Control Engineering
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Lecture - 35

Let us take one more example to illustrate the Routh algorithm. As I told you earlier, you should learn to make up your own examples. So, I am going to choose a polynomial which will have 1 root in the right half plane and 2 roots in the left half plane. Consider this polynomial s minus 1 multiplied by s plus 2 multiplied by s plus 3.

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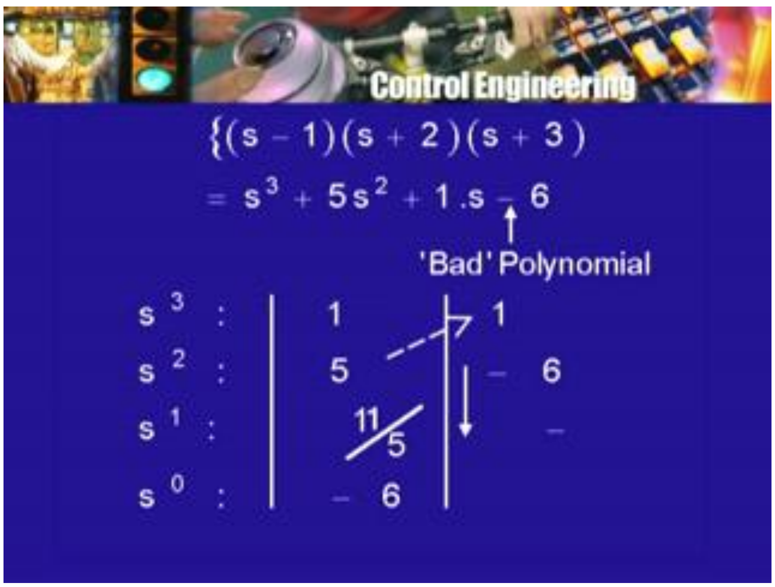

So there is a root 1 which is in the right half plane and the other 2 roots are minus 2 and minus 3 which are of course in the left half plane, are there any roots on the j omega axis, is there any purely imaginary root, no, there is it. Now, let us expand out this polynomial and I told you earlier that you should have some practice in multiplying out polynomials. So, for example what will be the highest power of s, it will be s into s into s, so it will be s cube, what will be the multiplier or the coefficient of s square. Well, as you can see it will come from minus 1 plus 2 plus 3 so it will be 5 s square.

So that is the second term, 5 s square, what about the s term the s term, as you can see will come from multiplying the coefficient 2 by 2. So here is minus 2 here is minus 3 that is minus 5 and here is 6 that is therefore plus 1. So plus I will write it as 1 into s and the constant term you of course the product of the constant term, so that is minus 6. Now, right away we can say that this polynomial is a bad one, even if you did not know the factorization. Suppose, I did not know the factorization I only knew coefficients or I knew the polynomial, I would immediately conclude that this polynomial is a bad one that is it has at least 1 root which is not in the strict left half plane.

Now, why is that so, why do we conclude that it is bad because if you look at the coefficients, they do not all have the same sign, three of them are positive but the 4th one is negative. So this polynomial is bad but suppose you want to go further and find out how bad it is that is, where are the roots how many roots are in the left half plane, how many roots in the right half plane, are there any roots on the j omega axis. So we start with the Routh array, so s cube first I will write down 1 and 1 and then, the second is the s square row which corresponds to the even part, so I get 5 and minus 6.

So the next row s to the 1 as usual 5 minus, minus 6 that is 11 divided by 5, so that is 11 divided by 5. There is no entry here, I am writing a dash not a 0 and then, s to the 0, this into this minus nothing divided by 11 by 5, so minus 6. Okay, so we were able to complete the table without encountering any 0 as a pivot element therefore, what is such a case called; it is called a regular case. It is not a singular case; we can go all the way up to the end because the pivot element has never become 0, so it is a regular case.

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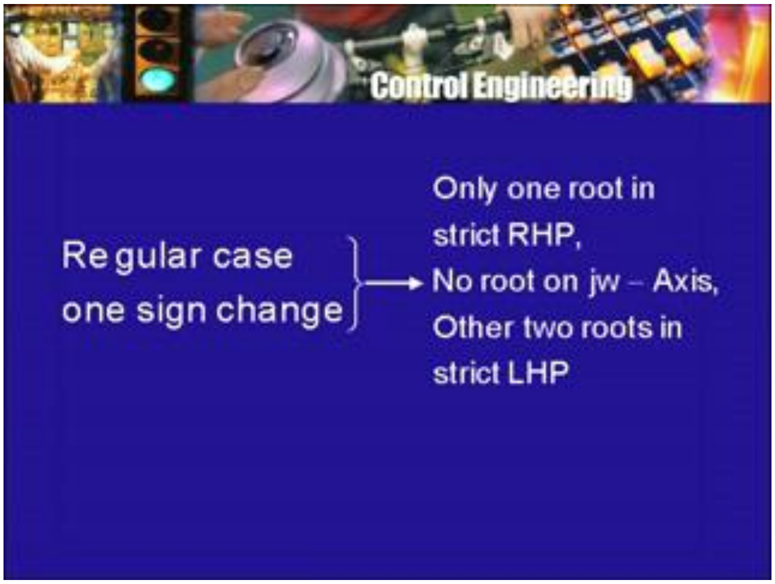

Now for the regular case, Routh theorem tells you that if you look at the first columns and look at the signs of the terms in the first column then, as you go down the first column, find out how many changes of sign take place. So plus, plus, no change, plus, no change but plus to minus, so there is only one sign change and Routh's theorem tells you there for that this polynomial will have exactly 1 root strictly in the right half plane and that is true. We have exactly 1 root strictly in the right half plane. Now therefore the conclusion will be that there will be 3 roots in the left half plane.

Now, why because the case is a regular one, the 2 singular cases have not arisen. The singular case will arise when there are 2 roots or more than 2 roots on the j omega axis. It can also arise when the roots occur in quadruplets that is 4 roots which have quadrantal symmetry. So these are the bad cases plus there is the possibility that the pivot element only becomes 0, the other elements may not be 0 but those are all bad cases, where there will be roots either on the j omega axis or in the right half plane. We are looking at a regular case only one sign change, so only one root is in the right half plane. But, let us confirm that the remaining 2 roots are in the left half plane then, how to do that.

