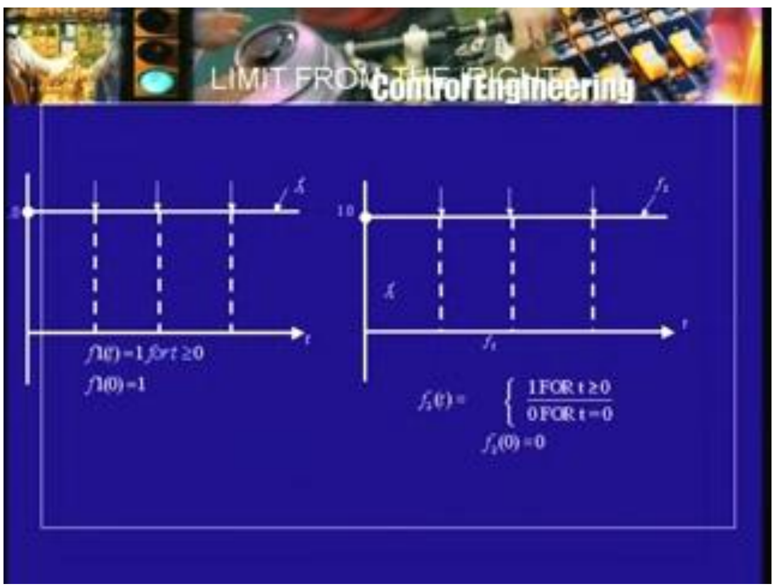
Control Engineering
Prof. S. D. Agashe
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture – 23

Let us spend a few minutes on this u 0 plus business. Consider the 2 examples that I am going to show you; here is a function whose plot is as shown for t greater than equal to 0. As you can see this function let us call it f 1 has the value 1 f 1 t is equal to 1 for t greater than equal to 0 that is for all time instants t, which are greater than or equal to 0 that is time instants, including the initial time instant and following it. This is the function f 1 on the other look at the function f 2 which is as you can see here. This is the function f 2, f 2 is given by f 2 of t equal to 1 but for t greater than 0 and 0 for t equal to 0 or in other words f 2 0 is equal to 0 whereas, f 10 is equal to 1.

(Refer Slide Time: 01:32)



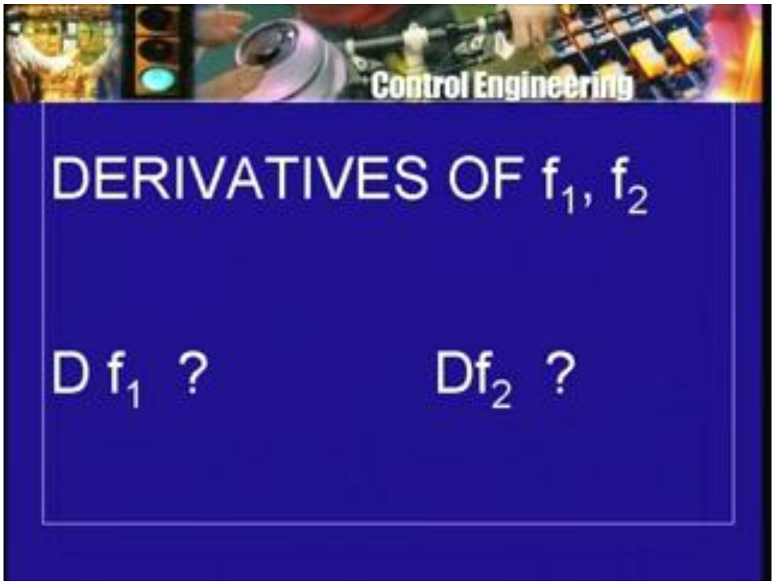
Now a dot here is supposed to show that the value of the function at t equal to 0 is 1 whereas, the dot here indicates that the value of the function at t is equal to 0 is 0. Now are these 2 functions the same or are they different. Of course, the answer should be obvious since I have used 2 different symbols f 1 and f 2, they must be different and they are different because although their values are equal for t greater than 0, f 1 t is 1 for t greater than 0 because f 1 t is 1 even for t equal to 0, f 2 t is 1 for t greater than 0. So, f 1 t equals f 2 t for t greater than 0 but for 0, f 10 is 1, whereas f 20 is 0. So the 2 functions are different and in mathematics 1 does distinguish between these 2 functions.

Now they are different of course, in a very trivial way. They differ in their values only at 1 point or at per 1 number, namely t is equal to 0 at the point or for that number 1 function takes the value 1 and the other function takes the value 0. But they have slightly different properties. For example, recall the definition of continuity of a function from your calculus core, what about the function f 1 and it is continuity. You can not only see from the figure but you can also prove it rigorously using the definition of continuity that the function f 1 is continuous for every value of t, whether 0 or greater than 0. The function is continuous here, the function is continuous here that means at this value of t, at this value of t the function is continuous here and the function is continuous at t equal to 0 because the function is not defined on the left hand side, I told you that in control usually we do not know anything about, what has happened before t equal to 0.

So, for f 1 the limit from the right equal to 1 and the value of the function at 0 is also 1. So the limit from the right at t equal to 0 or at the point t equal to 0 and the value at t equal to 0 are equal. So the function f 1 is continuous not only for every t greater than 0 but also for 0, whereas the function f 2 is continuous at every value of t greater than 0 but it is discontinuous at t equal to 0 because the limit from the right at t equal to 0 is 1 whereas, the value of the function at t equal to 0, I have deliberately defined it to be 0.

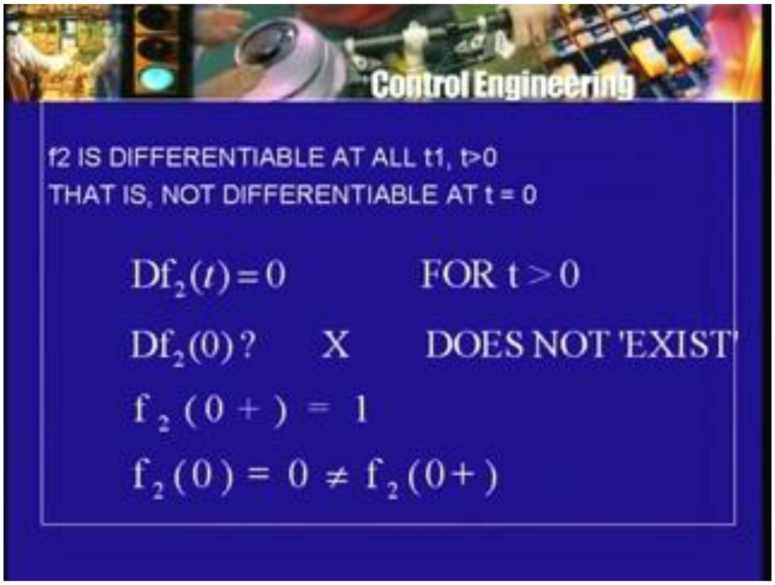
Now unfortunately, it is the function f 2 which is known as the unit step function because it looks as if there is certain jump at t equal to 0 and so, it is common fact is to call the function f 2 as the unit step function, whereas the function f 1 normally in mathematics course, will be called the constant function f 1, the function f 1 is constant for its domain namely, t greater than equal to 0. However, the function f 1 has a f 1 is continuous at all points where as the function f 2 has point of discontinuity.

(Refer Slide Time: 06:37)



So this is 1 difference between the 2 functions but from the point of view of our derivative property, there is another difference, the function f 1 is not only continuous at every t but it is also differentiable at every t that is we can talk about the derivative of f 1 at every number t, whether it is greater than 0 or equal to 0. Of course, in the case of 0, we have to talk about derivative of the function from the right but the function f 1 does have a derivative from the right and so, I can legitimately talk about D of f 1 for all times t except at t equal to 0. I have to think of the derivative from the right only because the function is not defined before t equal to 0, where as if you look at the function f 2, it is not differentiable at t equal to 0.

(Refer Slide Time: 06:40)

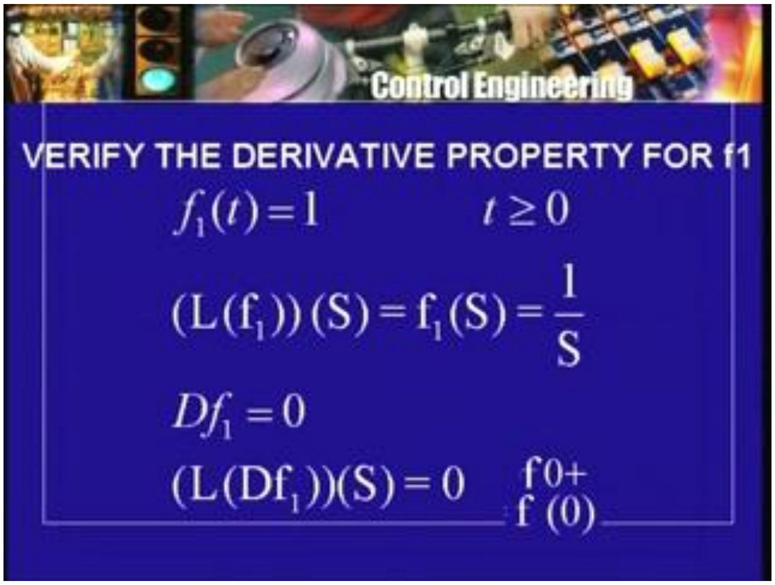


So, I can talk about D f 2 for every t greater than 0 but D f 2 0 simply, does not exist there is no way, I can talk about the derivative of the function f 2 at time t equal to 0 whereas, for f 1 I can talk about its derivative from the right at 0 and therefore, the derivative from the right could also be written as D f 10 plus. Now this is what we mean by that 0 plus by the 0 plus we mean the limit of the value of a function from the right.

Similarly, 0 minus indicates limit of the function at t equal to 0 from the left and the derivative property of the Laplace transform or Laplace transformation applies, as you can realize for a function which is differentiable even at the point t equal to 0 and then, as we wrote last time the Laplace transform of the derivative of f its value at s wherever, that integral is defined equals S times the Laplace transform of the original function f, it is value at S and this product from that you subtract f 0 plus. This is how you would probably find it in your mathematics book. For us of course in place of f, if you are thinking of an input we will have the letter u because that is our preferred symbol for any input.

So that is the derivative property of the Laplace transformation and it applies only under certain conditions. Obviously, the function f has to be differentiable the function f has to have a limit from the right at the t equal to 0, it has to be differentiable at t equal to 0 therefore, the function must be continuous at t equal to 0 also. So in fact, I could even replace this f 0 plus by f 0. Now, let us apply that or verify that derivative property for a simple function and we can verify it for the function f 1 that I showed you earlier namely the function f 1 is the constant function 1.

