Control Engineering
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Lecture - 31

We have seen the2 simple rules, as far as the root locus branches are concerned namely, the root locus branches start at the poles and end at the 0s unless there are asymptotes. If the number of poles is greater than the number of 0s then, there are asymptotes for the 0s as the gain K goes to infinity or increases without limit whereas, if the number of poles is less than the number of 0s then, it is the other way round. The root locus branches some of them start from infinity and then move towards the 0s and we are now looking at the next rule for the location of the root locus namely, portions of the root locus which belong to the real axis or the other way round as I said portions of the real axis which belong to the root locus.

Now the rule is little more complicated but not too much more complicated and the easiest way to remember it is as follows. Think of any point on the real axis, for example on our pole 0 diagram here we have a number of poles and number of 0s, some of them are the poles and 0s of G the others are poles and 0s of H and as far as the root locus is concerned, it does not matter whether there are of G or H. We are looking at the polynomial which involves both G and H.

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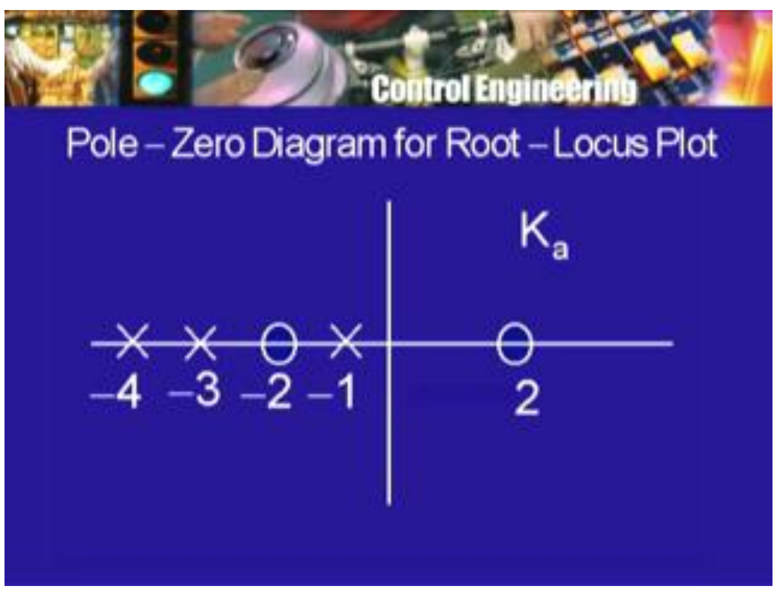

Now, if you consider a point on the real axis there are various positions that it could have. For example, the point could be here the point could be here or here or here or here or here. In other words, the point could be on the positive real axis to the right of all the poles and 0s or it could be on the negative real axis to the left of all of the poles and 0s or it could lie in between 2 poles or in between a pole and a 0.

We are excluding the poles and 0s from our consideration at the moment because we already know that the root locus branches start at the poles and end at the 0s. The finite branch that is those which are not asymptotic do that, now the rule is as follows. Any given number, real number that is any point on the real axis will belong to the root locus and what does it mean to say that a point will belong to the root locus, it means that point or that real number will be a root of the characteristic equation. We have looked at the characteristic equation earlier a number or a point geometrically is on the root locus, if the number is a root of the characteristic equation or a root of the characteristic polynomial.

Now the rule says that a particular point or any particular real number is on the root locus, if the number of poles and 0s to its right on the real axis is an odd number. So I will write it here, number of poles and 0s to the right are odd in number. Let me repeat, a point on the real axis belongs to the root locus, if the number of poles and 0s to the right of that point lying on the real axis is an odd number. For example, consider this position here which is to the right of all the poles and 0s on the real axis. This point here will it belong to the root locus or not.

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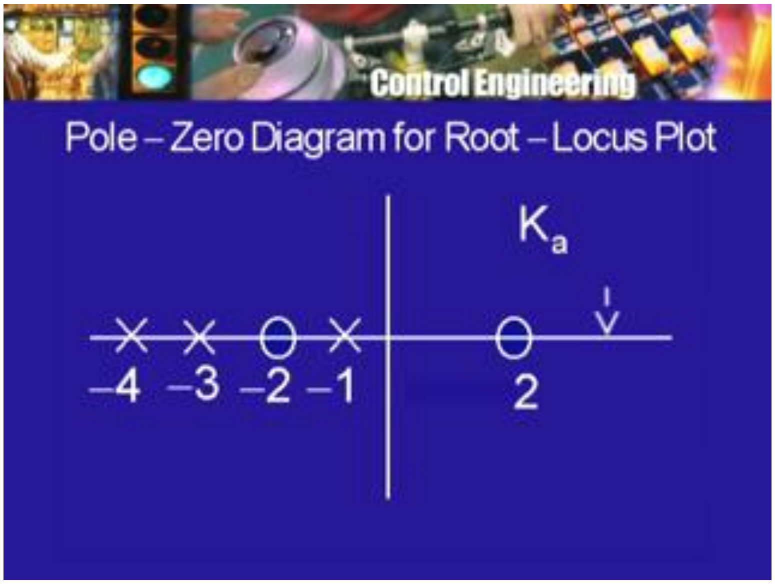

Well, look towards its right, are there any poles and 0s towards its right, now. So the number of poles and 0s towards its right is 0 and as you know 0 is regarded as an even number therefore, it is not an odd number therefore, this point does not belong does not belong to the root locus, what about the point between, this 0 at 2 and this pole at minus 1. Let us say, a location here, now will it belong to the root locus. Once again, look at the portion of the real axis to its right are there any poles and 0s in that portion, the answer is yes, there is only one 0s. So the total number of poles and 0s to the right of this location is 1.

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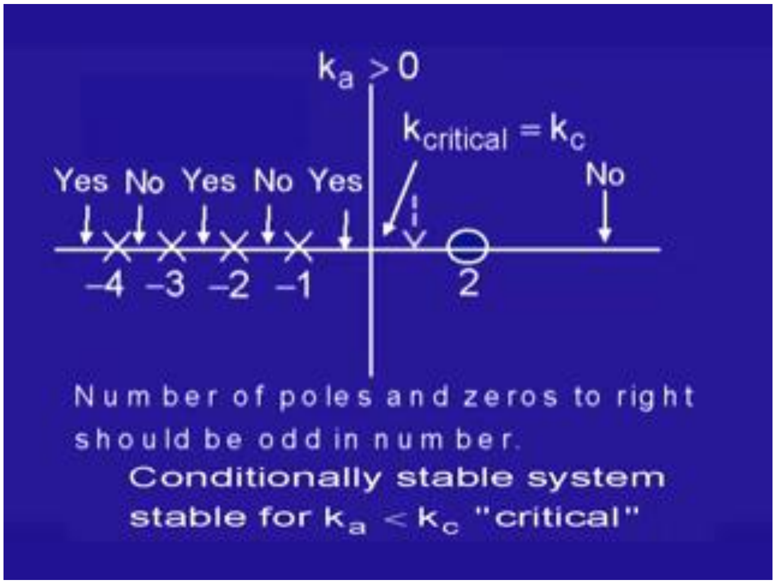

Now one is certainly an odd number and therefore, this point will belong to the root locus and by a similar argument then, all the points of the real axis lying between minus 1 which happens to be a pole and to which happens to be a 0. All the points lying on this portion of the real axis, will belong to the root locus, what does that mean once again it means that each point or the corresponding real number will be a root of the characteristic equation or a of the characteristic polynomial for some value of the gain K a, for some value of the gain K a. Now in this respect there is something which I did not state explicitly but it was understood that the gain K a that I am talking about is a positive number, I will write this therefore explicitly, the gain K a that I am talking about is a positive number.

