Control Engineering
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Lecture - 21

We have had a quick look at one method of solving a non-homogeneous linear ordinary differential equation with constant coefficient of order greater than 1, what was that method called, it was called the method of variation of constants or method of variation of parameter. It is a method which is not mentioned in many books and that is why, I spend some time on it also it is useful to be aware of a number of different methods that can be used and one should not restrict oneself to using only one particular method for solving any kind of problem.

We are going to take a look at the method of Laplace transformation but depending on the problem, you may find the alternative, some of the alternative methods also good or even better and so one should be aware of number of methods of solving problems like this. So before, I get on to the method of Laplace transformation, I will spend some time talking about one more method, which again is not normally mentioned in most of the text books but which is also quite simple to understand and it can actually be used without much difficulty.

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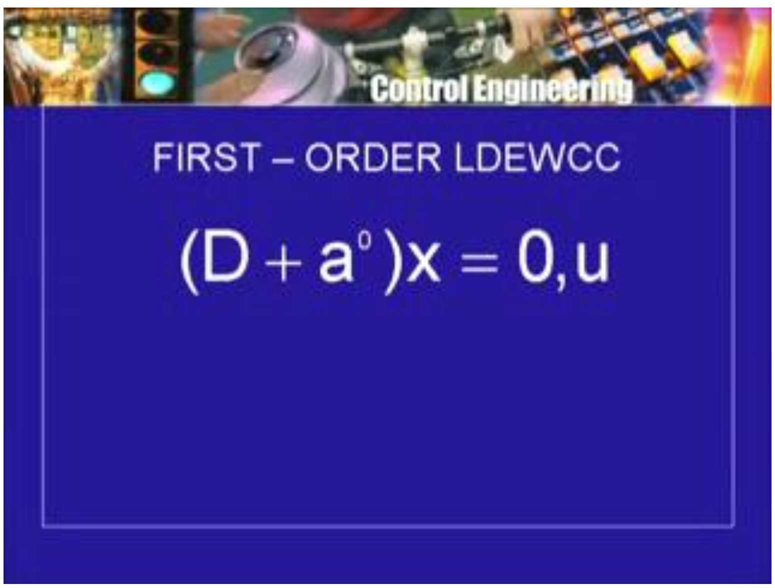

We saw that, in order to solve the first order differential equation whether homogeneous or not, was not very difficult. The homogeneous equation was of course very easy to solve because by, we knew the exponential function would work. The non-homogeneous equation that we solved with the help of the method of variation of parameter, is when the degree of the equation is more than one for example, the equation of degrees 2, 3 and so on that there are something more to be done and we saw that, if you have the equation which was of second degree then, we have to look at a polynomial associated with the polynomial differential operator and this particular equation was called an indicial equation and then, we have to solve the problem finding this m or in other words finding the roots or 0s of the polynomial and then, one has to use appropriate exponentials and so on.

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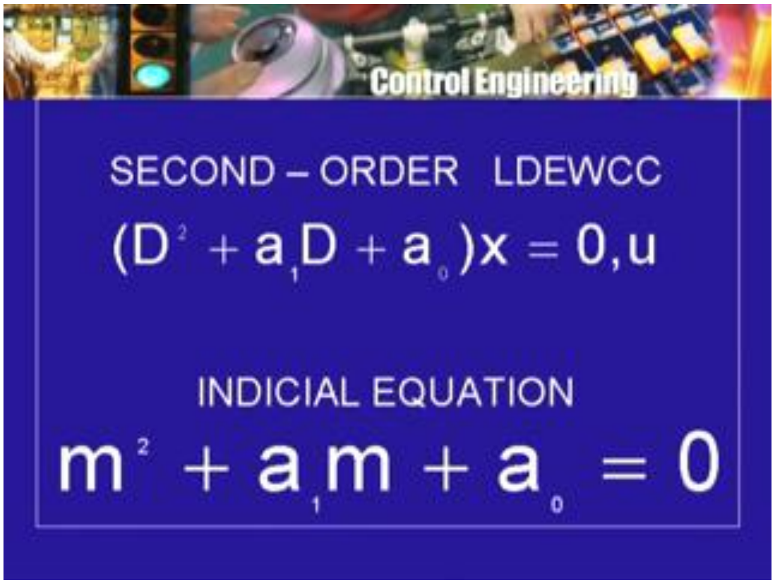