(Refer Slide Time: 07:52)

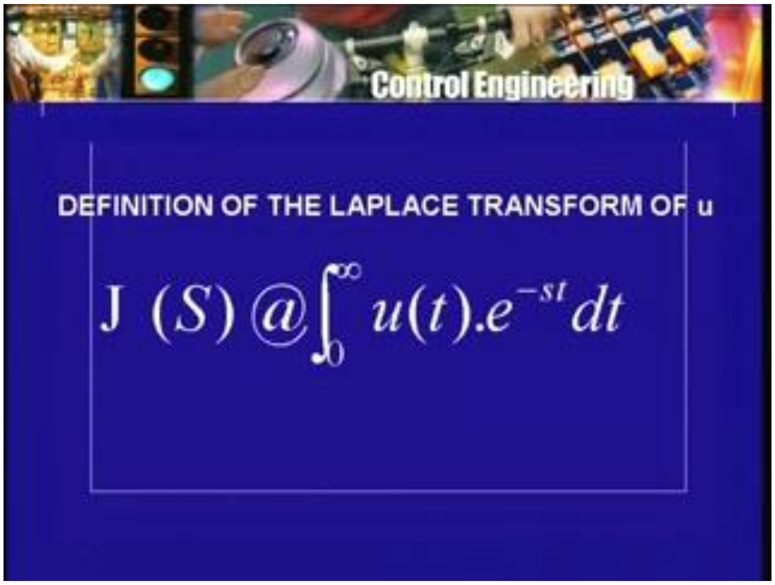


Now, what is the Laplace transform of the constant function 1 from the table the Laplace transform of the constant function 1 is given by 1 by S is the function whose value at S is1 by S. So, if I write it down as capital F 1 then, capital F 1 of s is 1 by s, small f 1 denotes the constant function 1. Now what is the derivative of the constant function f 1, the function is constant. So the derivative is 0 or the derivative is the 0 function and it is easy to verify that the Laplace transform of the 0 function, is the 0 function because in the integral I have 0 multiplying e raised to minus s t and integrate it that is not going to give me anything non-zero, it will just give me 0.

So the Laplace transform of D f 1, it is value at S equal to 0 and now, we can verify that 0 equals S into 1 by S minus f 10 plus but the limit of f 1 from the right at 0 or also its value at 0 is simply 1 and so, this gives us 0 on the right hand side and so, we can verify that the derivative property holds for the function f 1 and its derivative. Now as I told you these derivative property is what makes it useful for the solution of linear differential equations and the other 2 properties, additivity or homogeneity make it useful for differential equations with constant coefficient and therefore, the Laplace transformation turns out to be useful for solving or at least for studying ordinary linear differential equations with constant with coefficients.

Now let us see, how but before that let me show you how the derivative property can be used to calculate, I am using the word calculate in a difference sense, I am not going to calculate the integral 0 to infinity of u t, e to the minus s t d t for a given u t that will be strictly the calculation of the Laplace transform, using the definition of the Laplace transformation or the Laplace transform. But using the derivative property we can evaluate or obtain the Laplace transform of many functions without actually going through the integration and in fact, we already seen just few minutes ago an example of it. The Laplace transform of the constant function 1 is 1 by S according to the table but I can obtain it from the derivative property also because look at the earlier equation, the function f 1 is 1 for all time t greater than equal to 0.

(Refer Slide Time: 11:06)

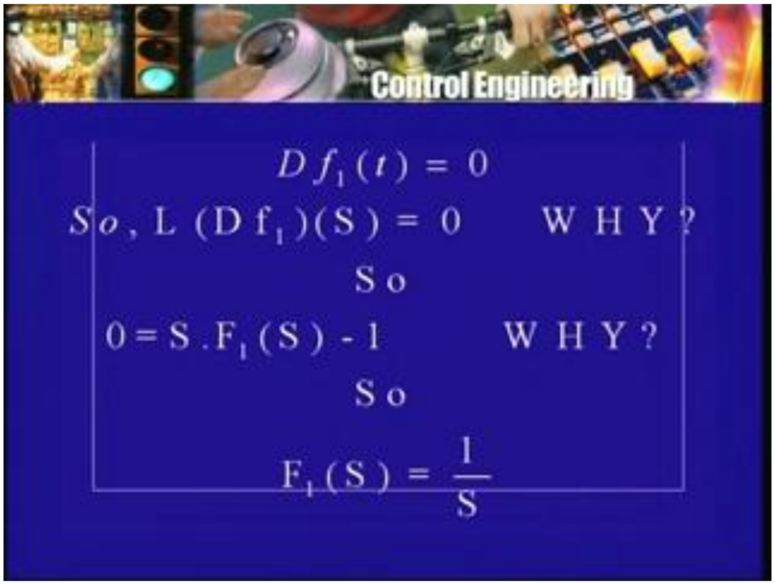


So the derivative of f 1 for all time t equal to 0. Now using the derivative property the Laplace transform of the derivative of f 1 will be 0. So I have 0 equal to S times and let us suppose that I did not look up the tables, so I did not know what the Laplace transform of this function f 1 was. So I will write down S times f 1 S, f 1 is not known to us right. Now minus the limit from the right, the limit from the right is 1. So what does this equation give us this equation gives us f 1 of S equal to 1 divided by s, this is exactly what the table has.

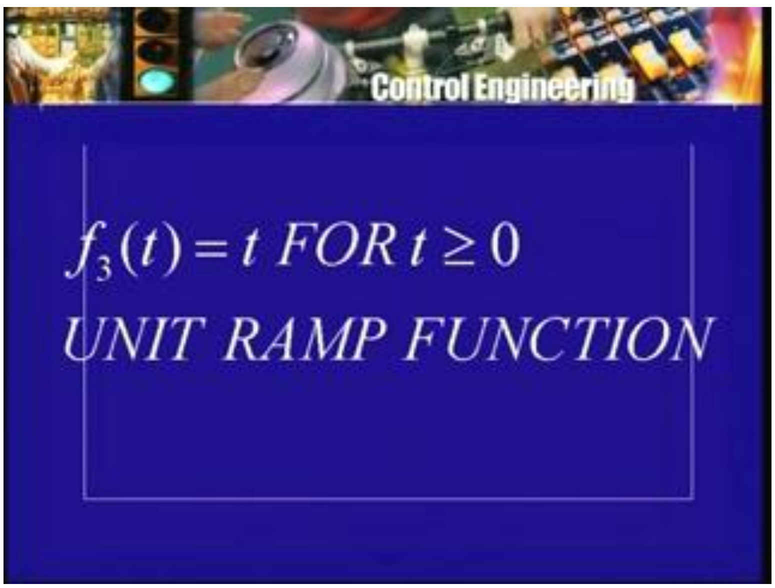
So, in other words knowing that will the Laplace transformation has those 3 properties and knowing the derivatives of some functions. For example, the constant function has the derivative 0, I can obtain the Laplace transform of the function without evaluating an integral. Of course, this is too simple an example but this can be done for many other functions. Let us take a slightly a more complicated function complicated in this sense, the calculations will be a little more complicated.

So, let us a call it let us a f 3 of t equal to t, for t greater than equal to 0, this function is known as the unit ramp function and its graph is something which very simple to draw and therefore, the graph looks like this. This is called unit ramp, ramp because it is looks like a ramp or a sloping or an inclined surface which is called ramp. It is a unit ramp because the slope is 1 because if, you differentiate then the derivative of f 3 of t, when f 3 it is t is 1.

(Refer Slide Time: 12:02)

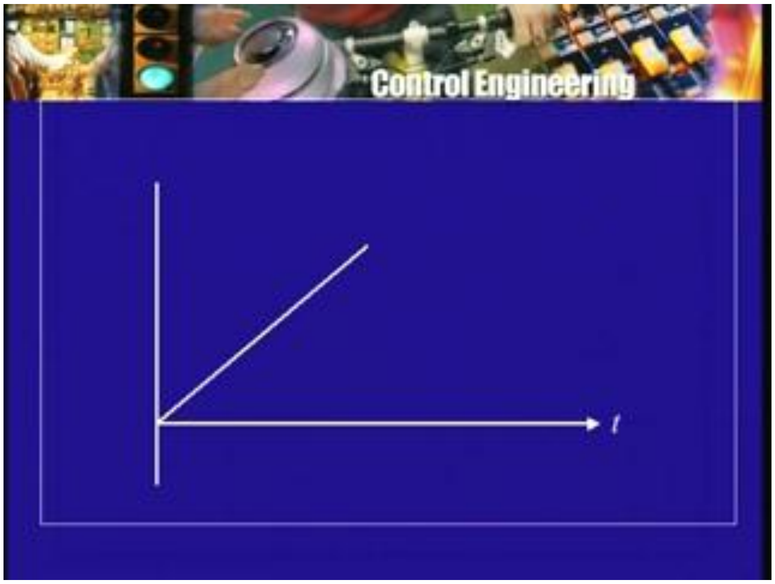


(Refer Slide Time: 13:09)

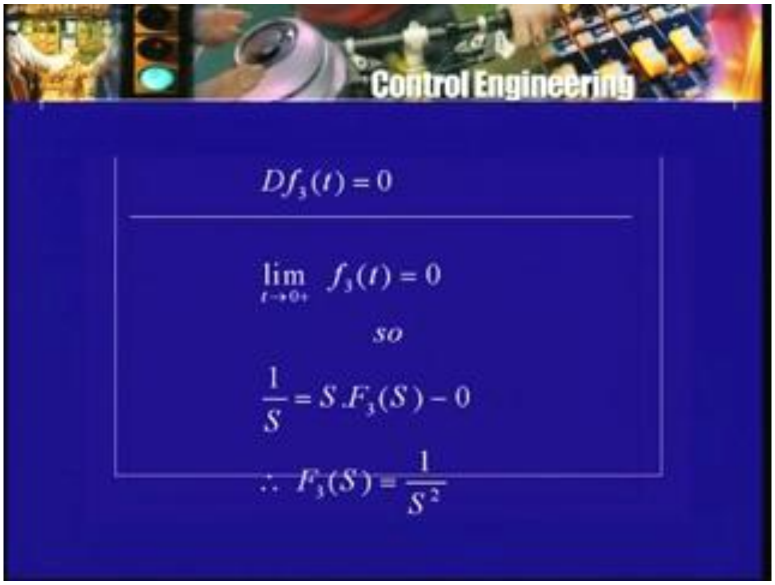


So the derivative of the ramp function is the constant function 1. Now, let us apply the derivative property to this function f 3, on the left hand side the derivative of the function f ,3 which is the function constant function 1, its Laplace transform we already found out, its 1 by S equal to S times. The unknown Laplace transform of the ramp function f 3 of S minus the initial value or the limit a from the right of the function and what is the limit from the right, it is simply 0. So I have this equation from that I get f 3 of S equal to 1 by S square.