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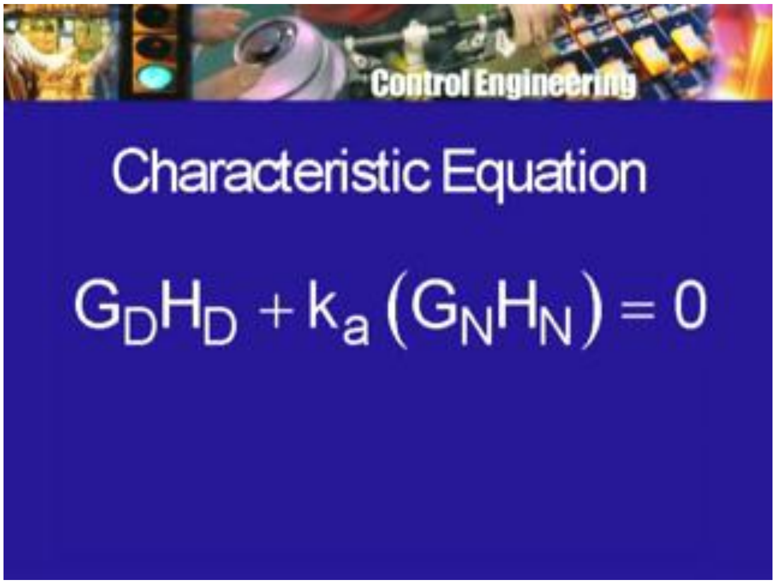

If you remember, the characteristic polynomial and I will write it again was G D, H D, the denominator parts of G and H those 2 polynomials plus K a, this gain into G N, H N, the numerator parts of G and H that polynomial equal to 0. This is the characteristic equation or the left hand side of it is the characteristic polynomial. It is understood now, we are looking at K a which are positive because normally when one talks about the gain of an amplifier, one thinks of the gain as being a positive number rather than a negative number. If I want to talk about a negative gain then I will put a minus sign in front of K a rather than say that K a is negative.

So, let it be understood that the gain coefficient K a in the characteristic polynomial is positive. If that is the case then this is the rule, a point on the real axis is on the root locus if the number of poles and 0s to its right lying on the real axis is an odd number and therefore all the points of the real axis lying between minus 1 and minus 2, all of them belong to the root locus for some value of K, not the same value for all the points. This point will belong to the root locus for some particular value of K, let us say K equal to10, another point will belong to the root locus for a different value of K but for some positive value of K and therefore, I will darken this portion to show that this is a portion of the root locus which is lying on the real axis and now, we know that the root locus must begin, one branch must begin at minus 1, another branch must also end at 2, it must begin at some pole say minus 1, a branch will end at a some 0, say 2.

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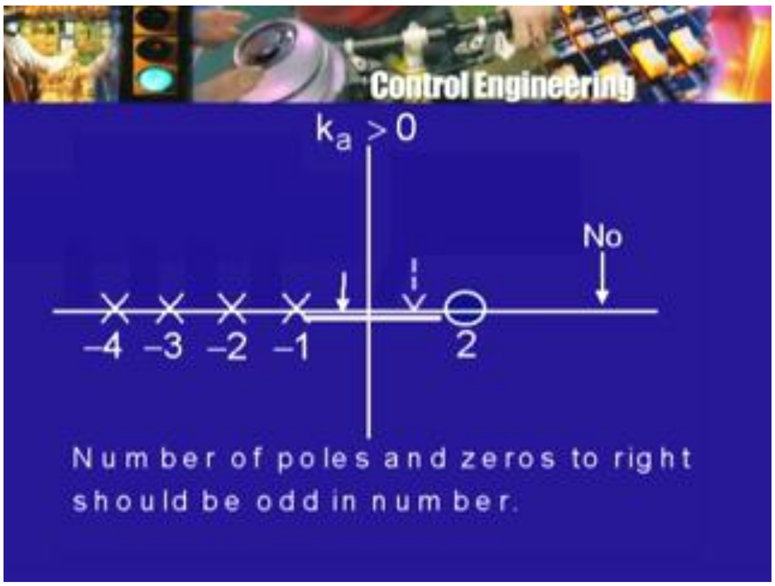

In this case, I am anticipating what we are going to find out later that the root locus will simply have a branch which goes from minus 1 pole to the 0 at 2, as K increases from 0 up to an indefinitely a large value and so, in this case I am saying ahead of what we are going to do that this part of the real axis is not only in the root locus but it is a branch of the root locus. Let us move now to the next region of the real axis, let us say we consider a point here is it on the root locus. Once again, I look at the portion of the real axis to its right and find out if there are any poles and 0s to its right there are how many of them total poles and

0s there is one pole, there is one 0. So total number of 2 of poles and 0s, 2 is not an odd number, 2 is an even number.

So this point will not belong to the root locus portion or this point will not be a part of the root locus for any positive value of K. Next, we move to this region lying between minus 3 and minus 2, consider a location here, repeating once again how many poles and 0s to its right,1, 2, 3, 3 is an odd number. Therefore, this part of the real axis will belong to the root locus that is each and every point will be a root for some value of K and once again, anticipating what we are going to do, when we look at some more rules, this portion lying between minus 3 and minus 2 will not only be a part of the root locus but it will be actually one of the branches of the root loci. Loci is the plural of locus because there is more than 1, so I am using the word loci or branches.

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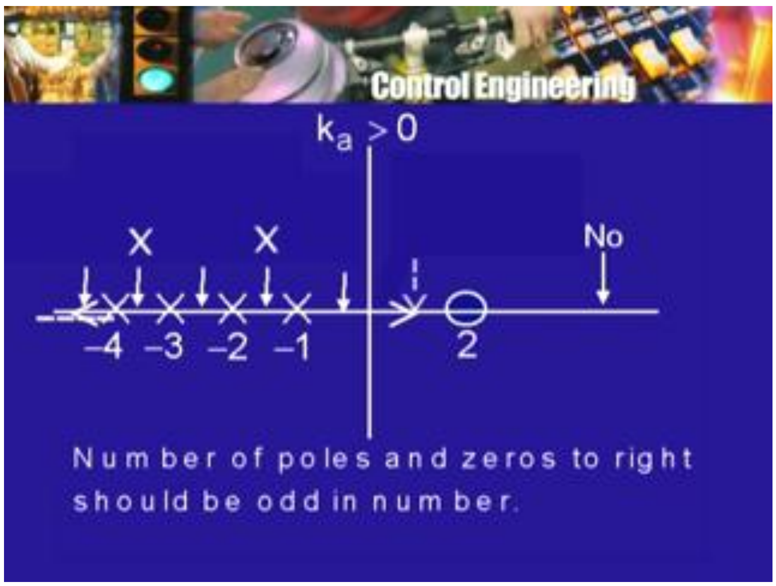

So that is here is the branch of the root locus which will start at minus 3 that is when K is equal to 0 or very close to 0, the root will be here, there will be a root very close to minus 3, with K very large there will be a root close to minus 2 and for values of K between 0 and a very large value, the roots will lie somewhere here. So this is another portion that belongs to the root locus. Next this region between minus 3 and minus 4, what about it is this point on the root locus, number of poles and 0s to its right,1, 2, 3, 4, 4 is an even number. So it is not an odd number, so this portion will not belong on each and every point between minus 4 and minus 3 will not belong to the root locus.

Lastly, then we look at a location here to the left of the left most pole or 0, namely minus 4 considering the points here, what is the number of poles and 0s lying to its right, 1, 2, 3, 4, 5, 5 is an odd number. So this will belong to the root locus, so also every point to the left of minus 4. Therefore, this entire region will belong to the root locus and once again anticipating, what we are going to do later on the third branch of the root locus will start at the pole minus 4 that is it will be near this pole for K very small and as K increases, the root will go on increasing and ultimately it will become very large in absolute value but a negative real number.

Now, therefore applying only the first 2 rules and this third rule which we are formulated today. We can more or less come to a complete conclusion about the root locus that is what will be the locations of the roots for various values of K because it seems that the root locus will lie entirely on the real axis but it will occupy only a part of the real axis and the conclusion is that, there will be one branch of the root locus which will move from minus 1 to 2, for K nearly 0, it will be there will be a root near minus 1 for K very large, there will be a root near 2, another branch of the root locus will move from minus 3 to minus 2, for K small, there with be a root near minus 3 for K very large, there will be a root near minus 2 a third branch of the root locus is this one it will start at minus 4 and it will go to infinity as K increases and so that is the asymptote. Here, we have 3 poles, so the 3 branch there will be 3 poles and 2, 0s 3 is greater than 2.

So there will be 3 branches of the root locus, all the 3 of them will start at the 3 poles, one at each pole. There are 2, 0s, so two of the branches will end at the 2, 0s and the third branch will go asymptotically away from the origin to an infinite distance and in this case a rule which I am going to state later on says that there will an asymptote in the direction of the negative real axis and so this is going to be third branch of the root locus. Now from this is there something that we can conclude, as far as let us say the practical utility of the scheme is concerned that is, if this is the pole 0 diagram for some system and K a is a gain which we have to choose then can something be said about the choice of K a. Now as you can see for K a small, the roots are all in the left half plane, there is a root very close to minus 1, another root close to minus 3, third root close to minus 4, all of them are in the left half plane.