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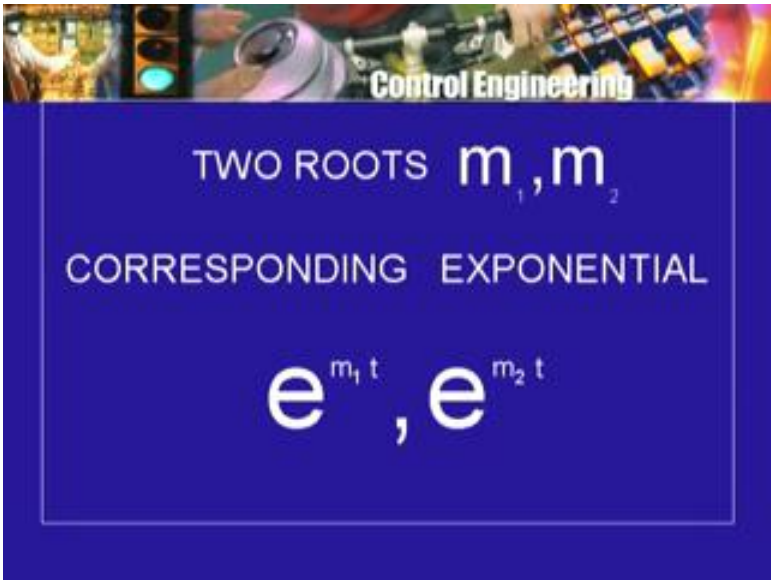

But we can do things a little differently using this ideal of a polynomial all right. But instead of finding the roots say m 1, m 2 etcetera depending on the degree and then saying that e raised to m 1 t, e raised to m 2 t and functions like this. We will build up our solution one can look at it in the following way. Instead of roots of the polynomial, one could talk about factorizing the polynomial, in fact that is how we really learn polynomials in school, most of us learnt first the idea of factorization of a polynomial then, the idea of a root or a 0 or a polynomial.

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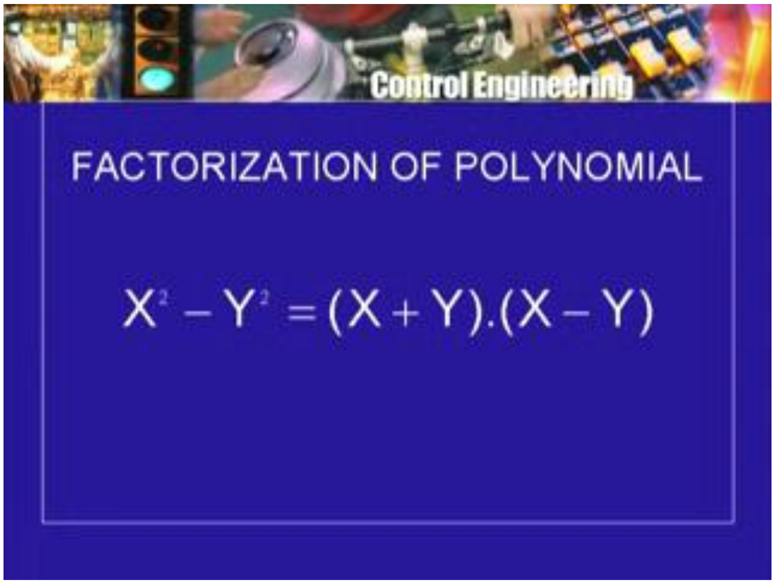

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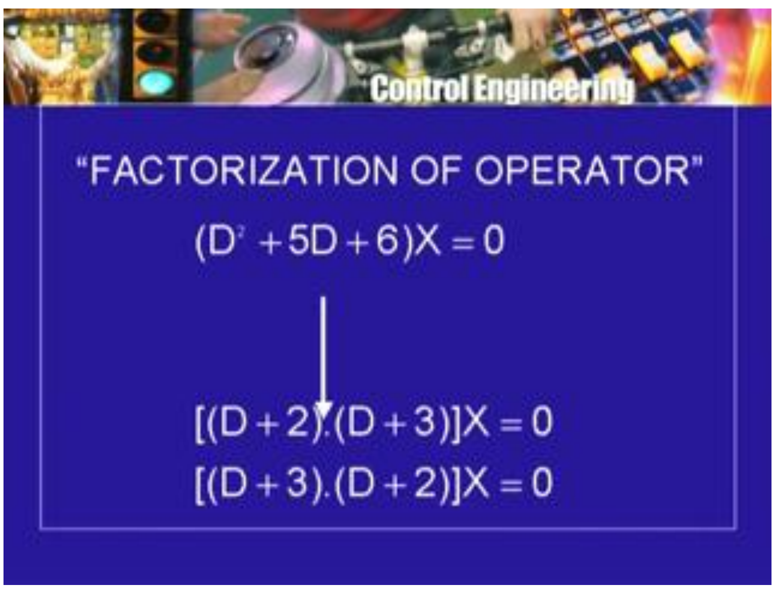

For example, all of us have learnt this particular factorization, X square minus Y square equal to X plus Y multiplied by X minus Y, looks very simple but many of us have had difficulty in understanding it and in remembering it in school and there are many, who just cannot remember it and therefore, I have to leave mathematics or algebra atleast. Now, this idea of factorization of a polynomial can be used as follows. Let me first take a numerical example that is an example where, there will be some particular numbers and I have written down this equation earlier as an example.