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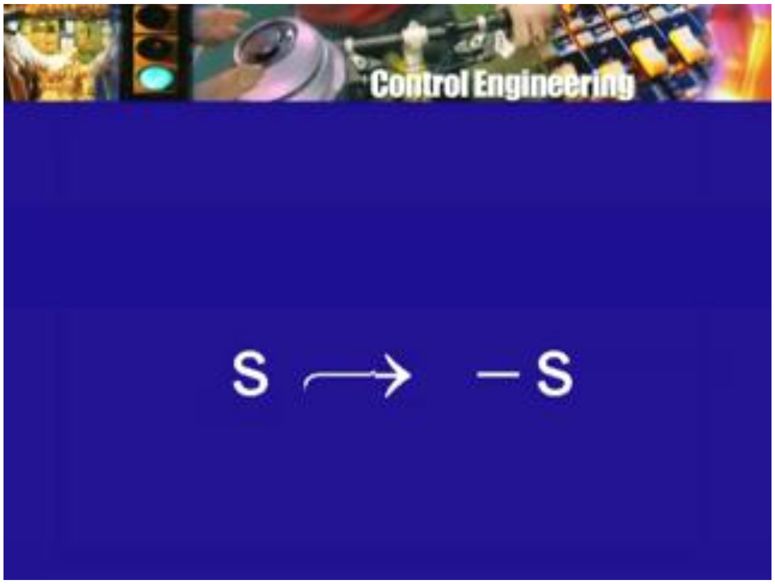

Well, in the given polynomial replace s by minus s, now what is this doing replacing s by minus s, if you think in terms of the complex plane then, it is a mapping of the complex plane which changes the flips, the plane around the j omega axis. So the positive right half plane will now become the left half plane, the left half plane will become the right half plane, the j omega axis however will remain unchanged.

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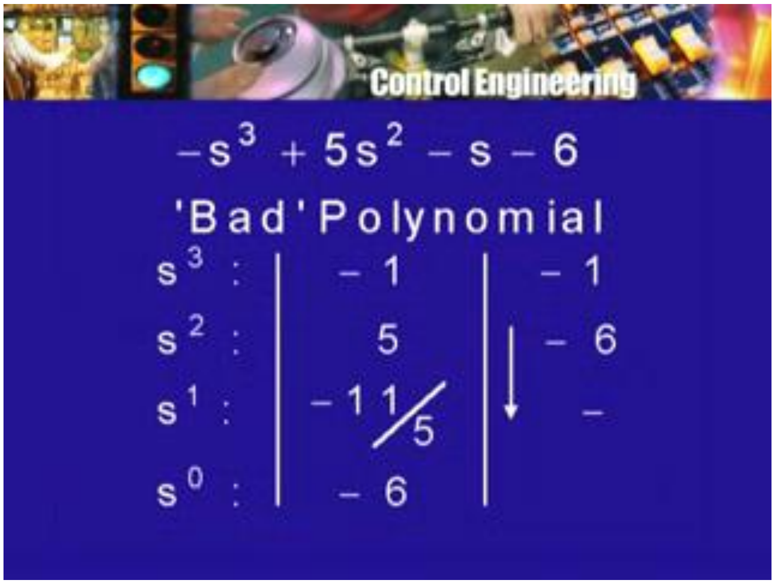

So it is called a mapping of the complex plane into itself. In your course on theory of functions of a complex variable perhaps you have looked at transformations or mappings of the complex plane it into itself such as a binary transformation and so on. Of course, this is the very simple one simply changing s by minus s, so what does our polynomial now become; it becomes minus s cube plus 5 s squared minus s minus 6. Now as you can expect this polynomial will also be bad because the original polynomial had 2 roots in the left half plane.

Now, this polynomial will have their negatives and therefore it will have 2 roots in the right half plane. But even without going in for the Routh algorithm look at the polynomial and look at the signs of the coefficients, they are not all the same, negative, positive, negative, negative. So this polynomial as also bad and we will try to find out how bad it is by using the Routh algorithm. Now, I could have multiplied this whole polynomial by minus 1 but I am not going to do that I am going to keep the entries as they are it is not necessary to get rid of any negative signs for the leading coefficient. So I will start once again s cube so minus 1 minus 1 s square that is the even part 5 and minus 6, right. So let us continue s to the 1.

So this is minus 5 minus 6, so that is minus 11 divided by 5, so this entry is minus 11 by 5, there is no entry here, s to the 0, this into this minus nothing divided by this, so minus 6. So the polynomial is also regular one, there is no hitch we proceeded without encountering any 0 in the first column that is no pivot element has become 0 and let us look at the entries in the first column now and go down the first column and look at sign changes, minus to plus. So there is one sign change plus to minus, so there is one more sign change minus to minus, no sign change. So the number of sign changes is 2 and so this polynomial has exactly 2 roots in the right half plane but we know, it is the negative of the original polynomial. So the 2 roots in the right half plane will be actually 2 and 3 and the root in the left half plane will be that third root which originally was plus 1, now it will become minus 1.

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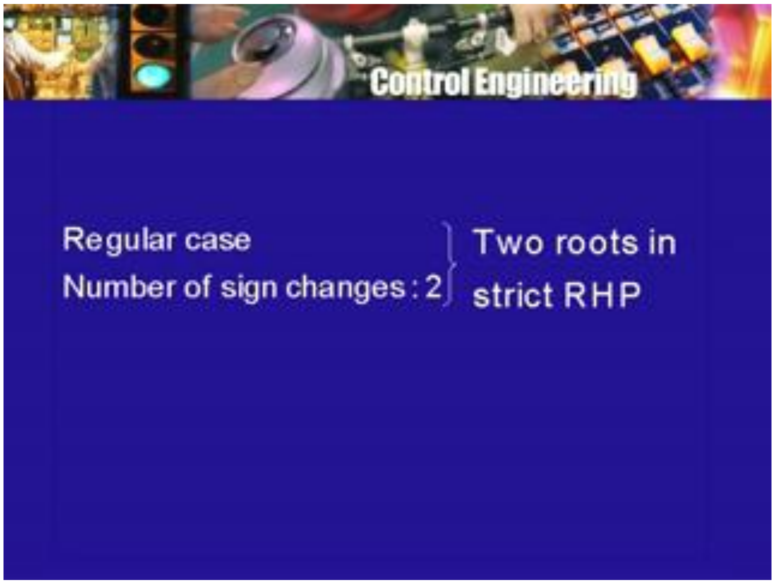

So this way, if an original polynomial is a regular one that is your table can be completed without any hitch then, we can verify by changing the s to minus s which will of course change the polynomial, go through the whole thing again and verify, that the number of sign changes will be such that you will get the roots in the left half plane. So in the regular case, the Routh algorithm completely gives you the break up between the roots which how many in the left half plane, how many in the right half plane and no roots on the j omega axis that becomes very clear.