(Refer Slide Time: 13:25)

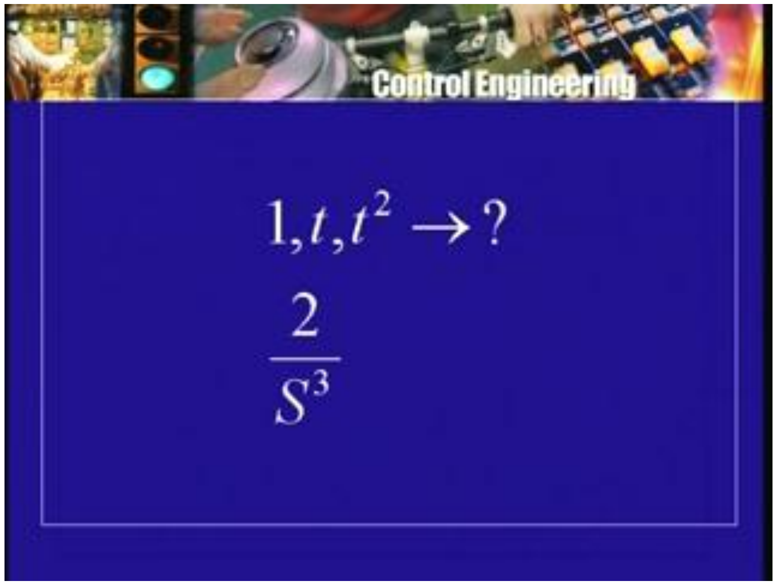


(Refer Slide Time: 13:39)

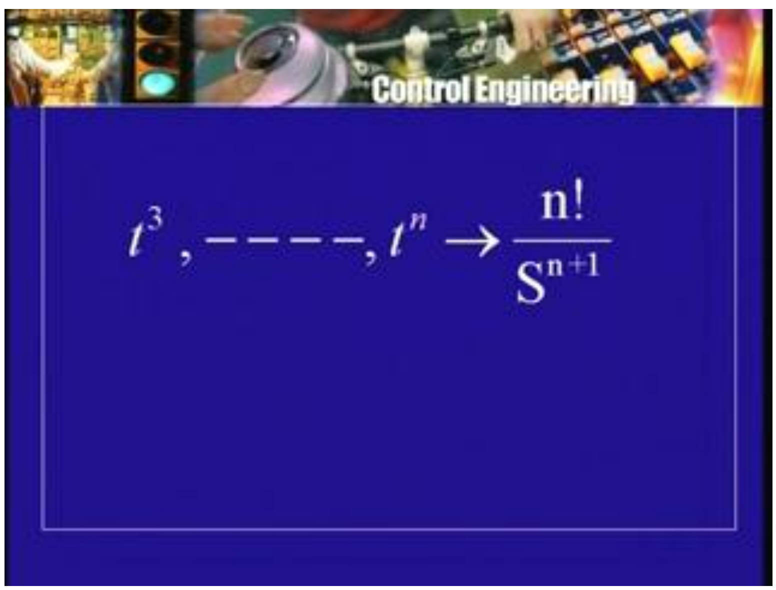


So without evaluating the integral but by using the derivative property of the Laplace transformation, I have found out the Laplace transform of the unit ramp function and now, you can see that you can follow this idea for other functions which are related to the ramp and then, to the next function and so on. So we started with the constant function 1, found out it Laplace transform then, found out the Laplace transform which of the function t, what would be the next function whose Laplace transform we should be able to find out then, well it will be t square.

(Refer Slide Time: 14:52)

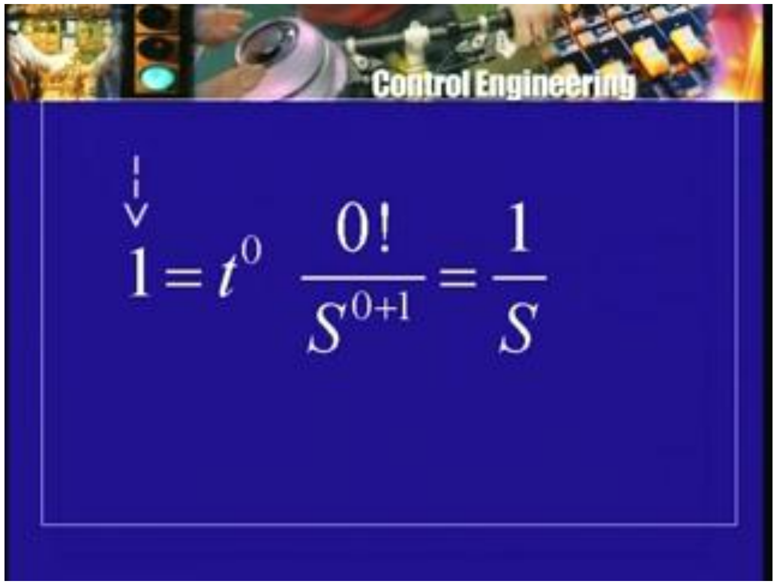


(Refer Slide Time: 15:43)

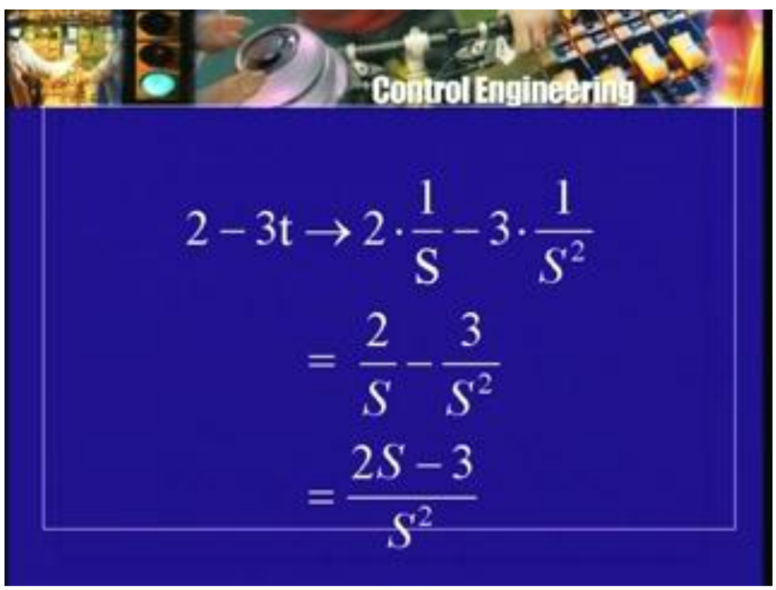


Now apply the derivative property to the function given by t square and show that its Laplace transform is 2 divided by S to the power 3 or 2 divided by S cube. Yes, take this as they unknown the function whose Laplace transform is unknown differentiate it, apply the derivative property to the derivative and the original function and see what you get. You should get 2 by s cube of course, this will tally with the table from the books and this can therefore be extended to function like t cube, t 4 or in general, if I have t raised to n then, one can show that the Laplace transform of this function t raised to n is given by in the numerator factorial n or n factorial in the denominator S to the power n plus 1 remember, that power of t is n here, the factorial is of n but in the denominator I have n plus 1.

(Refer Slide Time: 16:10)



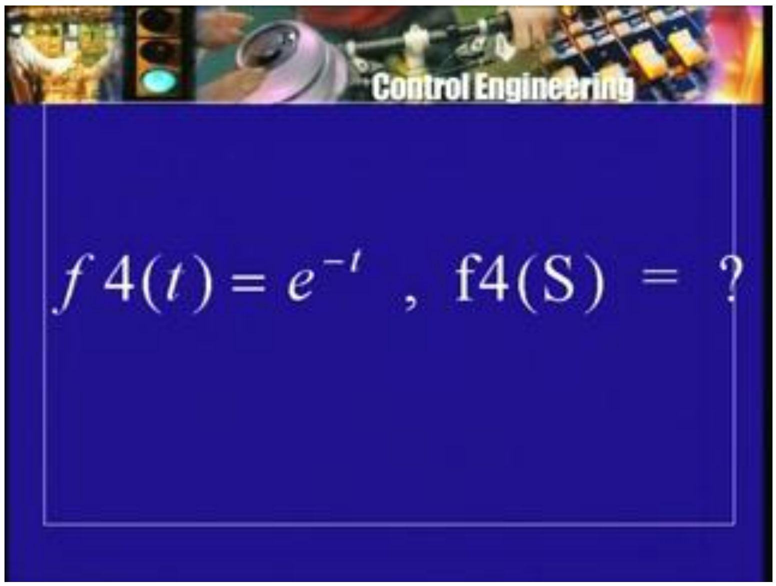
(Refer Slide Time: 16:49)



If you have forgotten, we will think of 1 a simply an alternative to t raised to 0. So the power is 0th power and then, the Laplace transform will be 0 factorial divided by S raised to 0 plus 1 and it is convenient to define 0 factorial as 1 and if you do that you get over S as the Laplace transform of the constant function 1 or the so called units step function 1. Fine then, we have the Laplace transforms of the functions 1 t, t square and so on, t to the end and therefore, if we have any polynomial function of time, we can calculate the Laplace transform of the polynomial function of time

How, suppose I have the function 2 minus 3 t and I want to find out the Laplace transform of this function 2 minus 3 t then, I use now not only the derivative property or the fact that I know the Laplace transform of t but I have also use the homogeneity and additivity. So from that then it will follow that the Laplace transform of 2 minus 3 t is the Laplace transform of the constant function 2, which will be 2 times the Laplace transform of the constant function 1. So it will be 2 into 1 by S and the second function will be minus 3 times the Laplace transform of t, the Laplace transform of t is 1 by S squared and so we have the Laplace transform of 2 minus 3 t as 2 by S minus 3 by S squared and so, in general for polynomial function we can find out the Laplace transform. But we can this for some other functions too. Let us take our exponential function.

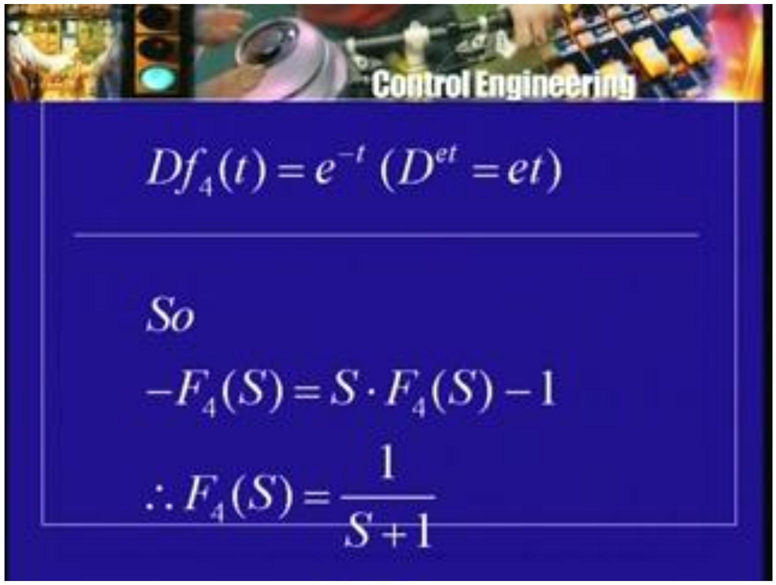
(Refer Slide Time: 18:09)



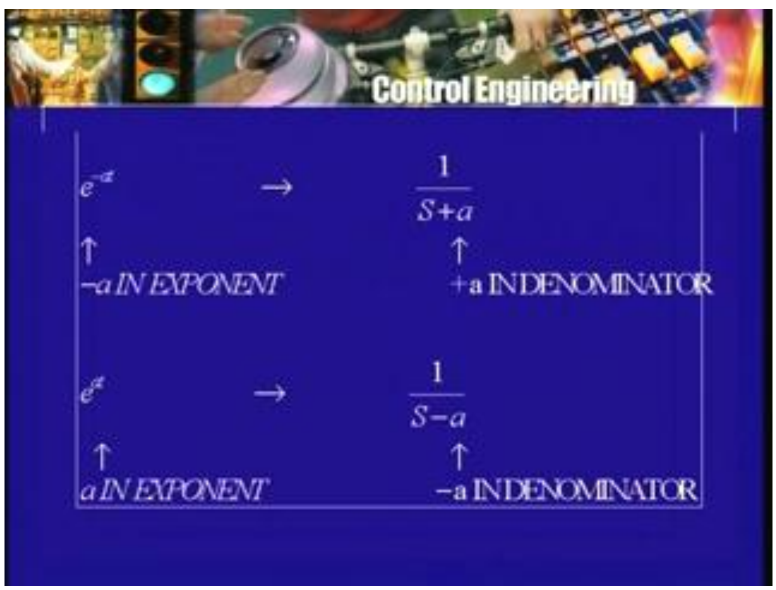
So let me call it now f 4, f 4 of t equal to e to the minus t, the familiar to exponential function and suppose, I did not know the Laplace transform of this function or I had forgotten it, but I have not forgotten the properties of the Laplace transformation and of course, I know what derivative is and how to differentiate. So f 4 is e to the minus t I want to find out capital f 4 of S. The Laplace transform of f 4, what is it, derivative property okay. So what is the derivative of f 4 the derivative f 4 is minus e to the minus t. This is of course, a derivative which 11 must know the derivative of the exponential function e to the minus t is minus times e raised to minus t, the derivative of the exponential function e to t, its simply e to the t all right.