Suppose, I write D square plus 5 D plus 6 operating on X equal to 0, this is the homogeneous equation that I want to solve. Then, because this D square plus 5 D plus 6 is a polynomial of course it is in D and not in the familiar X, I can think a factorizing it. Factorization of the polynomial, of course requires finding the roots but the emphasis is not on the root as such the emphasis is on the factors. So, what is a factorization of this of course, I have chosen the numbers which are so nice that with a little bit of effort, you can find out the factorization without even finding the roots. For example, D plus 2 into D pus 3 is a factorization of this polynomial as you can check again D square plus 2 D plus 3 D.

So that is 5 D plus 2 times 3 that is 6, so this polynomial D square plus 5 D plus 6 operating on X, factorizes into D plus 2 into D plus 3, operating on X. So that equal to 0 that is the homogeneous equation that I have to solve. Of course, I could have written the factors in the other order D plus 3 into D plus 2 X equal to 0. Now, there is a difference between a factorization like X square minus Y square equals X plus Y into X minus Y, where X and Y are suppose to be representing numbers. Here, D square plus 5 D plus 6 here D is not going to be replaced by a number because D is an operator; D represents the operation of differentiation.

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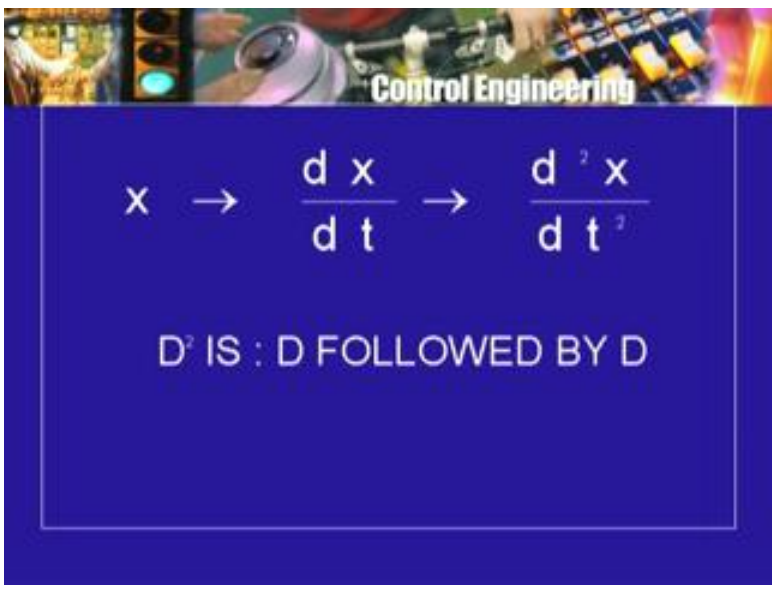

So, then what do we mean by factorization into D plus 2 into D plus 3 or what, what is meant by saying that D square plus 5 D plus 6 is D plus 2 into D plus 3. Now, what is meant is very similar to what is meant by D square, what the D square mean, did it meant that some number D was being squared, no D squared is not like 5 squared. A 5 squared is simply 5 into 5, so carry out the arithmetic and get the product. So, what was D square, why did I introduce D square because I had D square X by dt square, this was the traditional way of writing the second derivative but even there I see this D square and dt square.

So am I squaring D and am squaring t, no what this means is I am applying the differentiation operation twice in succession. I will start with X, find its derivative, find the derivative of that derivative, so it is D operator applied in succession or D followed by D. So that is what D square means then, what is the meaning. So D square means D, followed by D, apply D to it, whatever X apply D to the result that is what D square means, if that is the case then what is D plus 2 bracket followed by D plus 3 X, what this means is 2 X apply D plus 3 first, apply the operator D plus 3 first, which means what calculate the D X plus 3 X to result, you apply the operator D plus 2 and the whole final result should be equal to 0. This is the problem of the homogeneous equation.