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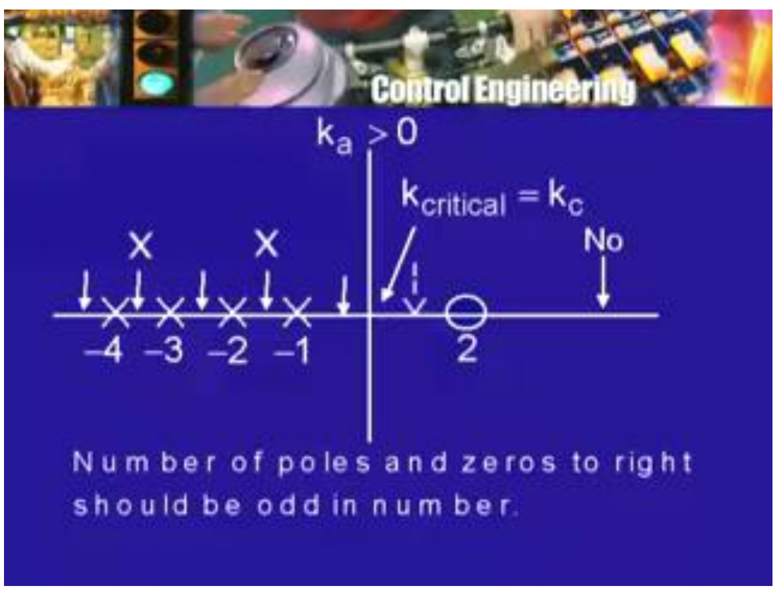

So, if you recall this means that the system response will have exponential terms which go to 0 and this is one kind of stability that one talks about that is the system response due to initial conditions will go to 0 as t tends to infinity or if the input is an input like a step input. The output will not increase in amplitude indefinitely but it will reach some steady constant value. For very large of K however, one of the roots is going to be near minus 2 that is in the left half plane, another root is way out here towards the negative real axis on the left side that is also well in the left half plane. But the third root is going to be to close to 2, 2 is not in the left half plane, 2 is actually in the right half plane and therefore, we will instability for a very large value of K and as we can expect then, if we think of increasing in K slowly from 0 then, there will be a value of K for which this branch of the root locus which lies between minus 1 and plus 2 will give you root which is exactly at the origin that is not in the negative left half plane, it is on the imaginary axis but its purely real. But for slightly larger value of K there will be a root in the right half plane and therefore, there will be instability.

So whatever system we are thinking of for which this is the characteristic equation or the G and H are such that this is the pole 0 diagram then, there will be a value of K, we can call it a critical value of K. So sometimes you know, call it K critical or K c such that when, K a is equal to this critical value K c then, there will be a root at the origin which is not quite instability but it corresponds to a steady state component which is not 0. However, for K greater than this critical value there will be no steady state, the response will simply have a increasing exponential part and therefore it will go to infinity therefore, such a system would not be acceptable. So whatever, system it be for which this is the pole 0 diagram, we can conclude that this system is what is called conditionally stable. The condition being that the gain K should be less than K c, the gain K should be less than this critical value of K c.

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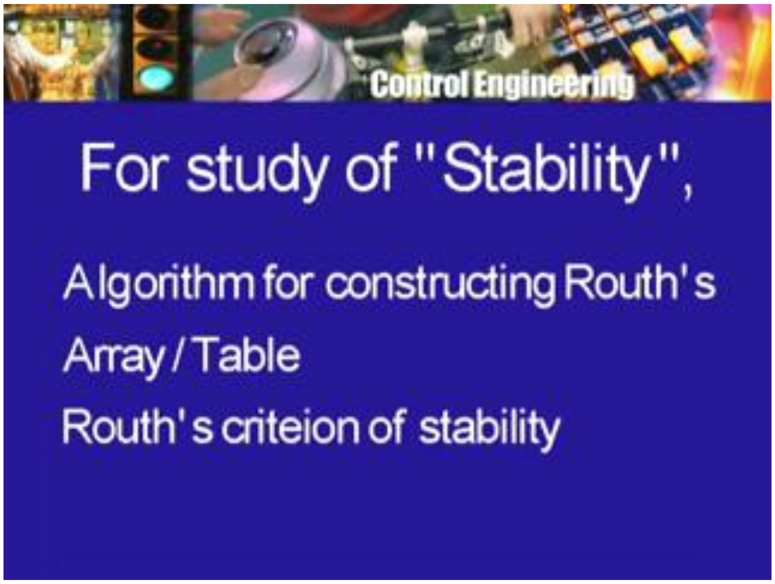

Now, it is possible to calculate this critical value of K c although it is not really a root locus rule as such however, Evans gave that also as one of the rules for the construction of the root locus and that determination of the value of this K c comes from another method or algorithm which is named after an English mathematician, who in the 1880s, almost 120 years ago gave a tabular method which enables you to determine this K c, the value of the critical value of the gain and this technique is known as the Routh array or Routh's array or it looks somewhat like a table, it is also called the Routh table and the algorithm or the procedure, where by you construct the Routh array is therefore known as Routh algorithm and if you say that it enables you to find out a condition under which there will be instability or not then, it gives rise to what is known as the criterion of stability and so, this is sometimes associated or called as the Routh criterion of stability or instability of a system.

So, just with the first2 rules about root loci starting at the poles, ending at 0s or starting from infinity or going towards infinity and this additional rule for real axis portions of the root locus, we conclude that for the system for which there are 3 poles as shown minus 1, minus 3, minus 4, there are 2, 0s, one at plus 2, another at minus 2. There will be a critical value of the gain K, namely K c such that for K greater than K c, the system will be unstable. For K less than K c, the system will be stable of course, we also see that if the value of the gain K is not K c but only a little smaller than K c then, the root of the characteristic equation will be very close to j omega axis or it will be a real root which is a very small and negative number because of that the corresponding time constant will be a very large number.

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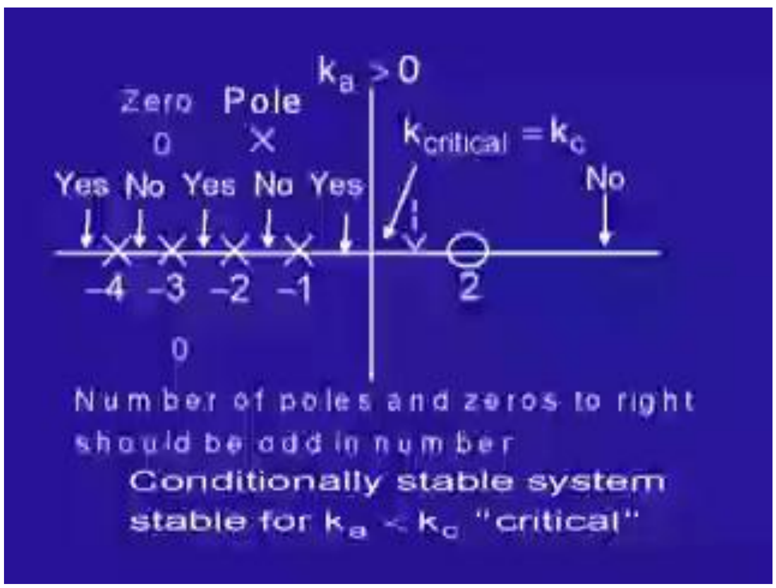

Of course, what is large depends on an application in some application one second may be large, in some other application 10 second may be large, in another application 10 second may not be that large. But, if it is very close to the real axis there is a possibility that the response is not going to be as short lived as you wanted to be because the time constant is not as small as you may want to be and therefore, one would like to avoid getting very close towards the imaginary axis and certainly not get in to the right half plane. Now in this third rule that I stated any poles and 0s of G or H which are not on the real axis have no effect. I could have for example, a pair of poles in the complex plane not on the real axis or I could have, had a pair of 0s in the complex plane and they could be in the left half plane or right half plane or they could be on the imaginary axis.

Let them be wherever they are poles and 0s which are not on the real axis, are not to be considered for the rule that I have stated. For determining the real axis portion of the root locus, for determining the real axis portion of the root locus, poles and 0s which are not on the real axis are not to be looked at, they do not matter at all, they are not to be counted literally. We only count poles and 0s, count them together to the right of a point on the real axis to find out whether that point is on the root locus or not. If the total count, number of poles and 0s to the right is odd then, that point belongs to the root locus. If it is 0 or even then that point does not belong to the root locus that is the third rule of the root locus that I have stated. Now, we do not have really enough time to go in to all the details. So you should look up your textbooks but some of you at this point may feel asking well, how do you get such a rule that is how do you come to such a conclusion that this is so. Now there was a particular way in which Evans got this rule and that was as follows.

