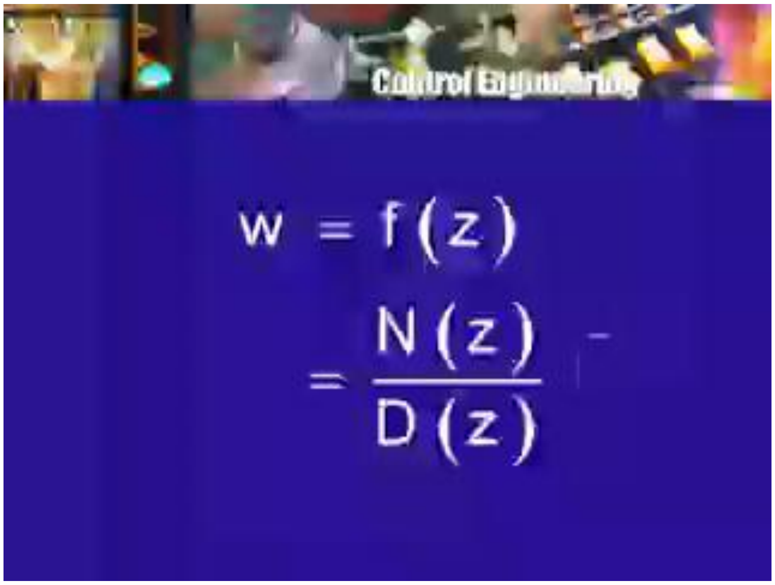

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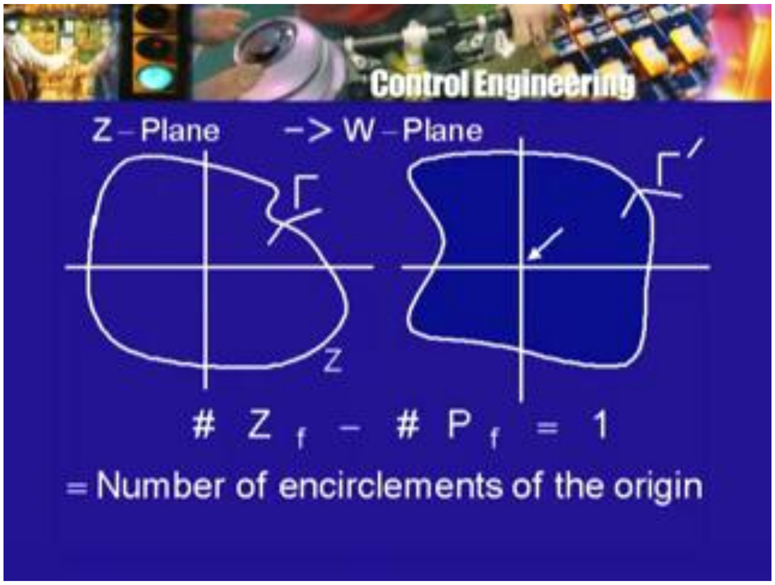

So this is the situation which also has to be excluded, the curve gamma should not pass through a pole of the function also and now we are ready for the general result namely, we have a function w which is a function of z which is given by a numerator polynomial n z, n for numerator divided by a denominator polynomial d z. These 2 polynomials could be of equal to equal degrees or they could be of different degrees one of them may be constant, as saw we earlier. We can therefore talk about the 0s of this function namely they are the roots or solutions of the N z equal to 0. We can talk about the poles of the function, there are the roots of the equation D z equal to 0. So in general this function rational function will have a number of 0s and a number of poles.

Now in the z plane you can choose any contour not necessarily a circle now, a simple closed curve, you can traverse it in either of the 2 directions, counter clockwise or clockwise. Let us say I once again select, the counter clockwise direction with only one restriction. The contour gamma should not pass through either a pole or a 0 of the function f, in other words the contour gamma should be such that for no point z on it and z will be equal to 0 and z equal to 0 corresponds to 0s of the function f.

So the contour gamma should not pass through such points also it should not pass through point z such that D z equal to 0. These are what these are the poles of the transfer function or the poles of the rational function. So suppose I choose gamma such that it does fulfill this condition, it does not pass through any 0 of f, it does not pass through any pole of f. Now what is going to its image in the W plane, now that question is not easy to answer, I have chosen a contour which is not a circle anymore. So it is not something as simple as that therefore, we cannot expect the image to be also something nice like a circle or a square and so on. But the following is true, this simple closed curve gamma will be mapped into a simple closed into a closed curve not necessarily a simple curve as we will see later on into a closed curve gamma prime.

I am going to show it as a simple closed curve but it could be a closed curve which is not simple, it may intersect itself a number of times as we saw earlier, only to illustrate I am showing it as a simple closed curve. But it will always be a simple closed curve. It will not blow up to infinity neither will it pass through the origin in the W plane because gamma does not pass through any 0 of f. So that is the first thing so it is going to be a closed curve, now if it is going to be a close curve I can talk and the origin does not lie on it, I can talk about the number of encirclements by this curve gamma prime and what about the direction on gamma prime that is determined by the direction of gamma and it can happen that when I do it in the counter clockwise direction here, gamma prime may be traversed in the counter clockwise direction but for some other functions, it may turn out that it is traversed in the clockwise direction.

So whatever is the direction of traverse, we have to stick to it for the sake of standardization, let us say we travels gamma in the counter clockwise direction. Now the following holds find out the number of encirclements of the origin in the W plane by the curve gamma prime, if traversed in the same direction as the contour gamma. Now gamma we are traversing in the counter clockwise direction. So, let us traverse gamma prime also in the counter clockwise direction. Now look at gamma prime only find out what is the number of encirclements of the origin by gamma prime. In this case, the answer is1 and you can easily see it. Gamma was traversed in the counter clockwise direction, gamma prime also we are going traverse in the same direction and now the number of encirclements of the origin in the W plane is 1. Then, the result is that this number equal to the following difference, the number of 0s of the function the f inside the contour gamma minus the number of poles of the same function f inside the contour gamma.

So it is interesting that this difference between the number of 0s of f inside gamma and the number of poles of f inside gamma, this difference is reflected in the number of encirclements by gamma prime of the origin in the W plane and this result goes back to the French mathematician Cauchy and it is sometimes called Cauchy's principle of the argument because encirclement has essentially something to do with the change in the argument of a complex number. Remember, we think of the rotation of a radius vector whose length may be changing but the radius vector changes its orientation.

So it is essentially connected with the argument not really with distance as such and therefore it is called the principle of the argument and this was first stated and proved by Cauchy. Now this principle of the argument is what we will make use of to determine the stability or instability or the closed loop poles of a closed feedback control system, lying all in the left half plane or not lying in the left half plane. So this Cauchy's principle of the argument is what we will use and this idea was used by the engineer whose name I already mentioned engineer cum mathematician Nyquist and so, this has given rise to a test or a method of checking stability known as the Nyquist criterion. So this is what we will now turn to.