Now, let me get back to the root locus. As I told you earlier, this is not all that can be said about the Routh algorithm, the 2 singular cases I have not said as to how to proceed further, when you encounter the singular cases. You should look up your textbook and the examples given there to know about it. Getting back to the root locus for just a short while now, before we get back to our design problem. Here is our old whole 0 diagram by this time you know it by heart and I have deliberately choosen an example in which there is a 0 in the right half plane and the remaining 4 poles and 0s are in the left half plane.

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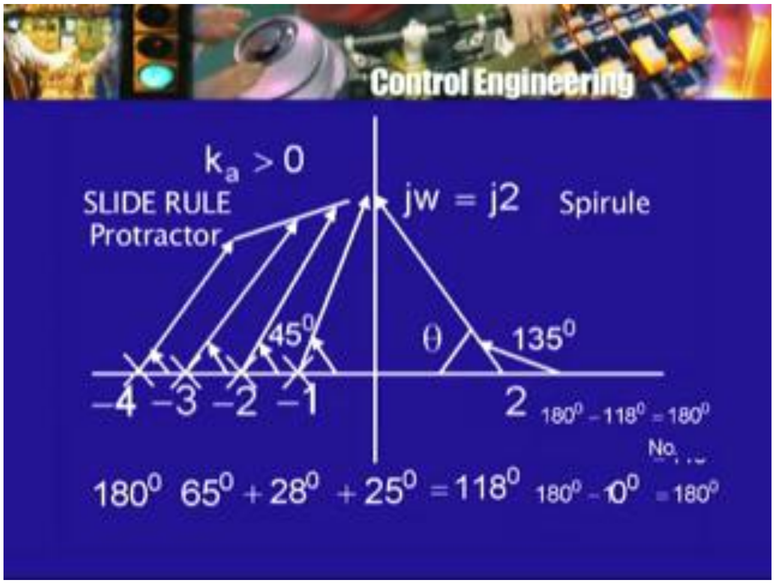

Now, we had looked at a number of rules for the root locus and we found out that the root locus entirely lay on the real axis. Now there are situations when the root locus will not be on the real axis only, there will be portions of the root locus which go out or move out into the complex plane. It is very easy to construct example, so let us construct one like this. Now remember, what was one of the rules about asymptotes of the root locus? The number of asymptotes is the difference between the number of poles and the number of 0s.

So, if the number difference is one, there is only one asymptote and what is the angle of that asymptote then, 180 degrees, if the difference is 2, there are 2 asymptotes and so, what are the angles made by the 2 asymptotes, 90 degrees and minus 90 degrees or 90 degrees and 270 degrees. Now, when the angle is 90 and 270, it means that the root locus must move out into the complex plane away from the real axis. So with this in mind let us construct an example.

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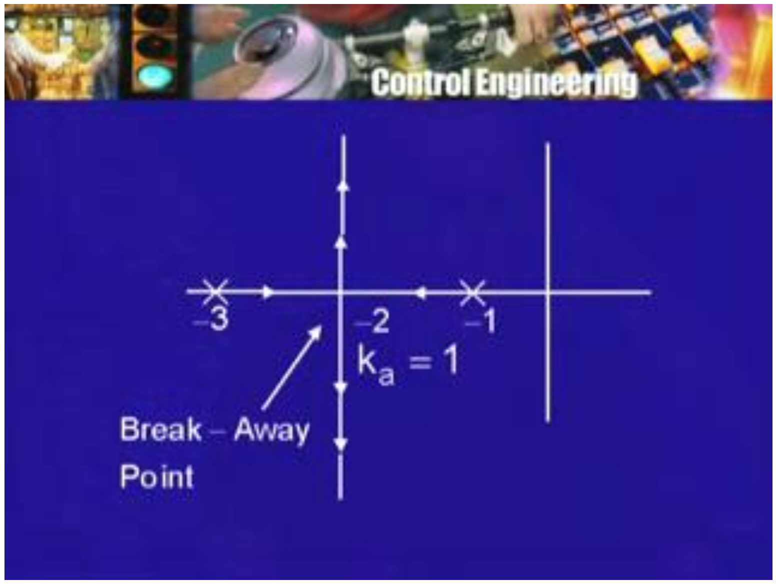

So, let me take a pole at minus 1 and another pole at minus 3. So what about the real axis portions of the root locus, there is nothing here the number of poles and zeroes to the right 0. So this part does not belong to the root locus, what about this portion between minus 3 and minus 1 the number of poles and zeroes to the right is 1, one pole only no 0, 1 is odd therefore this portion of the real axis will belong to the root locus, move further how many poles and zeroes to the right 2, 2o is an odd number or even number, even number. Therefore, no point here beyond minus 3 will belong to the root locus.

So this is the only portion of the real axis which will belong to the root locus. As we know the root loci start at the poles, so for very small k a, we expect that there will be a root very close to minus 3, another root very close to minus 1 and lying on this part of the real axis. But for large k, we know that there are going to be asymptotes, there are going to be 2 asymptotes. So which means that the roots will move out in to the complex plane and in fact, go to infinity infinity in such a way that the modules of the roots goes on increasing without any bound. Now, what about the point of intersection of the asymptotes, the rule was sigma p minus sigma z divided by p minus z sigma p is minus 4 p minus z is 2.

So, minus 4 divided by 2 is 2, so here are the 2 asymptotes. So the 2 branches of the root locus why, 2 because there are 2 poles no zeroes, 2 greater than 0. So the number of branches of root locus is exactly 2, they will start off at the poles but as k is increases eventually they will move away and go to infinity approaching these asymptotes. Now without doing any further calculations, it is not possible to say exactly what the root locus branches will look like. But suppose this is one possibility that these 2 branches which start at the poles as we say eventually go out like this. In other words, for some value of k this root and this root become equal or in other words, the polynomial has 2 equal roots, in fact it is easy in this case to calculate it, the polynomial that we have looking at is s plus 1 into s plus 3 that the pole part plus k a into there is no 0 part therefore, 1 not 0, 1 in the numerator s plus 1 by s plus 3 in the denominator.