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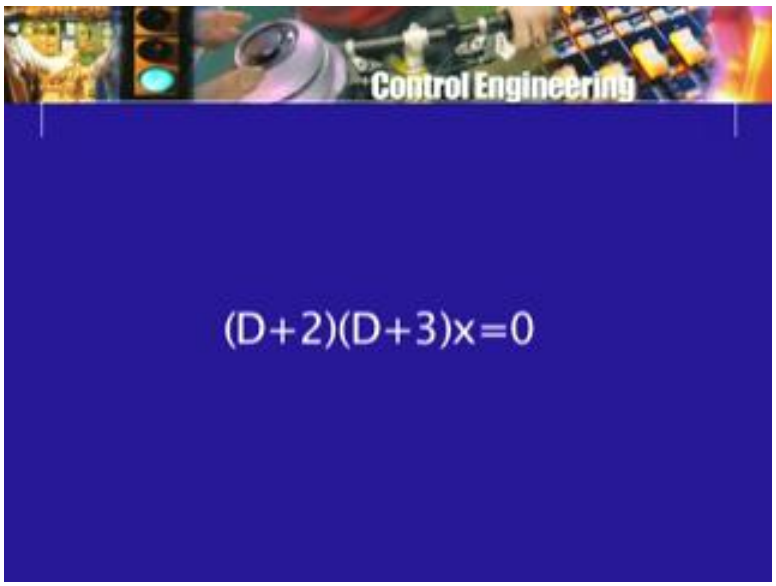

I am looking for a function X such that, when I apply the D plus 3 operator to it that is when I calculate D of X plus 3 X and then, to that result let us call that result y, I apply the operation D plus 2 therefore, I have find out Dy that is the derivative of y and add 2 to it y, the whole result should turn out to be 0. Find me such a function x that is the problem, now what has this factorization done.

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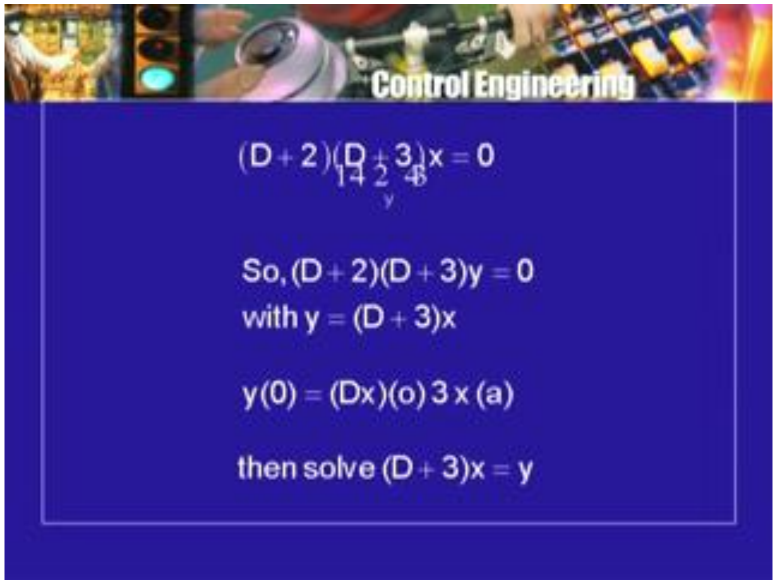

Now, if you had been attentive you can see that from this I get a method of solving the original equation D square plus 5 D plus 6 acting on x equals 0. Because as I written here D plus 3 acting on x, if I represent that by y, if I denote it by y then, what I have here is D plus 2 y equal to 0. Now, of course I did not know x, I wanted to find x but I do not know y also, so what is the difference? The difference is now, this y satisfies the first order homogeneous differential equation and we have already learnt how to solve it. We already know, how to solve a first order homogeneous equation, how do I solve it the solution involves the exponential and what else it involves the initial value, it involves initial value y at 0, but y is D plus 3 x.

So what is y at 0 because y is D plus 3 x, it is just another symbol or a name for a D plus 3 x, so y 0 is simply the derivative of x at 0 plus 3 times x 0. So, if I know x 0, if I know the derivative of x at 0 then, I know y 0, once I know y 0 I can solve the homogeneous equation D plus 2 y equal to 0 find out what y of D is but I have interested in x. So how to I get my x but what was the relationship between x and y, y was just another name for D plus 3 x. So, now I have the second equation D plus 3 x equal y, now this is not a homogeneous equation it is a non-homogeneous equation but it is a non-homogeneous equation of first order and we know already, how to solve it.

So to solve the second order homogeneous equation, what I end up doing is, I end of solving 2 first order equations, I solve one particular first order equation get its solution then, that solution becomes the forcing function or the input for the next differential equation to be solved of first order and that gives me the solution that I am looking for, the function that I am looking for. Now, what we can do for a second order differential equation, we can certainly do for a third order differential equation provided, we know how to factorize the polynomial into 3 first order factor or first degree factors.