So D 4 is it minus e to the minus t, apply the derivative property on the left hand side I have the Laplace transform of D f 4 but D f 4 is minus e to the minus t. So the Laplace transform on the on the term on the right hand side will be simply minus F 4 of S equal to on the left hand side S times the Laplace transform of the original function. So F 4 S minus the value of the limit from the right, the limit from the right of e to the minus t is 1. So I have minus 1 therefore transposing 1 term to the other side, we will get F 4 of S equal to 1 divided by S plus 1.

(Refer Slide Time: 18:19)



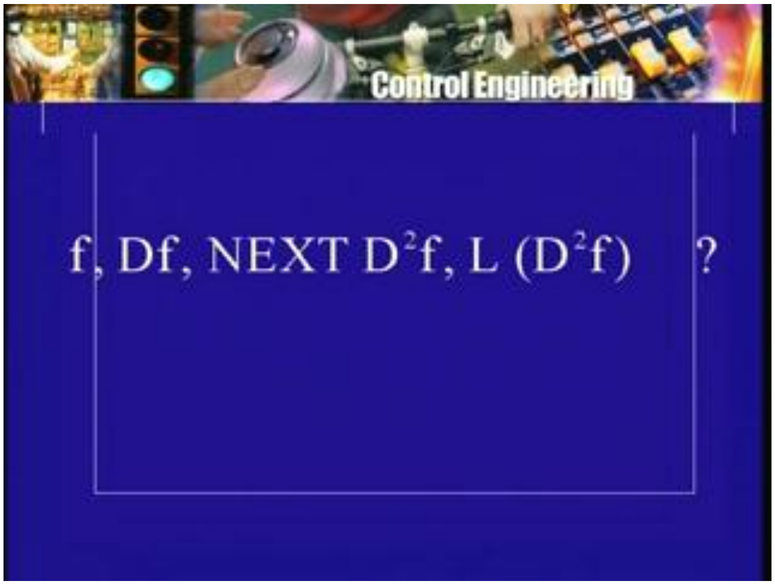
(Refer Slide Time: 19:33)



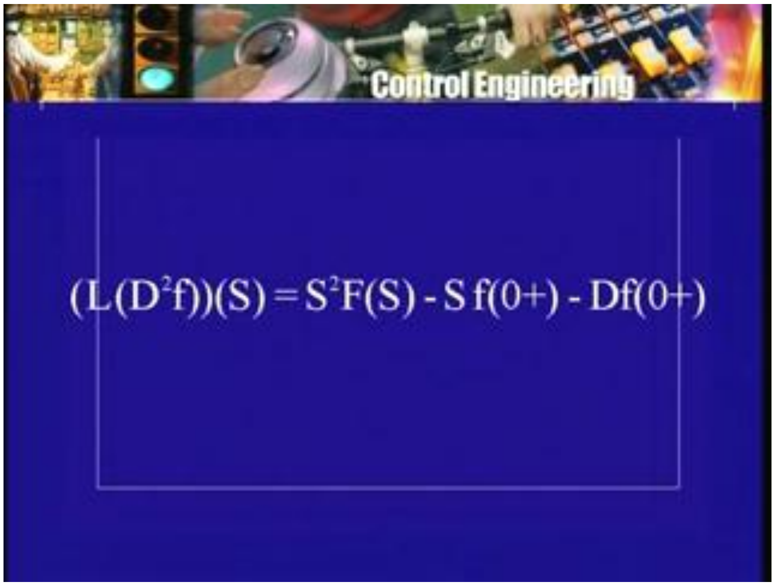
So I found have out Laplace transform of e to the minus t to be 1 divided by S plus 1 without evaluating any integral and of course, you can see now that I can apply to find the Laplace transform of e to the minus 80, it will turn out to be 1 over S plus a. Remember, minus a in the exponent plus a here in the denominator or if I have e raised to e t then, the Laplace transform will be 1 divided by S minus a, a in the exponent then minus a in the denominator.

Fine, then we have found out the Laplace transform of the exponential function, what else, can we find out the Laplace transform of the trigonometric functions this way and the answer is yes, only we have to go 1 step further. The derivative property applies for the first derivative or the first order derivative but we can extend that property and indeed therefore, there will be a theorem of the Laplace transformation which you can call the generalized or derivative property for higher order derivatives.

(Refer Slide Time: 20:31)



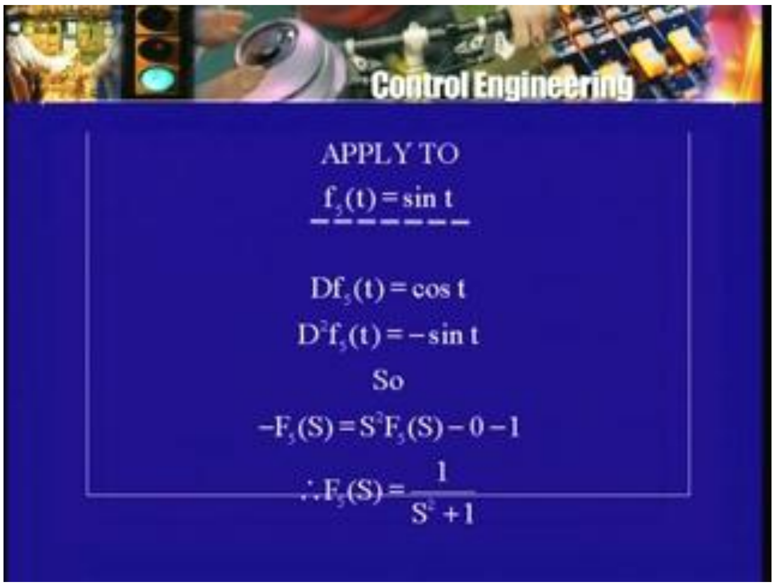
(Refer Slide Time: 20:50)



So if I have function f, I have its first derivative D f, I have its second derivative denoted by the D square f. By now, you should be familiar with this notation in preference to D f d t and D square f D t square then, what is the Laplace transform of D square f. The second derivative, then the following results holds its value at S equals, this time not S but S squared multiplying F of S. The Laplace transform of the original function S squared times F S and instead of 1 term subtracted on the right hand side, we have 2 terms, the first 1 is minus S f 0 plus minus D f 0 plus.

So you have S times the limit of the function from the right minus the limit of the derivative of the function from the right. So we have 2 terms subtracted from S square F S on the right hand side. In the derivative property for first order derivatives, we had only 2 terms on the right hand side now we have 3 terms and in fact, one can state and prove a derivative property for higher order derivatives higher than 2.

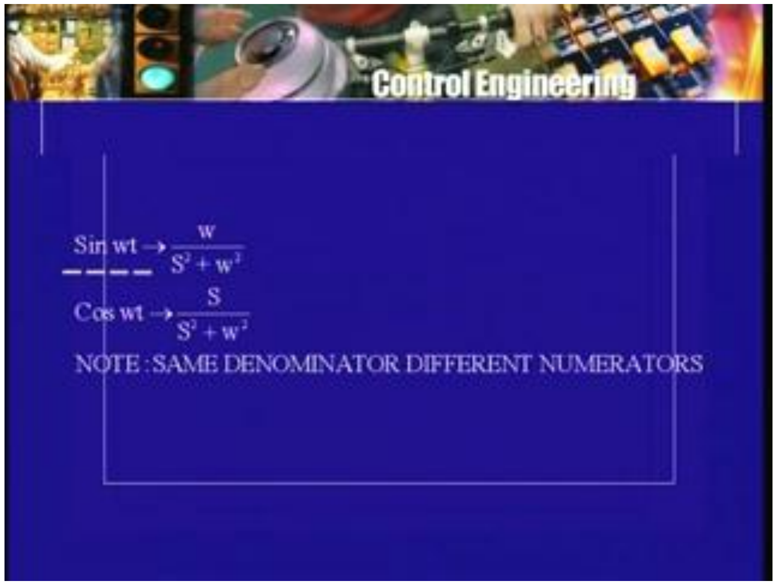
(Refer Slide Time: 21:54)



Now, let me show you how I can use this to find out the derivative of the sinusoidal function and once again, let us choose the simplest sinusoidal function. So let us think of f 5 of t as sin t the familiar sinusoidal function, we want to use the derivative property. We will see that the first order derivative property will not be enough because if I differentiate, what is the first order derivative or the first derivative of sin t, it is cosine t. Now cosine t is different from sin t, in fact quite different. So the derivative property for the first order derivative will not be so useful or cannot be used right away. But let us not give up I will differentiate1 more time. So what is the second derivative of f 5 of t that is the derivative of cosine t and therefore, it is minus sin t.

So although the first derivative is cosine t which is not related to sin t in a some simple way, the second derivative is minus sin that it is the simply the negative of the first derivative. Now, apply the derivative property for the second order derivative, what do we get. On the left hand side I have the derivative second order derivative is back to the minus sin t. So on the left hand side I will have minus F 5 of S equals on the right hand side S square times F of S minus S into f 0 plus the limit of the sin function at t equal to0 from the right is 0. So minus 0, minus the derivative of the sin function is cosine and what is its limit from the right at t equal to 0, 1. So I have this equation and from this equation, we can immediately calculate the unknown Laplace transform F 5 of S as 1 divided by S squared plus 1.