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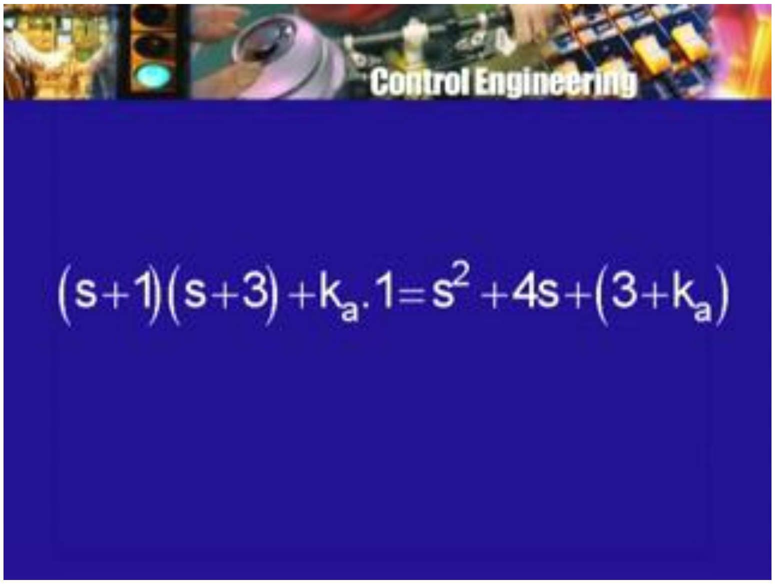

So the polynomial looks like s squared plus 4 s plus k a. This is the polynomial, when will this polynomial have equal roots, the discriminate tells you that this will have equal roots when 4 square is equal to 4 into k a or when k a is equal to 4. In that case, the polynomial is s square plus 4 s plus 4 which is of course easy to factorize, it is s plus 2 square. So for k a equal to 4 that roots are really equal or as we say there are 2 roots, equal roots are roots of multiplicity 2 and the root is minus 2. So the 2 branches of the root locus start of at the poles and we say that the come together or they meet at this point and then, as k a is increased beyond 4, the roots are going to become complex and so the roots will move out in to the complex plane.

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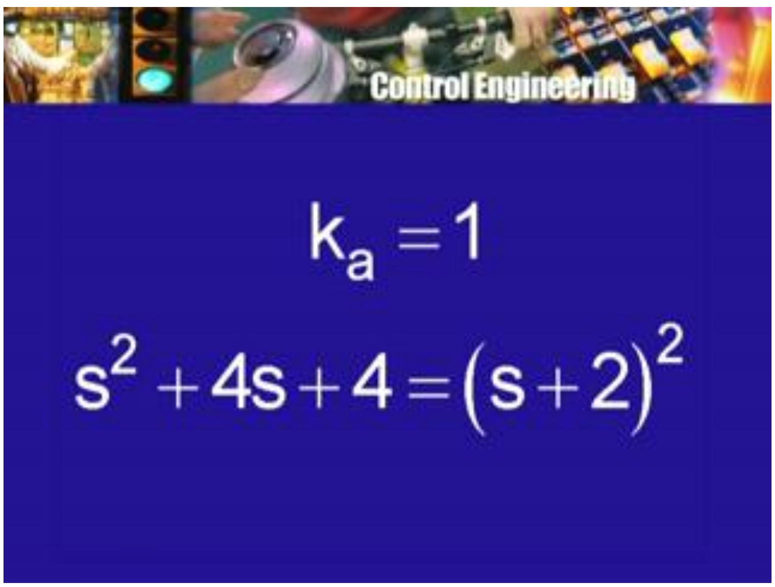

In this case, one can show that the roots actually move out along the 2 asymptotes. They not just approach the asymptotes, they lie on the asymptotes. Now such a point here, where branches come in for some value of k, for smaller value of k, they are not meeting, for larger value of k, again they are separate but for a particular value of k, they are together such points are known as break away points. So such points are called breakaway points of the root locus. In this case, there is one break away point, now people have given rules for finding out the location of the breakaway point. The breakaway point may not be always of the real axis it is not difficult to construct examples where the break away point will be in the complex plane.

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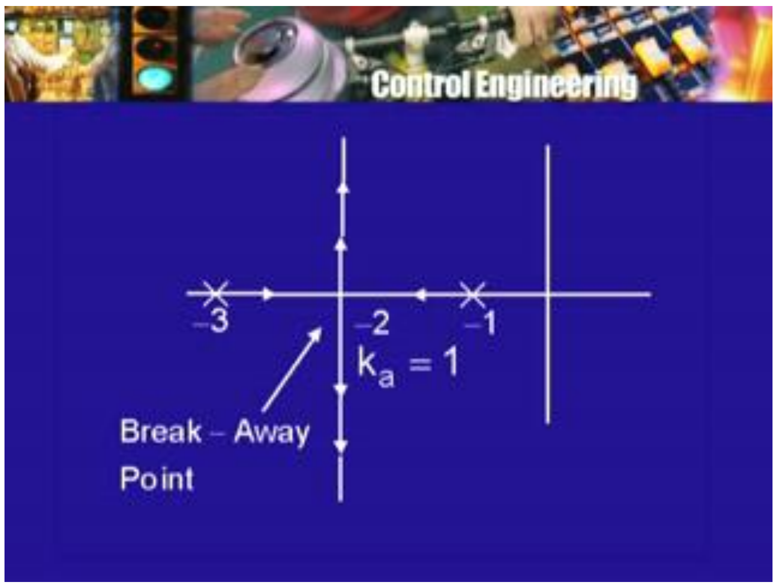

So some rules have been given, for determining, if there are any break away points then the location of the break away point and the corresponding value of the gain k a. Now, we do not have time to go in to all those rules but just as there are break away points you can expect that there will be points, where the root locus branches. Actually, in this case of the break away point it self as you can see 2 roots are coming together and then they are going away. Now the reason it is called a break away point is because as k increases, you are thinking of movement into the complex plane, as k increases from 0, the roots are coming together and then, for a larger value of k they are going away.

So actually although they are called break away points, there is the phenomenon of the roots coming together and there is then, subsequent getting away from each other. So really speaking the break away point is also a break in point. But in the root locus literature normally, this is called a break away point and what will be there for a break in point, if something like the following happens then, that will be called a break in point. For example, the root locus branches go towards the real axis and meet at some point on the real axis and then, may they move away from one another while staying on the real axis. So such a point in the literature is called a break in point. Notice that there is no really difference the roots are coming together and then, they are going away in both cases. But the real axis is given a special importance and that is why this is called a break in point and the other one is called a break away point and there are rules for determining the break in and the breakaway points and the corresponding values of the gain k a and you should look them up and work out some problems from your text book.