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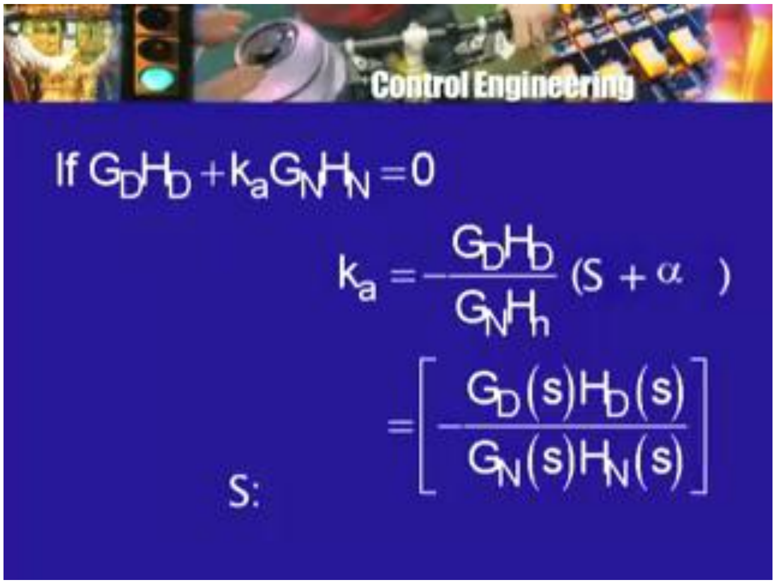

Let us once again look at our characteristic equation which is G D, H D plus K a times G N, H N equal to 0. Now we can rewrite this as Ka equal to minus G D, H D divided by G N, H N. Now this G D, H D and G N, H N are all functions of S they are all polynomial functions of S. So let me write that explicitly therefore K a is equal to minus G D (S), H D (S) divided by G N (S), H N (S) and by the as convention or the assumption that I have made G D, H D, G N, H N each of them look like S plus some, they consist of a product of factors each one of them looks like, say S plus alpha okay, that is what it is going to look like.

Now, on the left hands side we have K a, which is the gain, which is a positive real number. On the right hand side, we have a function of S. So for what complex numbers S will the point which corresponds to this complex number S belong to the root locus. It will belong to the root locus if this entire thing on the right hand side is such that it is equal to a positive real number, S may be complex, we are not saying that we are only going to look at real axis portion of the root locus of course that is the rule that I mention but in general, a point in the complex plane which corresponds to a complex number S will belong to the root locus, if this bracketed ratio of 2 polynomials is equal to a positive real number or if I get rid of that minus sign then, G D (S), H D (S) divided by G N (S), H N (S) this whole thing equal to minus K a and minus K a is a negative real number, I am excluding K a equal to 0, so this is negative. So what are the s (s) complex numbers S for which this whole thing is negative, now negative not only negative but purely real and negative.

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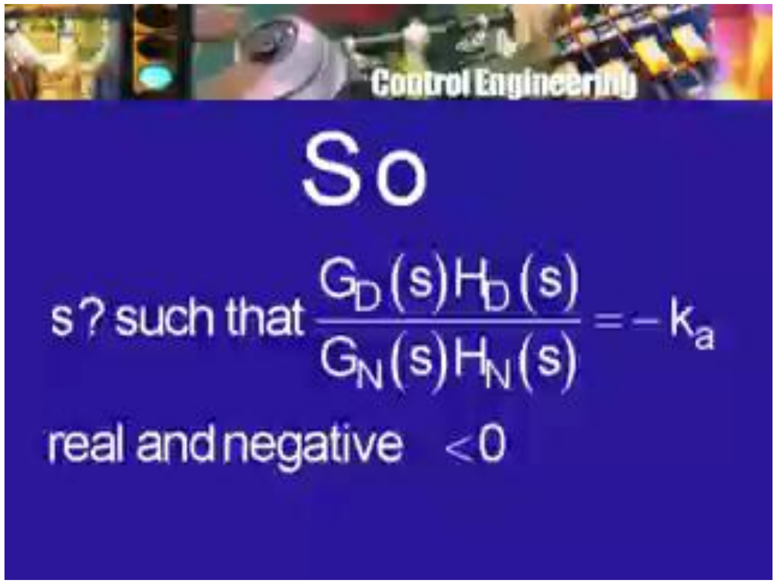

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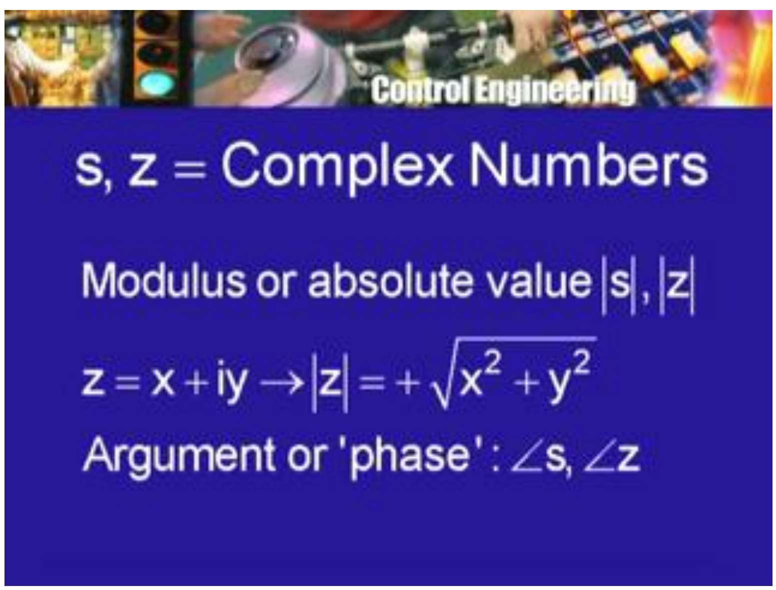

So from this one gets 2 conclusions or 2 conditions, one of which concern modulus or amplitude and the other concerns angle or argument. Remember, that when we have a complex number, a complex number say such as S or Z, we talk about modulus of S or modulus of Z, what is that? That is the positive square root of the real part squared added to the imaginary part squared, if z is equal to x plus I y then mod z is the positive square root of x squared plus y squared. So this is called the modulus of the complex number.

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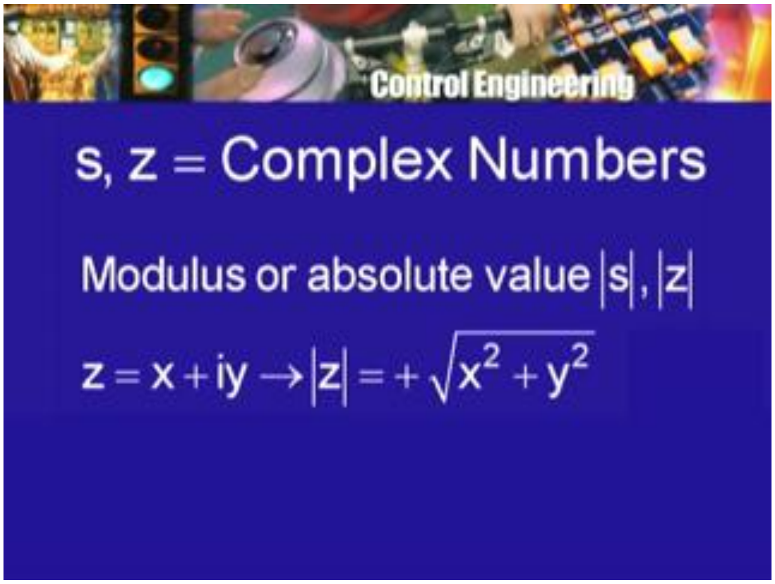

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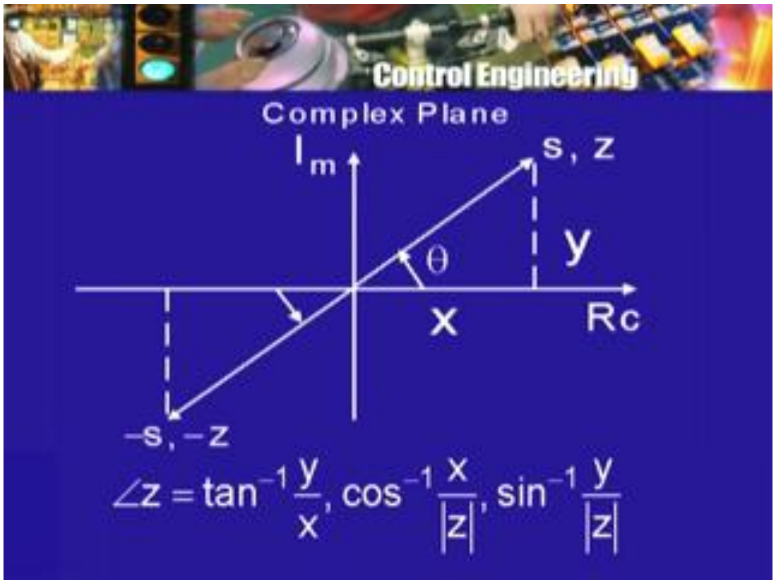