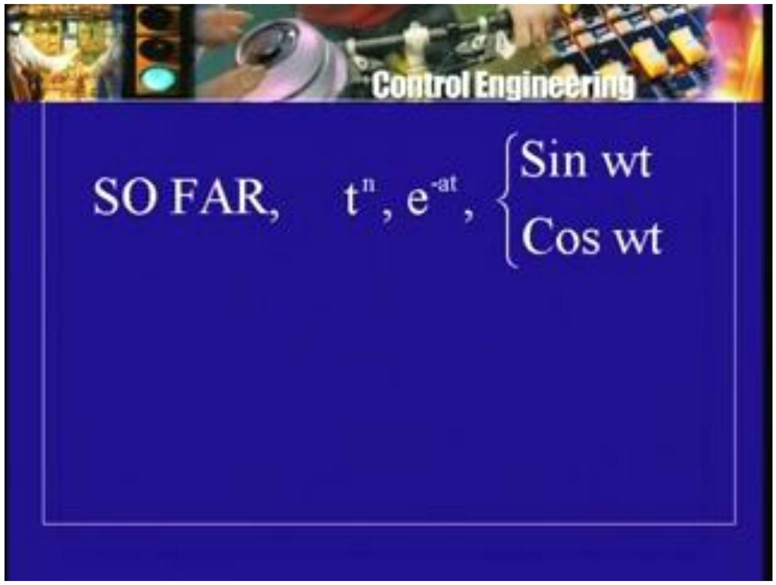
(Refer Slide Time: 24:00)



So the Laplace transform of the function sin t is 1 divided by S squared plus 1 and this, we have found out by using the second order derivative property rather than the first order derivative property or what I stated originally as the derivative property and you can then extend this, to find the Laplace transform, the function sin omega t and it will turn out to be omega divided by S squared plus omega square. The multiplier of t omega here appears in the numerator as it is and in the denominator as omega squared added 2 S squared and similarly, one can look at cosine omega t because although 1 differentiation of cosine converts it into the sin function, one more differentiation of that takes you back to the cosine function with change of sin at multiplication and so, cosine omega t Laplace transform that turns of to be S divided by S squared plus omega square, sin omega t and cosine omega t both of the them have the same denominator for the Laplace transform, S squared plus omega squared but the numerators are different.

The sin omega t, it is just omega for cosine omega t, it is S. These are the a few Laplace transforms it, it is not really very difficult to remember, but I have shown you just. Now, how you can find out these Laplace transforms if you have forgotten them provided you remember, some of the properties of the Laplace transformation namely, additivity, homogeneity and the derivative property, may be not only the first order derivative property but also the second order derivative property.

(Refer Slide Time: 25:35)

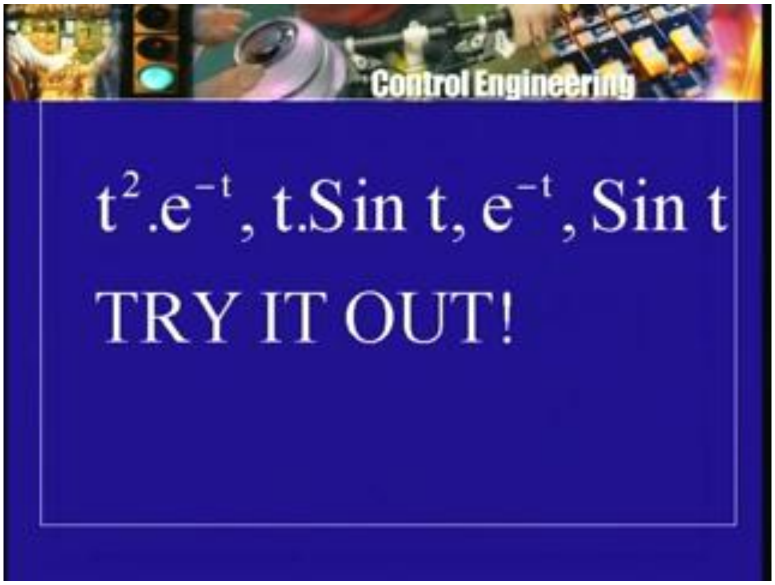


So, so far we have been able to obtain the Laplace transform of functions like t to the n exponential of minus at let us, I am writing it as e to the minus a t with the usual assumption that a is positive because usually, we encounter exponentially decaying function in control theory practice and of course, sin omega t and cos omega t. So these 3 functions or families of function but one can go further and apply the derivative property of course, it will involve more calculation. But no evaluation of integral, I have not evaluated any integral in all these calculations. I simply use the properties of the Laplace transformation of course, the properties of the Laplace transformation hold or are true because it is defined by an integral and in fact, they have therefore to be proved and many books do prove the properties of the Laplace transformation.

Now some more functions can be their Laplace transforms, can be found out using these 3 properties and they are not just these 3 functions separately, the power or polynomial function or the power function t raised to n, the exponential function e raised to minus at and the sinusoidal function sin omega t or cosine omega t and of course, therefore I can extended to sin of omega t plus theta, where theta is any angle so to speak but also to find out the Laplace transform of their products because one does encounter such product when working in control problems.

In fact, we have earlier encountered a product like t e raised to minus t, this is product of the function t itself which is the unit ramp function and the exponential function e to the minus t, what is the Laplace transform of this function. You can apply the derivative property, to find out what the Laplace transform is, try it on your own and then, checkup the answer with the table in the book and once you get confidence about this, we can go ahead.

(Refer Slide Time: 27:54)

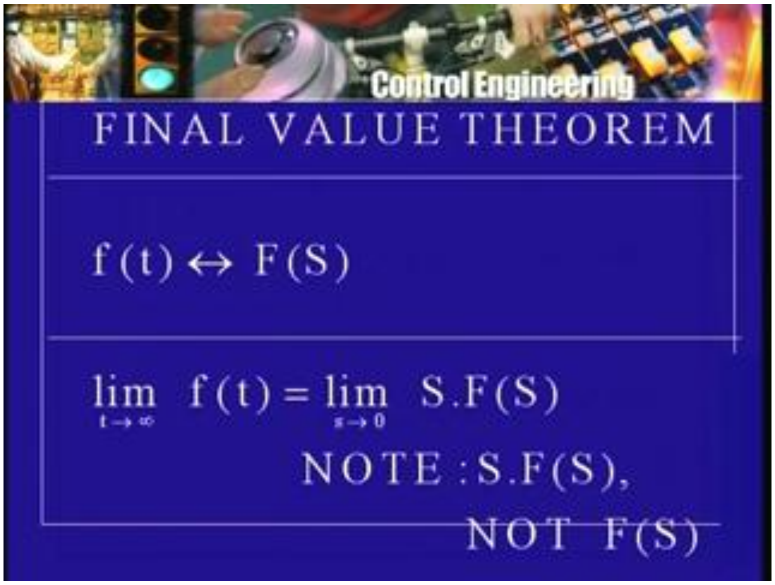


For example, then we can think of t square into e raised to minus t, what will be the Laplace transform of this function, one more t into sin t, what will be the Laplace transform of this product, one more why not e raised to minus t into sin t. This is of course is going to be much harder than the earlier example, but what is the idea take the function and go on differentiating and see what happens and see somewhere, whether you can use the properties of the Laplace transformation to get the thumb equation for the Laplace transform of this function e to the minus t sin t the product function.

Now, of course there are many other theorems of the Laplace transform which are useful in many applications but we will look at them only when we need them, as we go on I do not want to go into a very detailed discussion of the Laplace transform or the Laplace transformation. So this is enough except 2 more results which are not very difficult to remember and they are useful in fact, we will use them for our steady state calculations very soon and one of these results known as the final value theorem.

Now like all the other theorems or properties that I have stated, these hold only under certain conditions. So one should check before applying whether, those conditions are valid but the final value theorem tells you something like this. I have function f, I have its Laplace transform capital F of course, as we have been using the notation F is usually thought of as a function of time sigma changing in time, whereas capital F is function of a complex variable S f t is real, t is real S is complex F S also turns on to be usually complex.

(Refer Slide Time: 29:29)



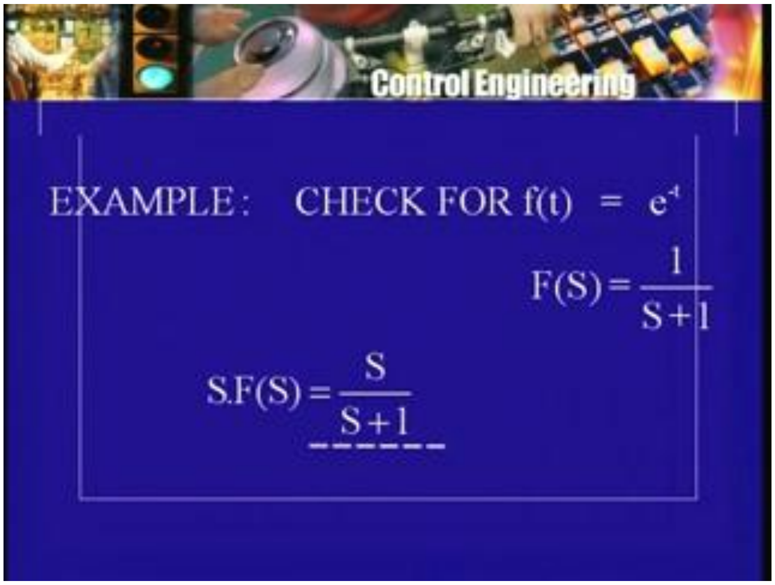
Now what about the limit as t increases without any bound or as we say limit as t tends to infinity of f t. The limit as t tends to infinity of ft, if f t where my speed as a function of time perhaps then, this is what I have called earlier the steady state speed meaning that if I weight for enough time, the speed will be very close to this value. So it is actually a limit that 1 is talking about limit as t tends to infinity of the function f, what is it. Now of course such a limit may not exist and there are many functions for which the limit does not exist. For example, sin t as t tends to infinity does not have a limit it keeps on oscillating, the ramp as t tends to infinity does not have a limit because the limit can be set to be infinite or the infinity but infinity or a number at all.

So it is only a symbol to denote something. But there are many functions which have a limit as t tends to infinity or constant function, what its limit as t tends to infinity, constant function one, it is not very difficult to see that limit is 1. The exponential function e to the minus t what is the limit as t tends to infinity 0, in fact that is why we called it such function an exponentially decaying function and talk to about time constants and so on earlier. Now there is relationship between this limit and another limit of F S, the Laplace transform but in this here it is not F S but S into F S.

So you take the product or you multiply the Laplace transform by S and then, take the limit as S tends to what do you get not infinity but 0, take the limit of S into F S as S tends to 0 then, the theorem says that under certain conditions these 2 limits are equal and you can see, why its call the final value theorem because its gives you limiting or final value of the function f as function of time in terms of a limit of its Laplace transform or something related to its Laplace transform. Let us quickly apply it without even writing down any equations. For this constant function 1, for the constant function 1, the limit is 1 as t tends to infinity the Laplace transform of the constant function 1 is 1 by S.

So S into1 by S is 1 and the limit of that it as s tends to 0 is 1 because it is not a function of S at all or it is a constant function of S that product. Let us apply it to e to the minus t what is the limit of e to the minus t as t tends to infinity 0, what is the Laplace transform of e to the minus t 1 divided by S plus 1. So I have to multiply that by S, so I have S divided by S plus 1.