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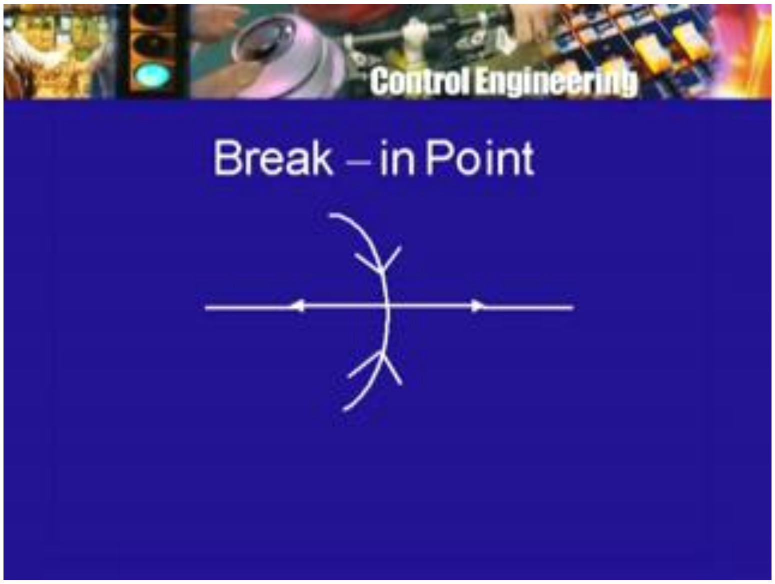

So for the moment for this will be enough as far as the root locus method is concerned to summarize very quickly, what is the purpose of the root locus method. The purpose of the root locus method is to try to determine what, the characteristic polynomial of the closed loop system which is given by this expression, I have called it p of s and where did this come from in the forward path we have the transfer function g preceded by gain k a, then there is this feedback difference element and there is this feedback transfer function h.

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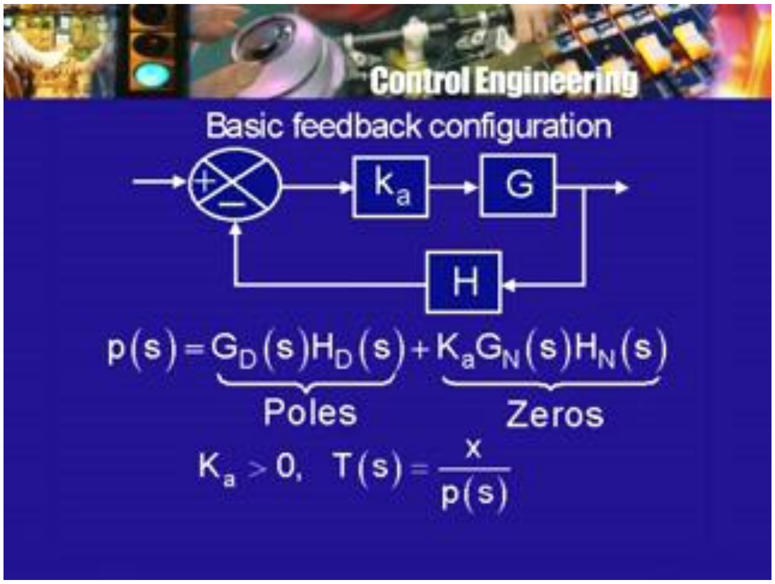

The overall transfer function was k a, g over 1 plus k a, g, h and when we look at that denominator term and write each as a fraction numerator or a denominator, we get this polynomial. In the denominator, in the overall transfer function which may be call t of s, this polynomial appears in the denominator and there is some polynomial in the numerator and the response of the system both the 0 input response and the 0 state response is very much dependent on the factors of p s or the roots of p s and so, you would like to find out where the roots of p s are for various values of k a.

The root locus method is a method which was introduced by Evan to enable you to answer this question at least qualitatively and to some extent quantitatively. Usually, this gain k a is positive there is no phase reversal here and that is called the direct root locus. It can be extended for k a less than 0, it is called the inverse root locus then the rules are a little different. Now the root locus method tells you that first of all, you should start with the pole 0 diagram that is show the poles of g and h on the diagram and show the zeroes of g and h on the diagram and then, there are a set of rules which are to be applied, which enable you in many cases to sketch the root locus that is get some idea, qualitative idea as to what is going to happen, as k a is changed.

So, what are the rules very quickly, first of all you have to look at the number of poles, you have to look at the number of zeroes. So first thing to identify is the number of poles and the number of zeroes the find out whichever is the larger of them or they may be equal, in that case either of them. The number of branches of the root locus is given by the larger of the 2. So that is the simplest rule number of branches is larger of p and z, though p and z may be equal in that case it is at number. So, immediately you know how many branches of the root locus will be there. So that is one thing.

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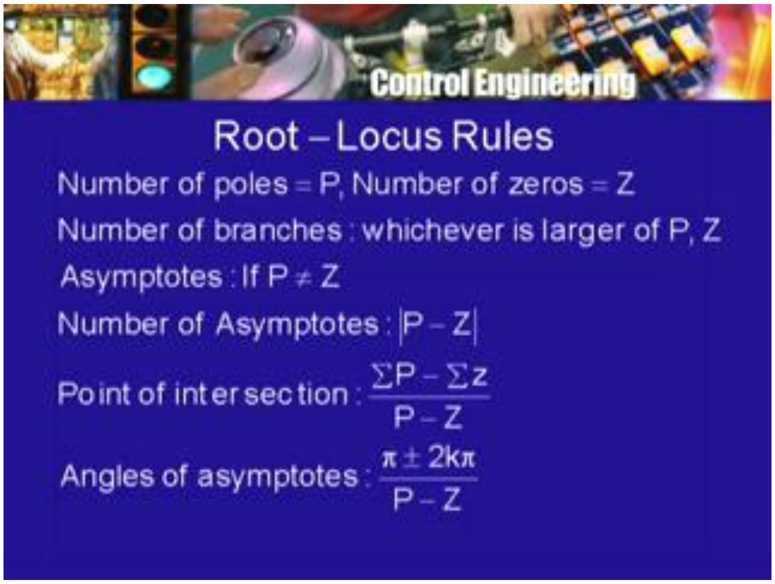

Then, the second thing perhaps that one can find out is the asymptotes, when p is not equal to z there will be asymptotes and one can find out the point of intersection of the asymptotes or what is call the centroid and the angles made by the asymptotes. The centroid is given by sigma p minus sigma z divided by p minus z and the angle of the asymptotes is obtained from the angle condition and what is it? It is given by phi plus minus 2 k phi radians divided by p minus z, give various values of k, if there is more than one asymptote, you will get a number of values for the angles made by the asymptote. Then, if p is greater than z then as k increases, some branches of the root locus will go away towards infinity along the asymptotes, where as if z is greater than p and usually that is not the case then some branches of the root locus will start from a point which is very far away from the origin that is we say that they start from infinity and then come towards the finite part of the complex plane.

So this is as far as asymptotes are concerned. With the pole 0 diagram, with the asymptotes sometimes one can immediately figure out, what is going to happen qualitatively but we have more rules and what are some of the other rules, well real axis portion of the root locus. A point on the real line or real axis belongs to the root locus, if the number of poles and zeroes to its right along the real axis is odd. So, here is a point on the real axis, is this point on the root locus for some positive value of k. Well, I will look at all the poles and zeroes, poles and zeroes to its right and count them and see if that number is odd, if that number is odd then, this point will belong to the root locus. Remember it, for a point to belong to the root locus, number of poles and zeroes to its right should be odd.