Similarly, we talk about the angle or the argument of the complex number, what is the angle or argument of the complex number? Well, the most convenient way of thinking about it is to plot the complex number in the complex plane, if the complex number is here let us say then, what is the argument the argument is obtained by drawing the radius vector from the origin to the point and then, measuring the counter clockwise angle starting with the real axis, positive real axis. So this is the angle through which I will have to turn the radius vector to become equal to the radius vector pointing towards the complex number S. Then, this angle is called the argument of the complex number. One can write an expression for it, except that there is an ambiguity which always arises when you talk about the inverse trigonometric functions.

So for example, I can write angle of z equal to tan inverse y divided by x or I can also write it as cos inverse x divided by absolute value or modulus of z, I can also write it as sin inverse, imaginary part y divided by absolute value of z. The only thing is when we think of the inverse trigonometric functions, there is an ambiguity, tan inverse of a number does not specify the angle uniquely not even within multiples of 180 degrees. I suppose, you all remember your trigonometry well enough. For example, here is the angle, say theta such that the point S is in the first quadrant and here is another point which is in the opposite direction, which is actually minus S that is in the third quadrant.

Now, the angel or the argument so the 2 complex numbers are not the same. In this case, the argument is theta in the other case I have to go counter clockwise, the angle is actually 180 degrees or phi radians plus theta radians. So these 2 complex numbers have different argument but tan inverse y by x is the same for both of them, for this number S y by x, y is positive x is positive, for the other number y is negative, x is negative. So, y by x is again positive and in fact equal in value. So tan inverse has an ambiguity, tan theta is the same tangent of phi plus theta equal to tan of 2 phi plus theta and so on.

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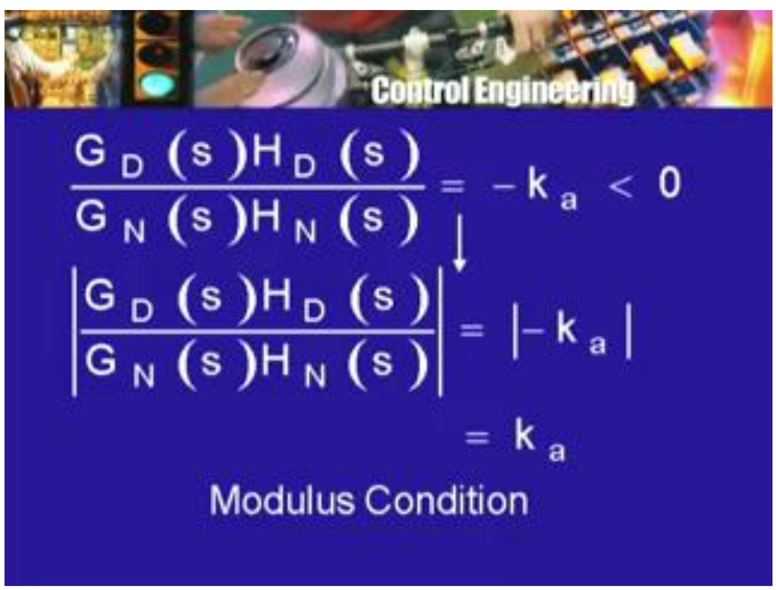

Similarly, there are ambiguities concerning cos inverse and sin inverse. So we define it the argument as the counter clockwise angle measured from the positive real axis towards the radius vector. So that being done then from the condition that we have the modulus, we have the complex G D (S), H D (S) divided by G N (S), H N (S) equal to minus K a which is real and less than 0, from this we get the following 2 conclusions. If I take the modulus of this big complex number then, this modulus is equal to what this modulus is simply equal to modulus of minus K a but modulus of minus K a is simply K a. Remember, K a was positive, so this is called the modulus condition and actually it is used or it can be used to calculate K a for a given S, if that S is on the root location, if the point S is on the root locus then we can use this to calculate the value of the gain K a to which it corresponds and there is a geometric interpretation of this which we will come to soon.

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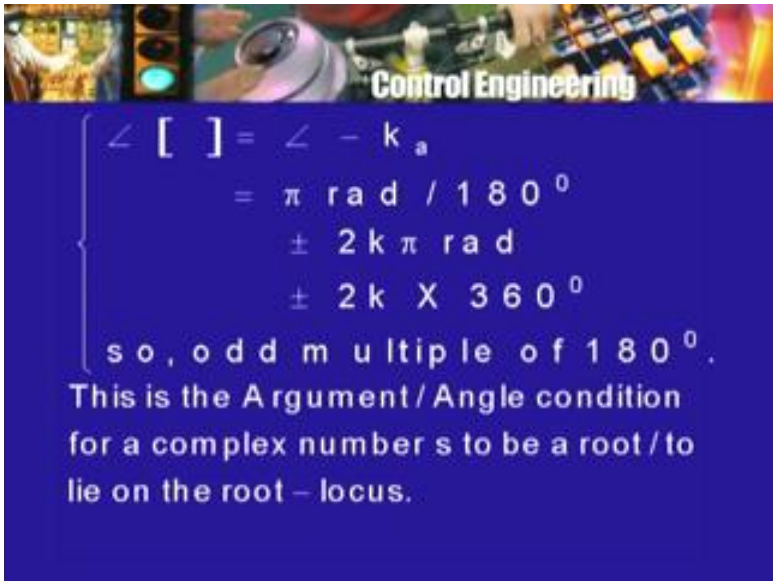

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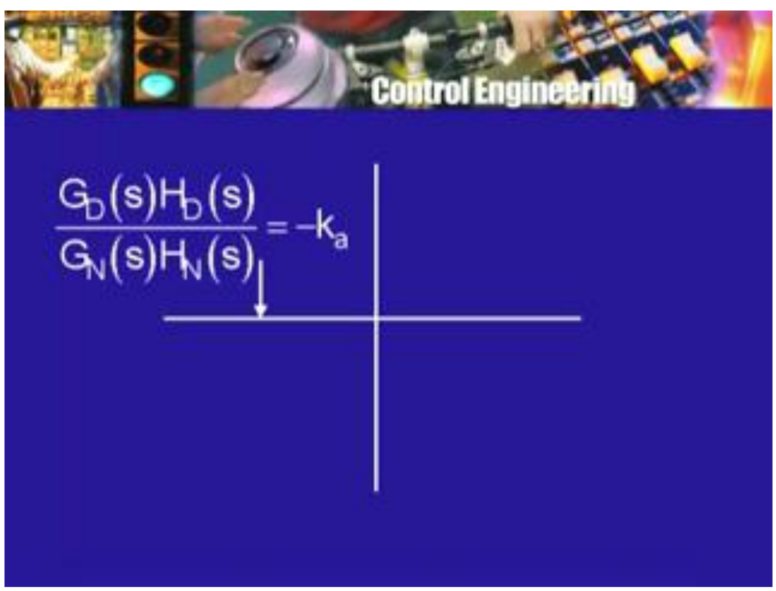

On the other hand, I can take the argument of this fraction, what about the argument of that fraction that is equal to the argument of the number minus K a but the number minus K a is on the negative real axis and so, what is its argument, its argument is phi radians or if you think in terms of degrees 180 degrees and of course, as you know all trigonometric ratios, inverse ratios are ambiguous within multiples of 2 phi that is theta 2 phi plus theta 4 phi plus theta minus 2 phi plus theta, all of these angles have the same given trigonometric ratio, tan theta equal to tan 2 phi plus theta equal to tan minus 2 phi plus theta and so on.

So one can always add, multiplies of 2 phi to this or subtract, multiplies of 2 phi. So I can write it as phi plus minus 2 K phi radians or what turns out to be it is an odd multiple of 180 degrees. This is because as we see, when we calculate the argument of this ratio, the sum may come out to be greater than 2 phi in absolute value or greater than 360 degrees in absolute value and therefore, we will have to think of it as an angle which lies between 0 and 360 degrees or 0 and 2 phi radians or alternately an angle that lies between minus 180 degrees and 180 degrees and that is minus phi radians and plus phi radians.