(Refer Slide Time: 32:53)

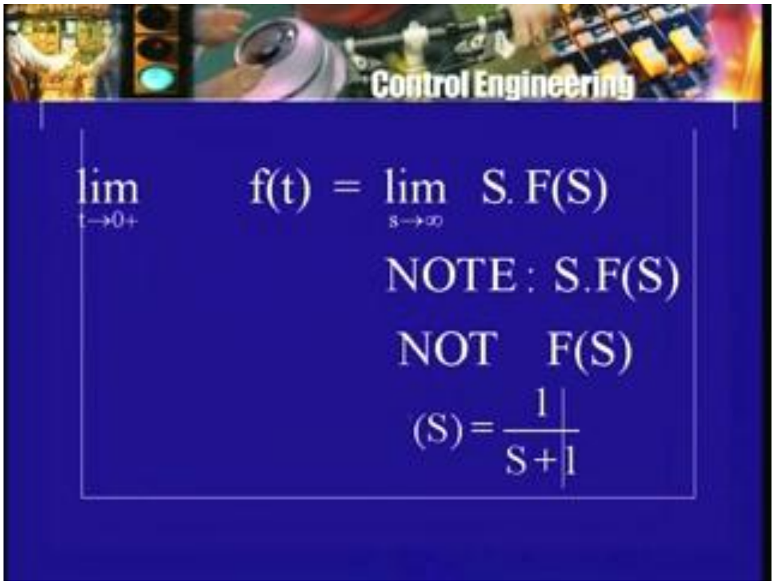


Now, I have to take the limit of S divided by S plus 1 as s tends to 0 and what is that limit going to be the numerator tends to 0, the denominator tends to 1. So the limit will be 0 and so we have verified the final value theorem, for the exponentially decaying function e to the minus t. So sometimes when you many not know the limit or it may be not so easy to calculate, a way out if you know its Laplace transform is to calculate a limit that involves the Laplace transform. Of course, you have to calculate a limit of the Laplace transform multiplied by S, now if you find that difficult then of course this of no use.

So this is the final value theorem and there is what is sometimes called a dual of this theorem which would be for what, this is what, for t tending to infinity, S tending to 0 .sS we can now expect the 2 to be interchanged. So suppose I want to look at limit of the function f, as t tends to 0 and 0 plus because nothing to be said about t less than 0. So the limit of the function from the right at t equal to 0, how is it related to F of S.

Well once again, I have S into FS and take the limit of that as S tends to not 0 but infinity that is as S increases in absolute value or modulus without bound, what is the limit of S times F S. These 2 limits are once again equal, this naturally is going to be called the initial value theorem and it is the dual of the final value theorem, applying it to the constant function 1, the left hand side is 1, the right hand side S into1 by S that is 1 is limit as s trends to infinity is 1, no problem, what is the exponential function e to the minus t left hand side is limit of e to the minus t as tends to 0 plus remember, the graph of the exponential function. It is 1, on the right hand side I have the limit of S into 1 divided by S plus 1 as S tends to infinity.

(Refer Slide Time: 34:00)

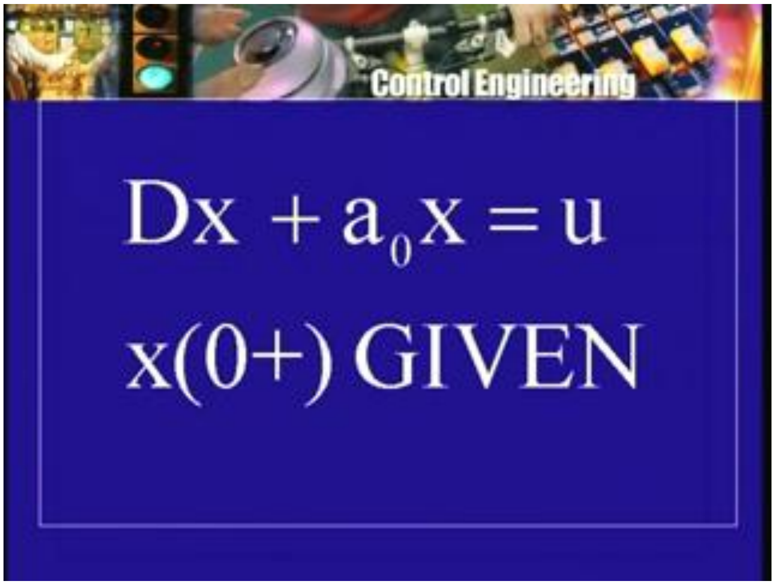


Now one can see that the limit of S divided by S plus 1 that fraction as S increases without bound in absolute value will be 1 and so, the result checks with the theorem, the initial value theorem. So these are 2 results especially, the final value theorem which is going to be also useful to us because final value indicates this steady state value for many of our applications and we have earlier have been looking only at the steady state behavior of the motor control system.

So of this much is enough about Laplace transformation and the use of the Laplace transform. Well, I have not really talked about the use of the Laplace transform but use of the properties of the Laplace transformation perhaps to evaluate or obtain the Laplace transforms of some function. Now the user or the application of the Laplace transform for the solution of a differential equation. For our motor control problem, I will be going back to it very soon. We may not be interested in solving so much a differential equation as you will see as obtaining some relationships between various things, the motor speed, the applied voltage, may be the load torque and so on, in a different way and I have mentioned already the concept of a transfer function.

So for the motor speed control problem, our main interest will be in obtaining transfer function. But the idea of the transfer function involves the Laplace transformation and that is why, we have to take a look at the Laplace transformation. But the solution of the differential equation is not to very far of from this. So let us do it for the first in order differential equation. So the first in order differential equation going back to our earlier notation, if x was the response u was the input then, the differential equation looked like D x plus a 0 x equal to u and as we saw earlier, I have to be given the initial value of x.

(Refer Slide Time: 37:22)

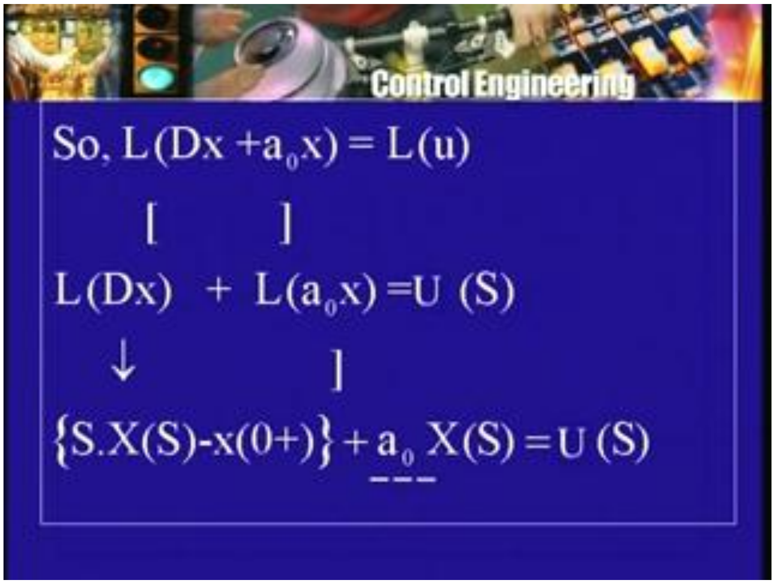


So I will write x 0 or I can write it has x 0 plus is given, so given the initial value of the response and given the input as a function of time. Calculate or obtain the response for all time t. We know it for the at the initial time only knowing the input and the initial value, calculate the response for all other instance of time, all future instance of time. This is of course of interest to us, we want to know what is going to happen if I start the system with this initial value and apply that input, how is my response going to change with time which is the basic problem of control theory because we want to know before and what is going to happen then, we will do somethings with the system introduce something and see what is the difference etcetera. Fine, now how do we apply the Laplace transform technique or the idea of the Laplace transformation to this.

Well, it is very simple. You any equation what does equation say on the right hand side I have function u that function u is supposed to be given hopefully, I will be able to calculate or obtain or lookup the Laplace transform of u, on the left hand side I have x I do not know x, I have a terms which is x multiplied by given or known coefficient a 0, I have another term which is not x but derivative of x. Well, let us apply the Laplace transformation to the 2 sides, the 2 functions are the same function that is what this equality means only 1 function, on the right hand side is the input but it is also equal or the same as D x plus a 0 x, on the left hand side.

So let us apply the Laplace transformation to the 2 sides and therefore, I am going to write L that means the result or applying the Laplace transformation to D x plus a 0 x is the same as L applied to u. This is stating nothing very big it is a just the simple fact which is obtained from the, our assumption that the function u equals the function on the left hand side. But, now we will start using the Laplace transformation properties. So we have on the left hand side L of a sum of 2 functions. So by the additive property or the additivity of the Laplace transformation, it will be L of D x plus L of a 0 x that is the left hand side.

(Refer Slide Time: 39:19)

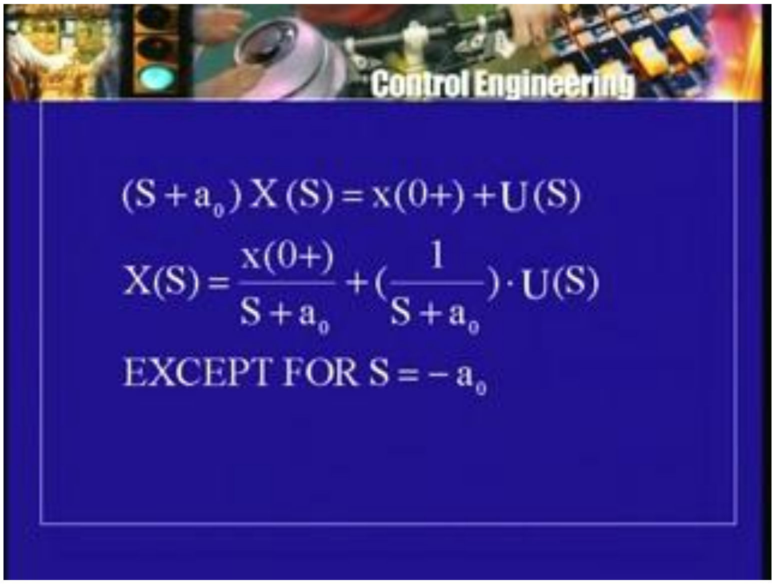


On the right hand side, I have L of you and I will assume that it is the Laplace transform I will can calculate. So I will right it as capital U or if you wish capital U of S just to remind us that it is a Laplace transform. So on the left hand side has still I have not got out of any trouble because I do not know, what L of a 0 x is or L of D x is. But I know the derivative property and the homogeneity property, if I apply the derivative property then L of D x at S is S times, the Laplace transform of small x of t that we are going to denote by capital X of S minus x 0 plus.

So that is what the Laplace transform of D x at S, at the complex number S is. We still do not know what capital X of s is but this is true nevertheless, what about the second term Laplace transform of a 0 x, it is just a multiple of x by a number a 0. So by the homogeneity property it is simply a 0 times, the Laplace transform of x and that at s is simply plus a 0 x of s. So, we have this equation. Now, this equation can be simplified that is terms can be rearranged. So that we can make some progress but before we do that what do I have here, I have the U of presumably the function u t is given, let us say it is sign t then, US is known it is 1 by a square plus 1, what else is given to me or known a 0, the coefficient a 0. I have to be told, what it is, if I want to first solve the equation otherwise, a 0 will remain as a parameter. Similarly, x 0 plus the initial value that will have to be specified or it will remain as a parameter.