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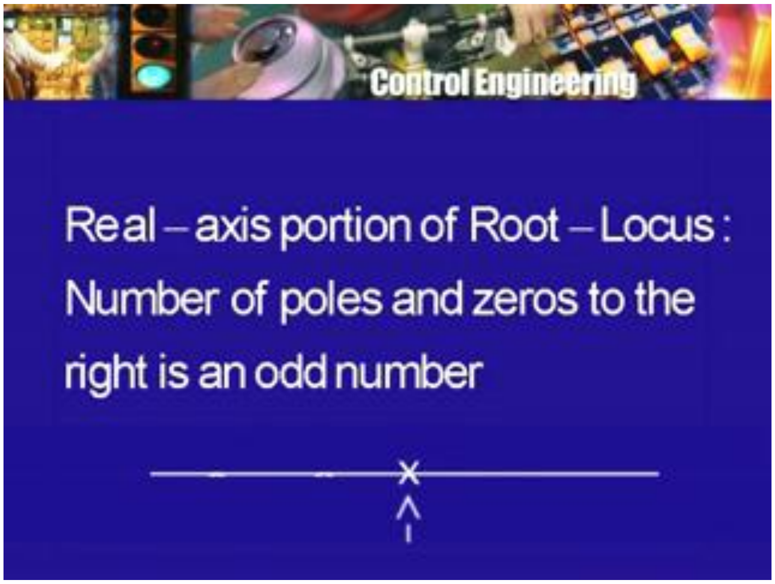

So with this we can determine the real axis portions on the root locus. Then, there is a rule that gives you the angle of departure and angle of arrival and I did not state it fully but I have worked out an example to give you some idea of how the angle of arrival and angle of departure can be calculated. The next rule was intersection with the j omega axis and as a part of it of course, one uses the Routh algorithm. Of course, by looking at the polynomial, by looking at its coefficients the characteristic polynomial of course will have k a in it sometimes one can figure out immediately that for all values of k a there is no sign change.

So the polynomial does not look bad but if for some value of k a there is going to be a sign change then for that value of k the polynomial is going to be bad. But to find out how many roots in the left half plane and how many roots in the right half plane and perhaps, no root in the right half plane one can use the Routh algorithm. You have to construct the routh table, split the polynomial into its odd and even parts and then, carry out that successive division process. Remember that you are not just manipulating numbers, you are dividing one polynomial by another and writing down the reminder then, the 2 polynomials that are at the top of the table now, not all the way but last one and the previous one, take them as the dividend and divisor, work out the reminder, keep on doing this.

In the regular case, you go till the end when the number of sign changes is the number of right half plane root and if the case is a regular case then, there are no roots on the j omega axis and then there are 2 specific situations, singular cases which I have mentioned but I have not discus how to proceed further in that case. Finally, there are rules for the break away and break in points so this is the root locus method. There are program packages available today, which will draw the root locus virtually for you that is you specify the pole and 0 locations then, on the computer screen, on the screen of your monitor you will see the root locus drawn and you can even zoom in on some part of it, you can zoom out all kinds of things.

So, you have a tool which has been developed into a computer program but that does not mean that you should not know any thing about root locus of what is it that one is doing with the root locus method, why is it that one is looking at the root locus at all. So do not ignore this, simply because there is a program available which will draw the root locus for you, you must be able to understand what is the root locus showing, it is not enough to have a plot, you must understand what is it that it is showing, what it shows is locations of the roots of the characteristic polynomial for various values of the gain k a and this is to determine whether or how large the gain k a can be without things happening, what things happening, without the system becoming unstable.

So this is the main purpose of the root locus method and remember, it is a qualitative method not fully qualitative somethings can be determined, you can actually plot a few points on the root locus, you can actually calculate k a by using the angle condition first calculate the point, find out whether point is on the root locus then, calculate k a etcetera. But I would not recommend that to you because it is better to use a computer program if you want to factorize a polynomial. But for qualitative understanding it is a very good method although it is almost 50 years old, all right. So that is the root locus method and the Routh algorithm.

Now, let us get back to our control system problem namely, speed control of the motor. First with only proportional feed back and then, with proportional and integral feedback and what could be the problems arising there. Now, if you remember I hope you have not forgotten, in the open loop case of course there was a direct transfer function relating the armature voltage, applied armature voltage E a of s and the load torque to the angular speed, of course all these are Laplace transforms of the corresponding functions. We are talking about transfer functions, so we are talking about Laplace transforms and relationships between the Laplace transform.

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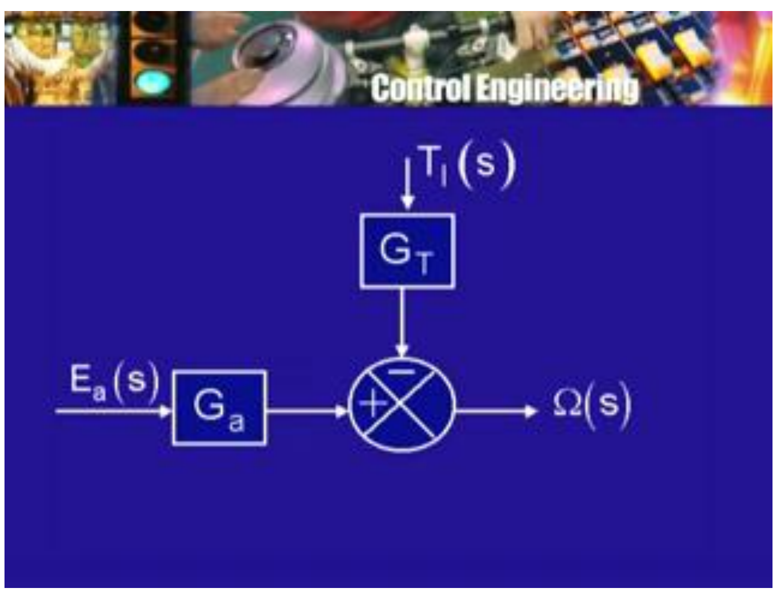