So this kind of a interpretation, we always have to do for angles associated with complex numbers. The second condition is known as the argument principle or argument condition. Now all this would not be useful if there was no simple geometrical interpretation of what is going on fortunately, there is a very simple geometrical interpretation. Now, as I told you we do not have time to go in to great details about what is happening. So we will take our particular example and I will illustrate this for that particular example and you should look up your textbook and go in to all the details.

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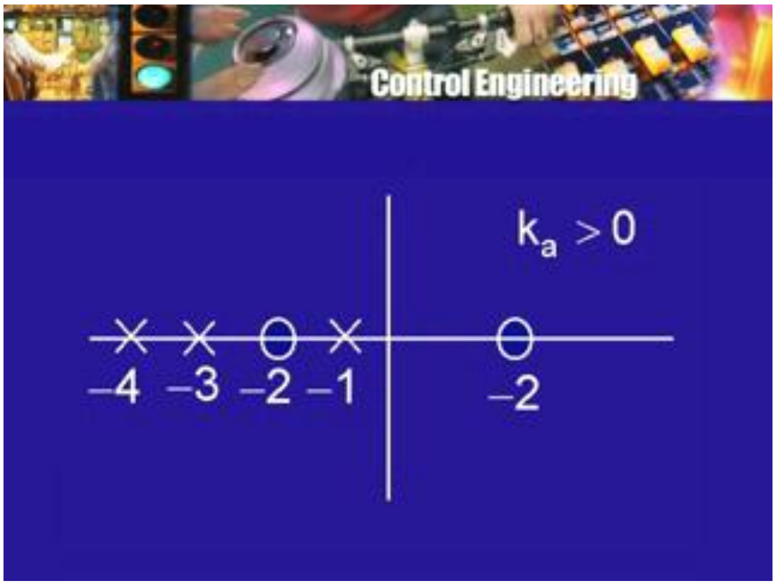

So here is pole 0 diagram once again, here is the 0 at minus 2 and here are the various poles and 0s, 1 pole at minus1, 0 at minus 2, a pole at minus 3 another pole at minus 4. So there are 1, 2, 3 poles, there are 2, 0s and there is of course the gain K a which is positive and what was the fraction G D, H D divided by G N, H N, then G D corresponds to the denominator or poles of G and H therefore I will have terms like S plus 4 into S plus 3 in to S plus 1 divided by G and H N, G and H N is given by there are 20s S plus 2 into S minus.

So this is the fraction that we are looking at G D (S), H D (S) divided by G N (S), H N (S) all right. Now, we are going to look at the argument of this that is for a given complex number S, this is a complex number what is going to be its argument. Fortunately, for us this S plus 4 can be given a geometric interpretation namely if I join this point S to the pole or the 0, in this case it is a pole to which this factor S plus 4 corresponds, S plus 4 corresponds to the root minus 4, which is a pole.

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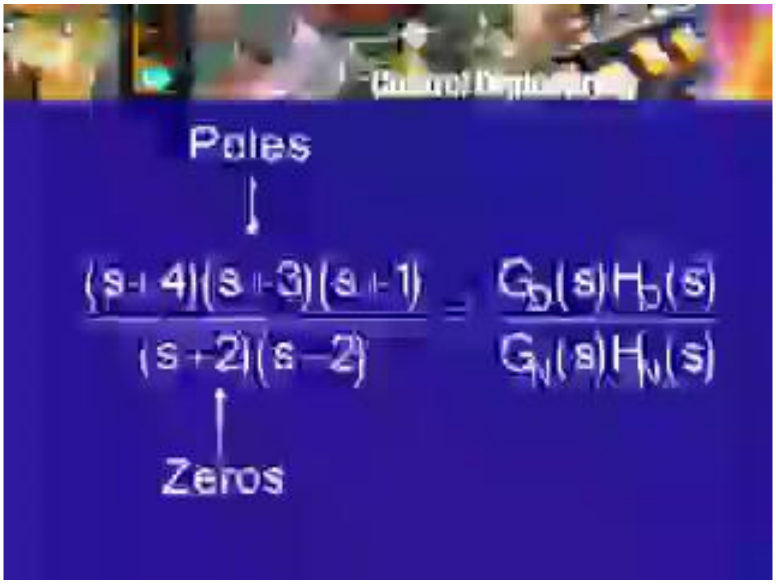

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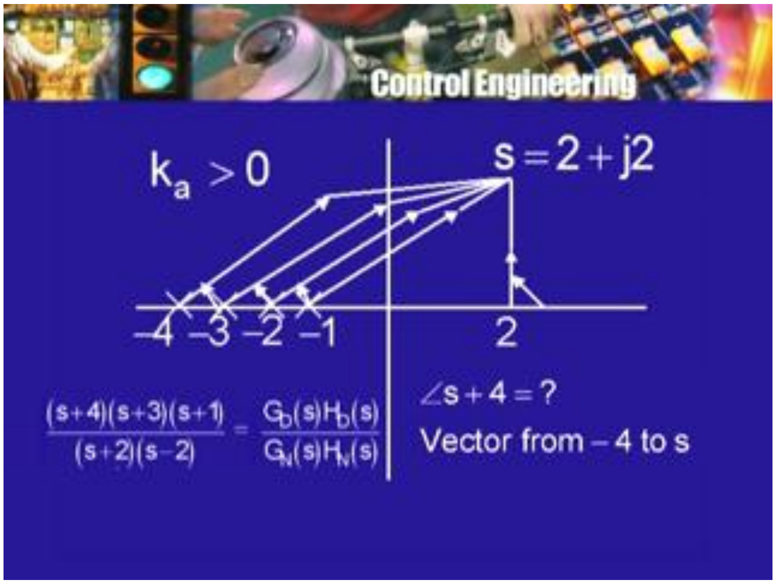

So, suppose I join this pole to this point S think of that as a vector in the plane pointing towards this point S, then this vector has a length, this vector makes an angle with the positive real axis if I consider the angle described in the counter clockwise sense. Now, you can easily see that for this complex number S, the angle of S plus 4 or the argument of S plus 4 is precisely the angle made by this vector with the positive real axis. So geometrically then I just think of the vector join the pole to this point S, think of that vector look at the angle that it makes and as I said root locus method is mainly for human beings for small problem I immediately see this angle of course, it is what it is I do not know exactly this is 30 degrees, 50 degrees, 40 degrees but it is an angle in the first quadrant.

So it is something like may be 20 degrees or 30 degrees. This is something which we can see immediately right that is what the factor S plus 4 or for the argument of S plus 4. For S here, the argument of S plus 4 looks like an angle which perhaps is about 30 degrees, it certainly an angle lying between 0 and 90 degrees. Look at the next factor S plus 3 once again, S is 2 plus j 2 I can calculate S plus j 3, then take the modulus and argument of that but instead I can draw the vector from minus 3 to the same point S. Here is a different vector, it makes a different angle with the positive real axis and that angle or argument will be precisely the argument or the factor S plus 3, what about the third one, S plus 1 I have one more vector, it makes yet another angle that angle will be the argument of the third factor S plus 1.

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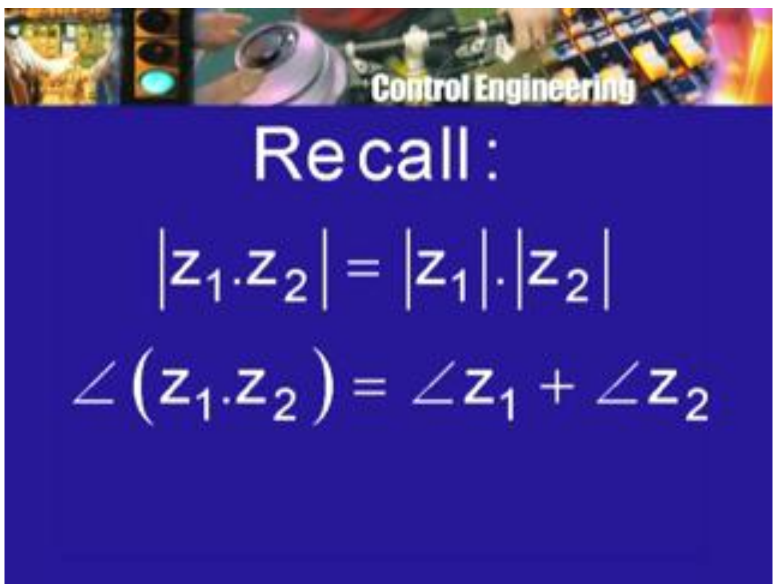

Now we make use of a simple fact from complex number arithmetic, if you have a product of 2 complex numbers, then what is the modulus of the product, what is the absolute value of the product. It is the product of the moduli, plural of modulus is moduli. If I have 2 complex numbers Z 1, Z 2 then modulus of Z 1, Z 2, the product is the product of the moduli mod Z 1, Z 2 is mod Z 1 into mod Z 2 modulus of the product is the product of the moduli that is something which can be proved and again, you all ready ought to be knowing it, what about the argument of the product of 2 complex numbers.