Now, instead of putting some specific numbers a 0 and x 0 plus, let us leave them as the as they are. So that they become parameters that is they will appear in the final solution and you simply replace them by their values, say a 0 is equal to 2 x 0 plus may be minus 5 or whatever. So let us go on with this equation, so what is it that we can do well there this xs here x s here. So I can combine the 2 and I can write it as S plus a 0 into capital X of S. Now there is the term minus x 0 plus which is as the number.

(Refer Slide Time: 42:09)



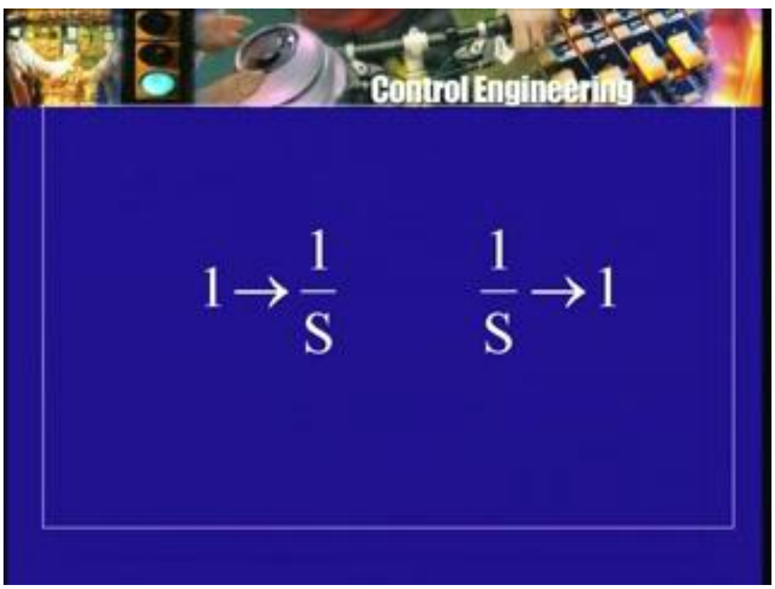
So I will transfer it to the other side and I write this as x 0 plus, plus capital U of S. So I have S plus a 0 multiplying the unknown Laplace transform X of S equal to this thing on the right hand side and so this is easy to solve for X of S and I will keep 2 terms separate on the right hand side also. So I will write this as x 0 plus divided by S plus a 0 plus and I will write the second term as 1 over S plus a 0 multiplying U of S. So the unknown Laplace transform X of S of the unknown function x t is given as sum of 2 terms and therefore, now this is where we apply the Laplace transformation property is in reverse.

So for I have been applying them to go from the function to the Laplace transform and now is the step to go from the Laplace transform to the function and so this is exactly really what the Laplace transformation method involves. You want to know, what the function of time is, by using the Laplace transformation approach or applying the Laplace transformation, we find out what the Laplace transform of the function is and so we have to go back to the time function. For this one talks about the inverse Laplace transformation and for an obvious reason, I can denote it as L raised to minus 1 it can be read as the inverse Laplace transform that is given the function of S, find a function of time whose Laplace transform is that function of s.

This is called the Laplace transform inversion and just as the Laplace transform was define by an integral, the inverse Laplace transformation can also be expressed in the form of an integral and that integral is more horrible than the 0 to infinity integral that we saw earlier. You should become familiar with it but we will not spend time on it. However, we can simply use the fact that we can recognize some functions as the Laplace transform of some known function and also a theorem which is known as the uniqueness of the Laplace transform or the uniqueness of the Laplace transform of the 1 to 1 as of the Laplace transformation as a function or as an operator that is, if I see 1 by S then, I can conclude that the function whose Laplace transform is 1 by S must be the constant function 1 and only the constant function 1. Of course, unfortunately in this statement 1 has to be more careful, I will not go in to the details that is there are functions which are not exactly 1 for all time t greater than 0, whose Laplace transform is still 1 by S or can be still 1 by S but they do not differ from 1 in a very big way.

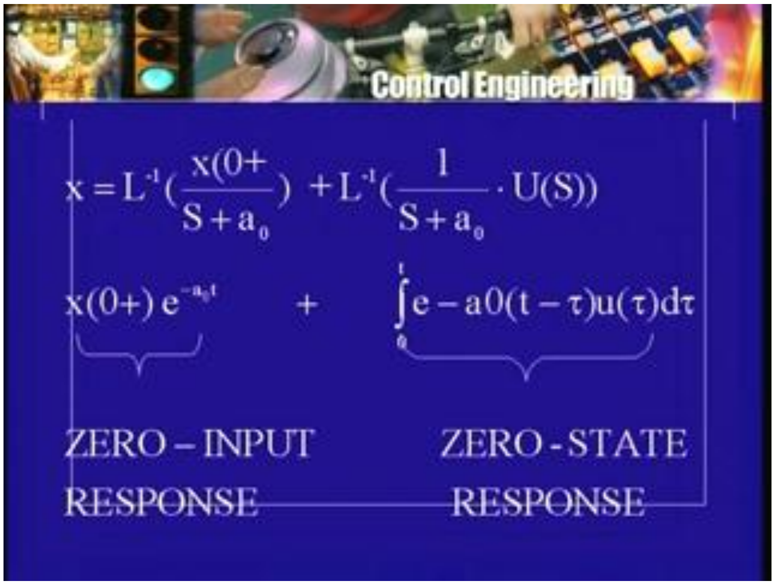
(Refer Slide Time: 43:43)

(Refer Slide Time: 44:47)



So it is reasonably good assumption to conclude that if the function is the Laplace transform of the function is 1 by S, the function must be 1. So once again there are some results which have to be used and you have to make sure that the conditions are satisfied etcetera okay. So we have x of s as a sum of 2 terms and therefore, I will write x which is the Laplace inverse of capital X but capital X is the sum of 2 terms and now, you can immediately see that just as a Laplace transform is additive the universe Laplace transformation will also be additive.

(Refer Slide Time: 45:38)



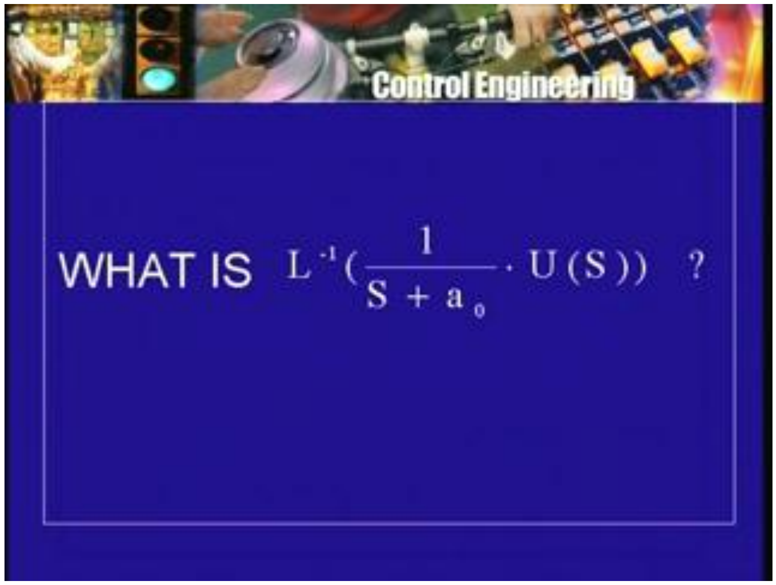
So, I will have Laplace inverse of x 0 plus divided by S plus a 0 as 1 term plus Laplace inverse of the product 1 divided by S plus a 0 multiplying U of S as the other term. So whatever is this function x it is a sum of 2 functions, 1 function whose Laplace transform is x 0 plus divided by S plus a 0 and another function whose Laplace transform of is this product or alternatively, I somehow find out the inverse Laplace transform of this x 0 plus divided by S plus a 0 and similarly, the inverse Laplace transform of the other function.

Now the first term is not a big problem because we have just now seen that the Laplace transform of e to the minus a t is 1 divided by S plus a and therefore, the Laplace inverse of x 0 plus divided by S plus a 0 can be seen to be x 0 plus into e raised to minus a 0 t. So the first term is found out easily, now go back to the solution that we had found out by 2 different methods. There was a method of variation of parameters then, there was an input and there was also a method of factorizing the polynomial differential operator whichever way.

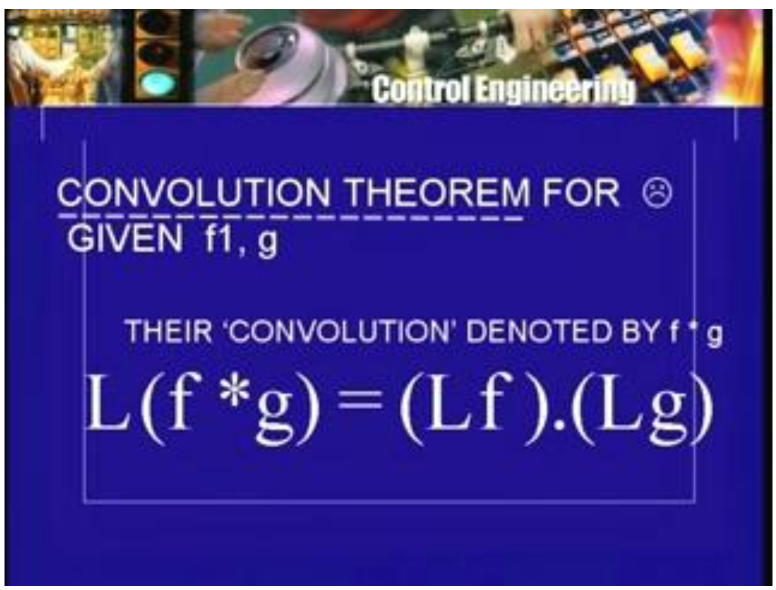
We had come across this term x 0 plus into e to the exponential of minus a 0 t and what is it called we called it the 0 input response. It is the part of the response which is produced by the initial condition and not by the input or it is the response that you get, when the input is 0 indeed if I put u t equal to 0 then, US is 0. So the second term here drops out altogether, so I will have only the first term.

So this Laplace inverse is simply the 0 input response term and therefore, what do you expect the second term to be. The second term you expect to be the 0 state response and what did that look like it was what is called a convolution integral and this case indeed you can show that this its value at any time t will be integral 0 to t of e to the minus a 0 t minus tau, u tau, b tau and there is one more theorem of the Laplace transform which one can say if one is using here. I have product S plus a 0 one over S plus a 0 into u of S, u of S is of course then Laplace transform u t is 1 over S plus a 0 the Laplace transform of anything of some function yes exponential of minus a 0 t.