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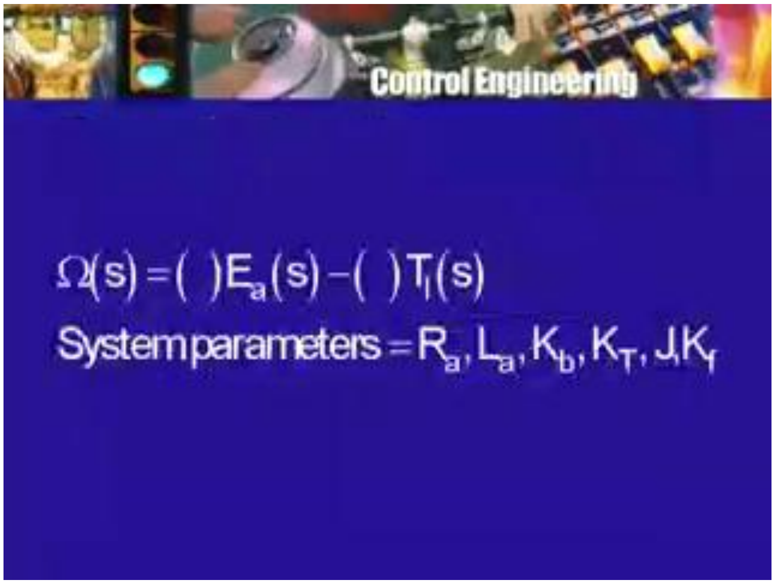

So, we had worked out omega s as some transmittance of some transfer function into E a of s plus another transmittance of transfer function into T l of s or if I put a minus sign here, I may prefer to put the minus sign here and these 2 transfer functions we were able to determine. Of course they depended on what they depended on the parameters r a, l a then k b, k t and j a and k f. These are the 6 parameters which appeared in those transfer functions and what the those transfer functions look like. Well, the transfer function which multiplies the armature voltage was something of the kind we could write it in the form s square plus a s plus b and the numerator was just some coefficient, let us say capital A that was the transfer function that relates the armature voltage to the angular speed or speed remoter. It was a divided by s squared plus a s plus b and the other transfer function relating the torque however, had a numerator polynomial in addition to the denominator polynomial.

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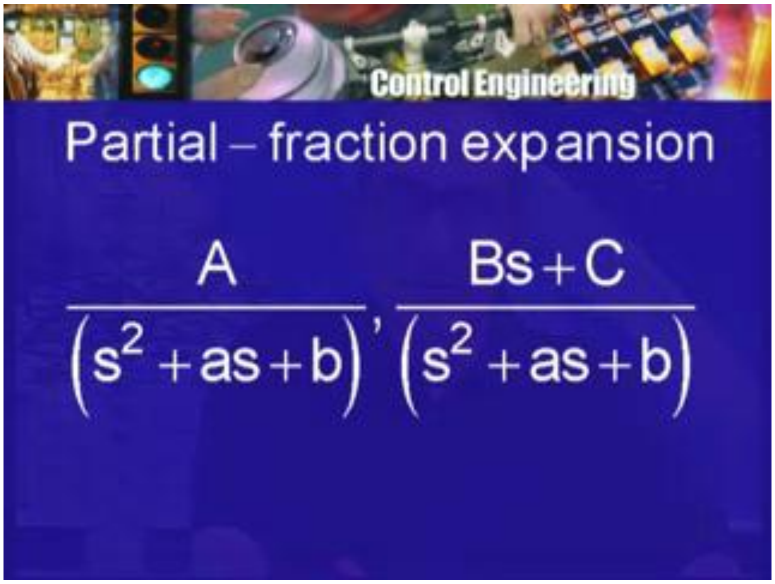

So it looked like perhaps b s plus c divided by s squared plus a s plus b, these were the 2 transfer functions then, of course we just have a quadratic so factorizing that is very easy, we do not need any root locus, we do not need any Routh algorithm or whatever we know how to factorize a quadratic and therefore, there will be 2 poles of this transfer function. Usually, these 2 poles are both real and negative, in fact this being a quadratic a and b, both positive because they involve the coefficients such a quadratic will always have roots in the left half plane.

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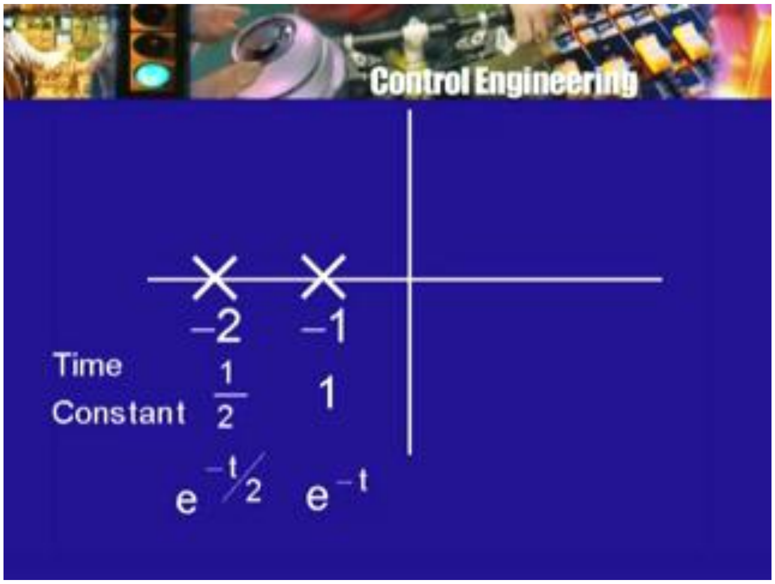

So there is no instability but in a very extreme case, the 2 poles instead being on the real axis like this which means what suppose this is minus 1 and this is minus 2, what does that mean. It means that there is time constant corresponding to it which is1 by 1 and there is a time constant corresponding to this which is 1 by 2. So we expect that functions like e raised to minus t and e raise to minus t by 2 will occur in the response. So there is a time constant 1 and there is a time constant which is a half or the larger of the 2 time constant is 1, one second let us a if the units are properly choosen.

So this means that after 5 seconds the transience what have died down virtually or 10 seconds and you will have a steady state and we saw that when the roots are in the left half plane, the system is stable that goes with this fact that the transient terms go to 0 as t tends to infinity.
So the open loop system has no problem it is stable. However, it may happen that depending on the values of the parameters, the 2 poles are not on the real axis but they are in the complex plane, still in the left half of the complex plane but complex which means what, the system is still going to be stable but what does this complex nature of the root poles mean that the transient response will be oscillatory.