Well, the angle or the argument of the product of 2 complex numbers Z 1, Z 2 is the sum of the argument, angle Z 1 plus angle Z 2, the angle or argument of Z 1 into Z 2 the product of

Z 1, Z 2 is the sum of their argument. The modulus of the product is the product of the moduli, the argument of the product is the sum of the argument, remember this. It is not correct that the modulus of Z 1 plus Z 2 is mod Z 1 plus mod Z 2, far from correct. It is also not correct that the argument of Z 1 plus Z 2 is the sum of the argument, it is not for the sum it is for the product somehow that this result holds, product of modulus and modulus of product, argument of product and sum of arguments of the 2 factors. This is the property of complex number this being the case then, if I look at now the 3 factors s plus s pus 3 s into s plus 1 then what will be the argument of this product. The argument will of the product will be the sum of the arguments of each one of them but what are each one of them, each one of them are these angles. So the argument of this numerator thing will be the sum of these 3 angles.

Now if I draw this diagram to scale I actually put the point s put the poles and 0s properly according to the scale, then by using a protractor I can actually measure these angles. You should try it out, do it for this problem with s equal to 2 plus j 2, the poles and 0s as shown actually on a graph paper, draw the pole 0 diagram, put the location of the point s, draw these vectors, with the help of the protractor measure the 3 angles. However, we see that these 3 angles are all going to lie between 0 and 90 degrees. In fact, they are perhaps around 20 to 30 or 40 degrees. So there is something that one can say about their sum also each one of them is a positive angle. So their sum is going to be a positive number, each one of them is less than 90 degrees certainly it looks like that.

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$$|z_1 . z_2| = |z_1| . |z_2|$$

$$\angle(z_1 . z_2) = \angle z_1 + \angle z_2$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\angle\left(\frac{z_1}{z_2}\right) = \angle z_1 - \angle z_2$$

So their sum will be less than 270 degrees. This is a conclusion which we can draw right away by simply looking at the figure. So the argument of the numerator when s is this complex number will be a number that lies in terms of degrees between 0 and 270 degrees, it is probably somewhere around 90 degrees or 100 degrees or 80 degrees or whatever it will turn out to be by making actual measurement. Now what about the factors in the denominator in the denominator we have s plus 2. So I have to again draw a vector from minus 2 to s and I will have its angle or argument, for the factor s minus 2 I have to draw the vector from the location is 2 to s.

So this vector corresponds to the complex number s minus 2, the argument of this vector is the argument of the factor s minus 2, the modulus of the vector or the length of the vector is the modulus of s minus 2. Now I have a product in the numerator, I have a product in the denominator but denominator is one thing and numerator is the other thing. So I have a product divided by a product. Now that rule about modulus and rule about argument extends to not just product but also to a ratio of 2 complex numbers, mod of Z 1 divided by Z 2, if Z 1 and Z 2 are 2 complex numbers such that Z 2 is not 0 of course, then the mod of Z 1 by Z 2 is equal to the ratio of the moduli, mod Z 1 divided by mod Z 2 and the angle or argument of Z 1 divided by Z 2, the complex number Z 1 divided by complex number Z 2 is argument of Z 1 minus the argument of Z 2.

So it is the difference of the 2 arguments, argument of the numerator complex number minus the argument of the denominator complex number. So if you look at our figure once again, the3 numerator factors give rise to 3 vectors and 3 angles, the 2 denominator factors give rise to 2 vectors and 2 angles. So what is the total argument of this whole ratio, it is the sum of the angles corresponding to the numerator which is what angles corresponding to the vectors going from the poles to this point under consideration minus because we have in the denominator now. The sum of angles the vectors which go from the 0s to S and that is the total argument of this ratio and this the argument principle says is phi radians or 180 degrees plus minus multiples of 360 degrees and therefore, the argument principle or the argument condition is stated as follows.

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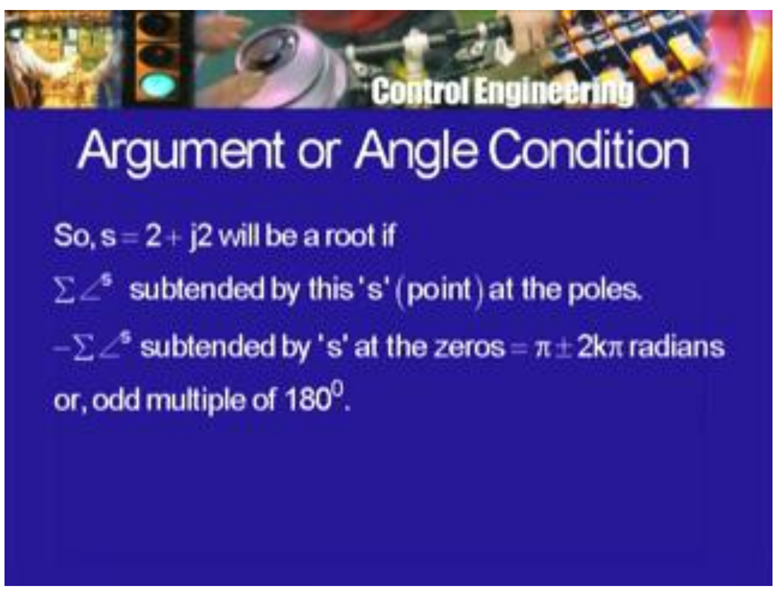

Instead of saying the angle made by the vector going from a pole or a 0 to a particular point in the complex plane S, the angle measured in the counter clockwise direction from the positive real axis. Instead of saying all that one simply calls that angle, the angle subtended by the point S at a pole or the angle subtended by the point S at a 0 and therefore the argument principle simply can be stated as, the sum of all angles subtended by S, the complex number under consideration at the poles minus the sum of all the angles subtended by the same complex number and all the 0s. This difference of 2 sums should be equal to phi radians plus minus 2 k phi plus minus any multiple of 2 phi radians.

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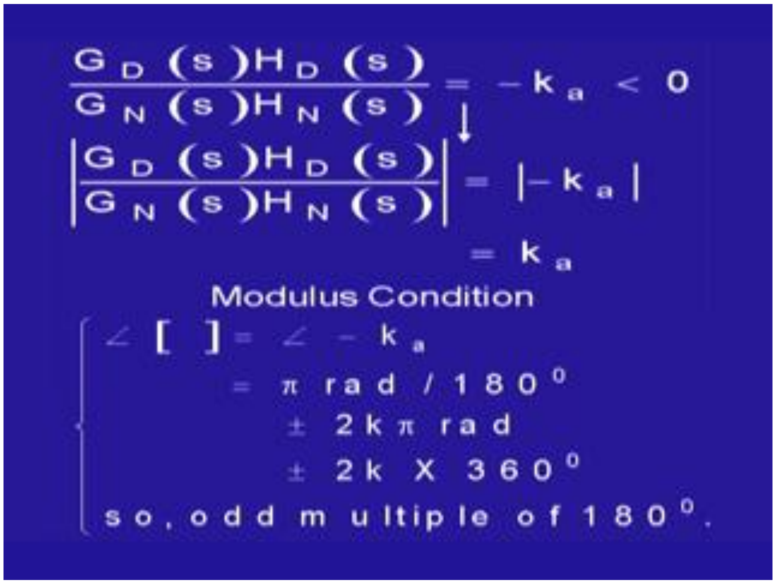