(Refer Slide Time: 48:27)



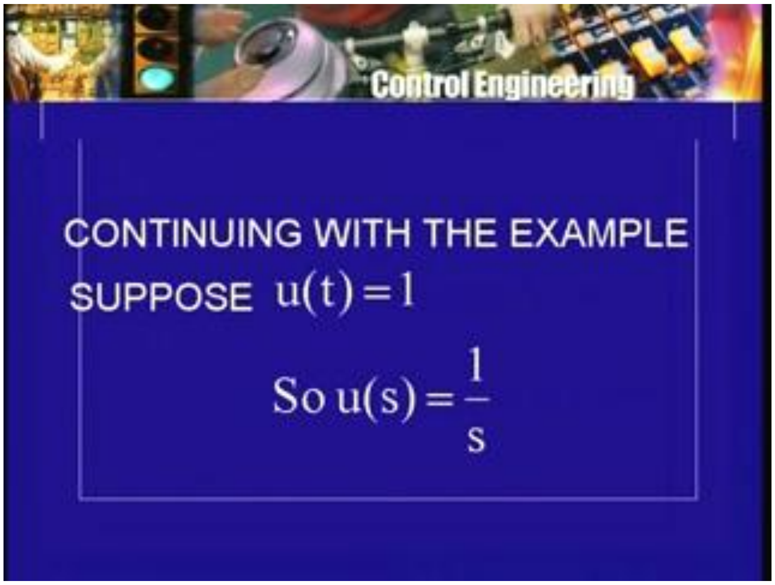
(Refer Slide Time: 48:43)



So there is the theorem of the Laplace transformation known as the convolution theorem which says that if you have 2 functions of time, you take their convolution which is another function of time then, the Laplace transform of the convolution is the product of their Laplace transform. In symbols if I have 2 functions, let us say f and g both functions of time, their convolution is usually denoted by putting star between the 2, it is read as f star g or the convolution of f with g or f and g then, the convolution theorem states that the Laplace transform of f star g is simply the product to the 2 Laplace transform L f into L g.

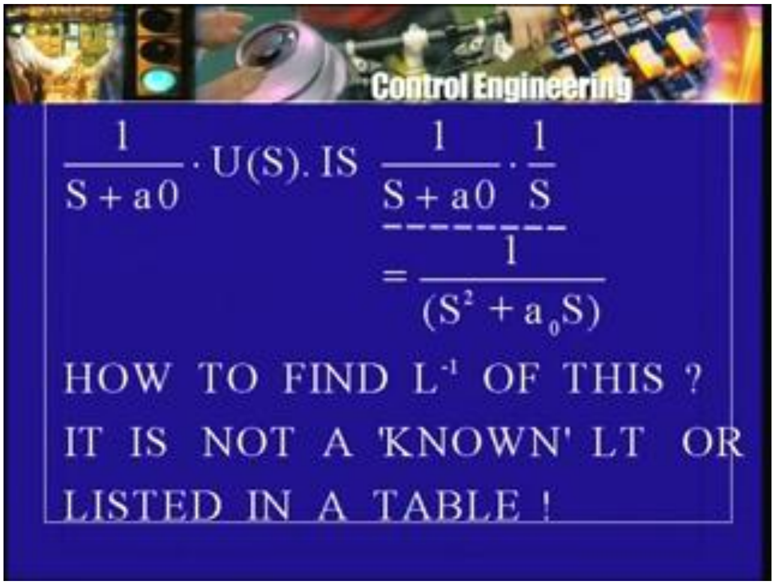
So I can say that I am using this convolution theorem of the Laplace transform to get the fact that the Laplace inverse of this is indeed the convolution of e to the minus a 0 t with the input function u and so, this is how one can solve the first order linear differential equation with constant coefficient with an input that is non-homogeneous and with perhaps a non-zero initial value or initial condition and what has it involved, it has simply involved additivity, homogeneity, derivative property of the Laplace transformation plus the idea of inverse Laplace transformation and a new result now, the convolution theorem. However, it is not necessary to use the convolution theorem in fact one does not use it because calculation of the convolution integral is also quite involved.

(Refer Slide Time: 50:30)

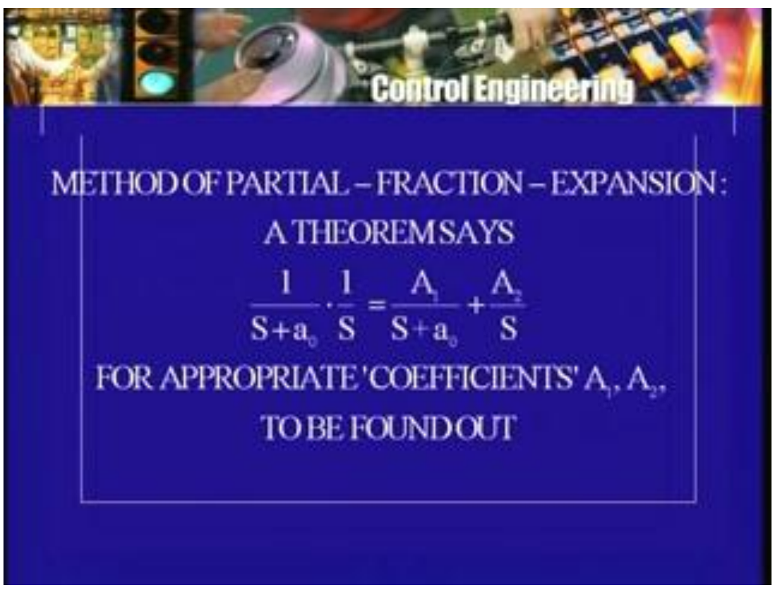


So let me illustrate that so suppose, we have an example u of t was simply the constant function one, I said that it is useful if you know what the function is. So it is the constant function one fortunately, the Laplace transform of that is also very easy. In fact, we have calculated is as 1 divided by S, so what is the problem. Now we simply have to find out the Laplace inverse of 1 divided by S plus a 0 into 1 divided by S.

(Refer Slide Time: 50:48)



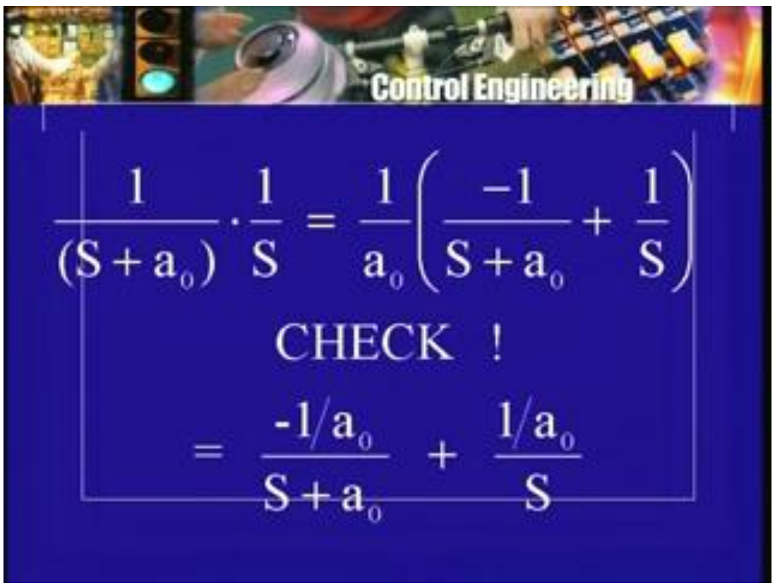
(Refer Slide Time: 51:30)



Now of course, if I try to make it look simpler it will look like 1 divided by s squared plus a 0 S and then, you might wonder, what is the Laplace inverse of this function. In fact, you will not find it in any of your table. Now of course people have applied their brains to such problems in fact, not for Laplace transformation but in other areas of mathematics and they have found out ways of tackling this. Now for doing this one uses a technique which again, you must have encountered in your algebra course and that is called partial fraction expansion, it is very important to know this technique.

We will find it useful later on when we talk about use of feedback and so on in the transient, for the transient performance, partial fraction expansion, what is the partial fraction expansion? We will have a product like this 1 over s plus a 0, linear term in the denominator into 1 over S, another linear term in the denominator then, the partial fraction expansion theorem, if you wish says that this product 1 divided by S plus a 0 into 1 divided by S can be written as 1 term which has one of the factors S plus a 0 only in the denominator, the other term has only the factor S in the denominator and the numerators are some 2 numbers a 1 and a 2. The partial fraction expansion theorem or methods tells you that this product can be rewritten as a sum of 2 fractions. This is the product of 2 fractions, it can be written as the sum of 2 fraction and that is why expansion into fractions and partial because each of 1 of them is only a part of the total.

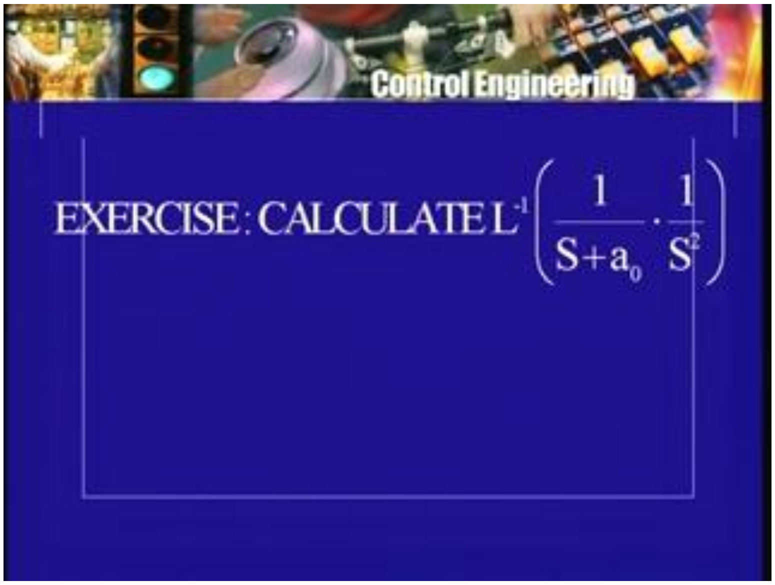
(Refer Slide Time: 52:51)



So that is why the name partial fraction expansion. In this case, you can verify that this product 1 by s plus a 0 into 1 by S is indeed given by minus 1 divided by s plus a 0 plus 1 divided by S but that is not all, the whole of it is multiplied by 1 by a 0 but I can write it as minus 1 by a 0 divided by S plus a 0 that is one term plus1 by a 0 divided by S, that is the second term.

Now look up your algebraic textbook, to find out how the partial fraction expansion is to be carried out, when there are more than 2 terms in the product and terms may be not only of a first order or degree like S plus a 0 but they may be of second order or second degree. We will find it useful as we go on later on because many of our systems, the speed control system in particular. When we want to look at the transient behavior it will turn out to be described by second order differential equation and therefore, we will get a quadratic expression and so one may have to use the partial fraction expansion technique, when the factors involve not just linear polynomials but quadratic polynomials. As homework, you can try to find out the Laplace inverse of the following using partial fraction expansion.

(Refer Slide Time: 54:15)



So in place 1 over S plus a 0 into 1 by S, I will just make a slide change, I will put it as 1 by S squared. Now what will the partial fraction exponential of this and use that to find out the Laplace inverse of this product.