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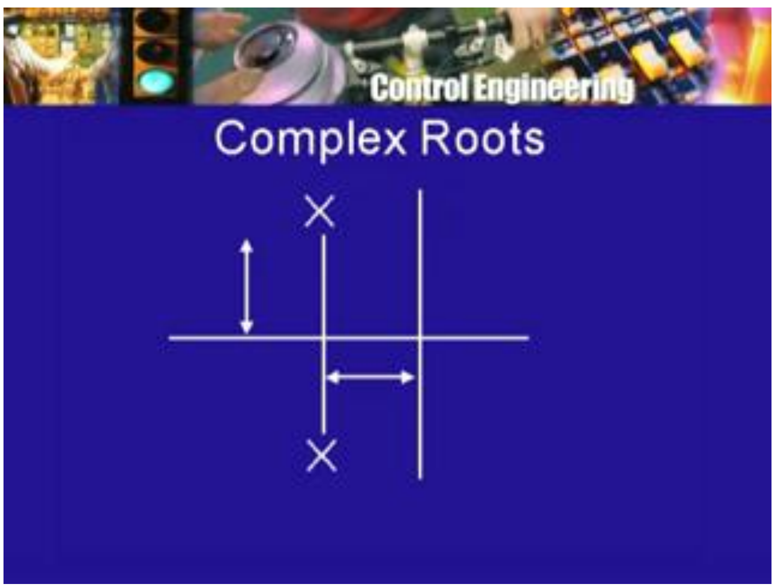

So, we can expect something like this there will be oscillation but the oscillations will die down. The dying down will be given exactly by the time constant corresponding to the real part whereas the frequency of the oscillations will be governed by the imaginary part and then, we have talked about damping ratio delta and so on, in that case. So frequency, actual frequency of vibration, natural frequency of the oscillation and so forth, if a number a square plus a s plus b. The natural frequency was square root of b and then delta was introduced the in terms of an square root of b but by completing the square we got the actual frequency of the oscillation.

So the system will still be stable the open loop drive is stable, if you make changes in ea sudden increase or sudden decrease changes in load torque, sudden increase or sudden decrease, there will be oscillation but the oscillations will die out, the system is stable. However, with the open loop case when the armature voltage changes of course the speed will change but with change of load torque the speed was going to change. Normally, the armature voltage will be kept constant one will try to keep it constant but the load torque is not under our control and so the bad part about that open loop system was that there will be steady state error, when the load torque changes and to remove this steady state error or to reduce it really, we introduced proportional feedback.

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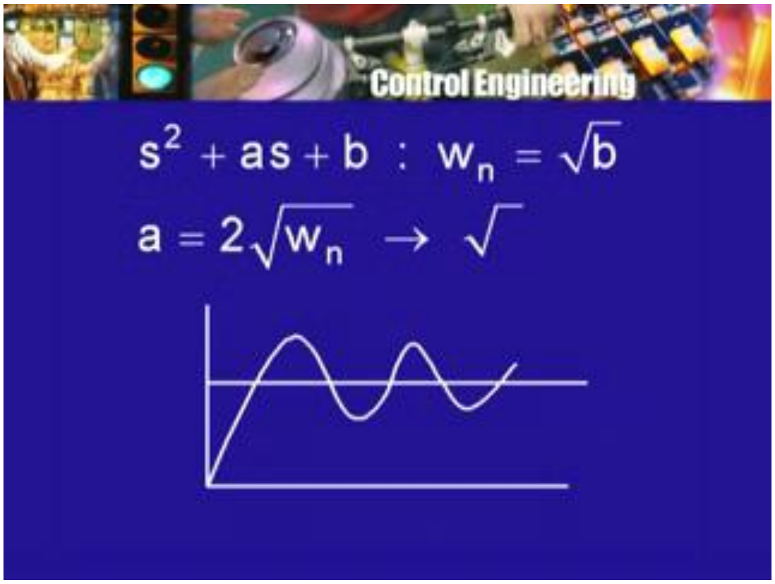

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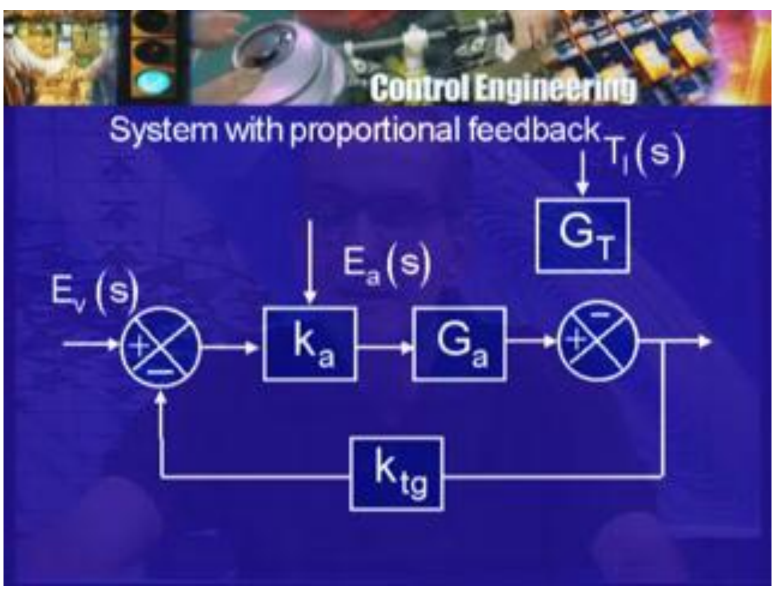

Now, when we introduce proportional feedback, the block diagram becomes modified and now, I will still show this older block diagram with the transmittances and I will show them separately. So here is plus here is minus here is this transfer function which I had called G a which is the part that multiplies E a of s and here is the transfer function G t which multiplies T l of s load torque, the 2 together produce omega of s and there is this speed back term k tacho generator which then goes into the difference device. So here is E r of s and then the output which is not really the error but the different signal between the reference voltage and the tacho generator voltage these to be amplified in general, so I had called it k a.

So this was the block diagram and now, we know that this is the situation as far as the transfer function from E r to omega is concerned, in other words we have put T l equal to 0 for the moment or do not look at that part. So it is k a multiplied by G a in the forward path and in the feedback path, I have h which is simply k tacho generator and of course, I want to determine or I want to see what happens as I change the gain k a. So this is exactly the situation for our root locus method, so here is my G now, which is G a here is my h, which is simply k it tacho generator.

So I look at the characteristic polynomial, I have a look at draw the pole 0 diagram and then, see the effect of change of k a on the roots of the system which means the change effect of the change of k a on the nature of the transient response. Steady state response, we had looked at separately and we found out that the steady state response could be made small by making K a large but we will see now, what is going to be the effect of making K a large on the transient response and I would like you to try this out before I do it for you in detail.

We have already calculated G a take it in that form some a divided by s square plus as plus b, plot the pole 0 diagram and then apply the root locus method to find out what is the root locus for varying K, as the gain K a is increased from 0 indefinitely, what is going to the happen to the roots of the characteristic polynomial and from that what will happen to the transient response of the system. So do this homework and we will carry on with it.