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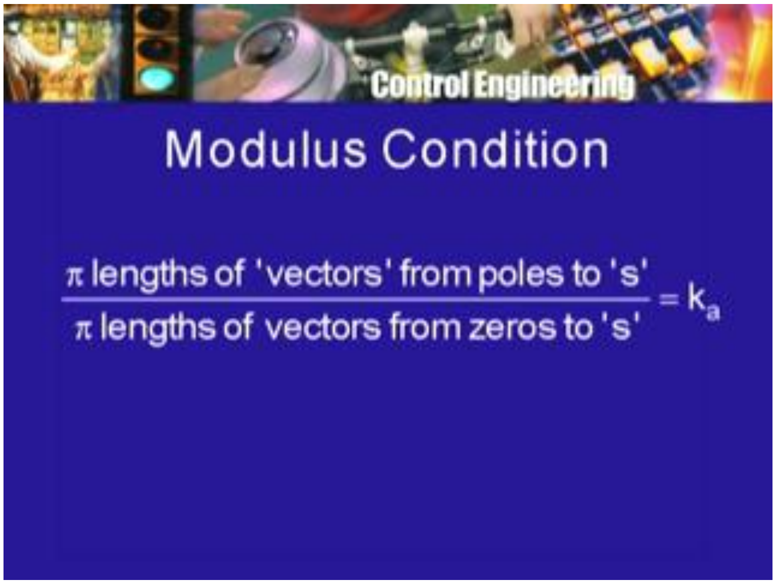

If a complex number S satisfies this condition then the complex number S will be on the root locus, for some value of k and what value of k that value of k is obtained from the modulus condition, as we saw earlier and the modulus condition was simply that the modulus of this complex number is equal to K a and as we can see from the property of the complex numbers that we have discussed that this modulus will be equal to, what it will equal to instead of some product of lengths of vectors from poles, lengths of vectors from the poles or joining S to the poles divided by the product of lengths of vectors to joining them to the 0s. This will equal to K a, the value of the gain K. This was Evans's idea of how to locate or how to find out whether the given complex number S belongs to the root locus or not and as we can see, it is a very nice graphical method and if you draw things to scale, one can easily determine these angles and measure these lengths or one can of course, calculate with the help of the calculator. All these angles and lengths and therefore for a given complex number S, we can find out whether the angle condition is satisfied, if it is then for that complex number we can use this formula to find out the value of the gain K.

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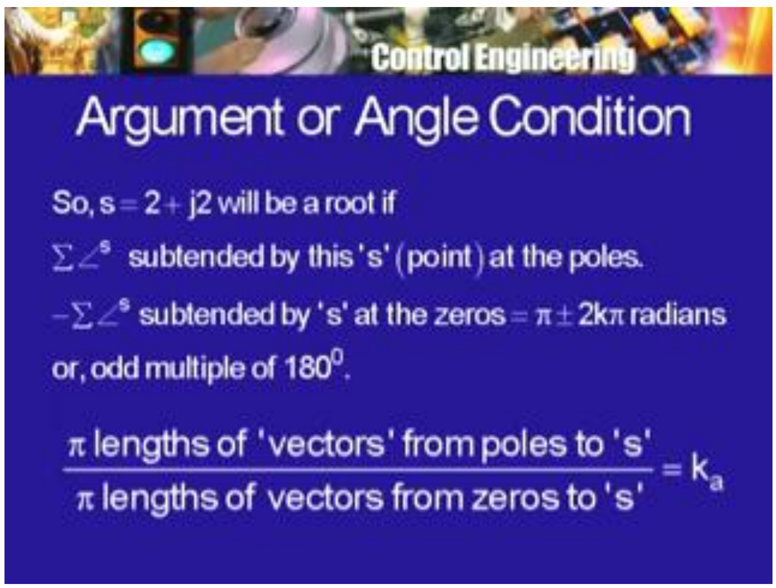

So the argument principle gives you a criterion or a test for finding out whether a complex number S that is a point in the complex plane is on the root locus or not. Once you have identified that it is or it is not then you proceed further if it is then, this rule product of lengths of vectors from the poles divided by product of lengths of vectors from the 0s gives you the gain K, value of the gain K a for which there will be a root at S. This is the basis really of the root locus method of course, as one goes on if one uses other principles like I mentioned Routh criterion or Routh algorithm is also used, so also some more ideas. In fact after Evans gave the method, a number of people gave some addition rules or made contributed to its development but this is the fundamental principle behind the root locus approach.

Now as homework, I would like you to once again repeat this with the point 2 plus j 2 that is S equal to 2 plus j 2, by actually plotting, find out if the angle condition is satisfied, if it is then find out the corresponding gain K. You can do that for one more point in the complex plane or choose any point on the real axis or on the imaginary axis, go through all these plotting and addition of the angles, measurement and addition of the angles. Alternately, you can do it purely with the help of the calculator by simply calculating the modulus and the argument of the real numbers that of the complex numbers that you get. Check whether 2 plus j 2 is on the root locus or not and if it is, for what value of K a but we can apply this immediately to the real axis portion rule and I will show you very briefly, what the geometrical considerations lead us to.

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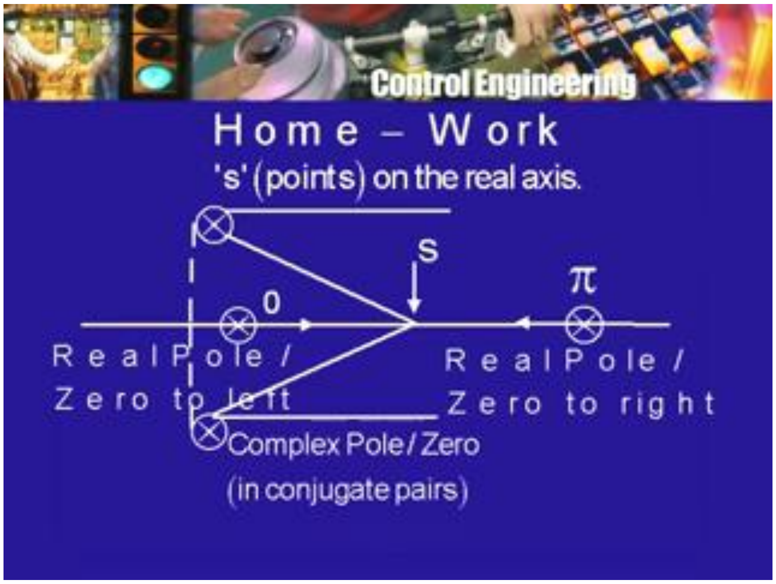

So, here is a point on the real axis, so this is the value of S, now suppose I have a pole on the real axis. Now what are the kind of locations that are possible we will exclude this point. So there it maybe a pole which lies here, if the pole lies here, what is the angle subtended by this number S at this pole, the vector simply points from the pole to the point S, it is in the direction of the positive real axis. So the angle is 0, so if I have a point on the real axis then a pole to its left contributes an angle of 0, instead of a pole it could be a 0, what is the angle subtended by S at this 0, it is also 0.

So, whether there is a pole or a 0 to the left of this point, its angle or argument contribution is 0, whereas if I have a pole to the right of this point, the vector from the pole to this point is going this way. So what is the angle subtended by this point S or the angle made by this vector with the positive real axis, it is phi radians. So a pole to the right will contribute phi radians to the argument a 0 to the right will also equally well contribute phi radians to the argument. Therefore, my argument principle now says sum of all angle subtended at poles minus sum of all angle subtended at 0s. If I look at poles and 0s on the real axis those which are to the left do not matter, those which are to the right only contribute, the poles contribute plus phi because they are in the numerator of that fraction, the 0s because I am subtracting contribute minus phi.

Now think about therefore, what effect this has. On the other hand, if I have a complex pole like this then what can I say, we have assumed that all the coefficients of the polynomial are real numbers. So if there is a pole like this, there will be a conjugate pole like this, the poles and 0s will occur in conjugate pairs. Now if I look at this point on the real axis, here is the vector from this pole to the this point, what is its argument, its argument because the angle is this way its minus may be 40 degrees or 50 degrees or plus 310 degrees, if I look at it this way, what is the angle made by subtended at this pole, it is this angle going this way. You can see that the sum of the 2 angles is going to be 360 degrees but 360 makes no difference to that sum because angle 30 degrees is equal to angle 390 degrees equal to angle minus 330 degrees adding or subtracting, multiples of 360 degrees makes no difference to the argument.

Therefore this pair of complex poles will make no contribution to that argument principle method or calculation because the total contribution of these 2 will be 360 degrees. If these are 0s then once again, the total contribution will be 360 degrees but it will be subtracted and that makes no difference, whether you add or subtract it makes no difference, if the angle is 360 degrees think about this and make sure that you can now derive the rule that I mentioned, for real axis portion of the root locus, why complex plane that is the poles and 0s which do not lie on the real axis why they do not matter why, only poles and 0s which lie to the right of a point matter and why, I can take their sum that is the total number how that rule comes about, think about it and then we will proceed further.