Now that is what one has been talking about in school factorizing of polynomial and subsequently, when one talks about finding the roots of a polynomial then, that amounts to do the same thing that is once we know the roots of the polynomial, we know the factors and when we talk about repeated roots, what you really mean is repeated factors.

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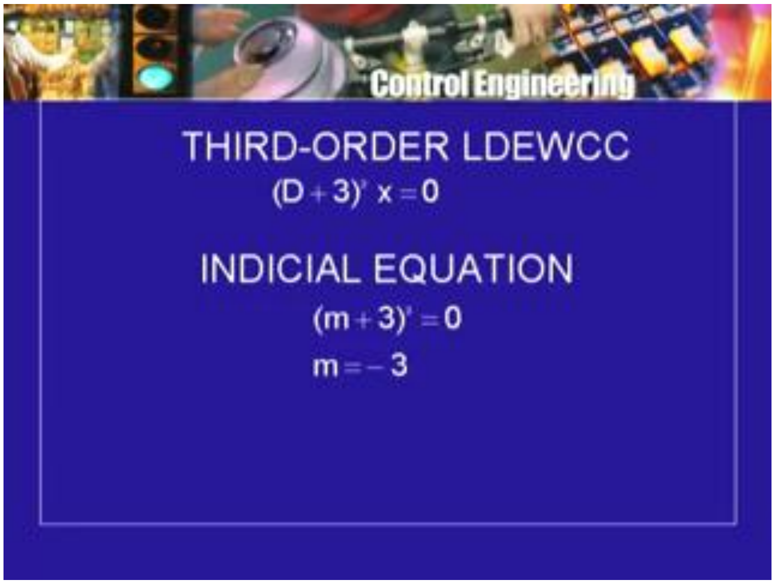

So for example, if I write down I wrote this earlier I had say D plus 3 cube equal to or acting on x equal to 0 that was the third order differential equation. For that the indicial equation was m plus 3 cube equal to 0 and for this no matter, how clever I am I can only get one root namely m equal to minus 3 but then, why do I talk about 3 roots, why do not I say that there is only one root, why do I say that there are 3 coincident roots or there is a root minus 3 rather of multiplicity 3 or there is a repeated root repeated 3 times, what really we mean is that in the polynomial the factor that corresponds the root is repeated 3 time, which it is m plus 3 is the factor but the polynomial is not m plus 3, the polynomial is m plus 3 multiplying m plus 3 multiplying m plus 3.

So the factor m plus 3 appears 3 times and therefore, we say that the root minus 3 is of multiplicity 3 or is a repeated root, repeated 3 times. So that talk about repeated root idea really refers to factors any way. Now of course, we just have to be careful here, if the factors are m plus 3 then, the root is minus 3. So this I the number that appears here appears as plus 3, the root is minus 3 because the root is what next is 0 and m plus 3 is 0 when, m is minus 3.

Now, in a hurry I have seen students making this mistake that they factorize all right. But when they are asked to find out the root because that will be needed later on in what is called root locus method. They will take the root for this polynomial m plus 3 cube as not minus 3 but 3. Now that is a kind of a mistake should not take place and one should be careful. But these are something which one has to be careful about, if you mean 2 you should not write minus 2 or if you mean minus 2, you should not write plus 2 as simple as that and so, we have a alternative method namely factorize the polynomial differential operator into linear factors, first degree factors and then, solve first order differential equations successively.

This is the method which is quite simple and it can actually be used for solving lower order differential equations by hand that is suppose, you are asked to solve forth order differential then, as I told you, now no matter what you do whether you use Laplace transforms or you use the earlier method or use this method, you have to find out the roots of a polynomial. Once, you have the roots of a polynomial, factorizing the polynomial is immediate and so, you can use that method. In fact, we will see that when we use the Laplace transformation method we end up factorizing polynomials also.

So, it is not that factorizations of polynomials is a very low grade method or anything like that it is involved in the Laplace transformation method of solution also. It is only little bit of a problem here and that is again a problem which we have come across earlier. When we looked at roots of a polynomial I said that there are 3 or 4 cases and in one of those cases the roots happen to be complex or as a 4th case purely imaginary. So much as you may not like them you end up using complex numbers.

Now, again in electrical engineering if you want to study any part of it, the use of complex numbers today is almost unavoidable and so, that is a concept which is very useful and therefore, one already becomes familiar with it in mathematics. When you learnt your mathematics, you would learnt what may be call theory of functions of a complex variable or complex number and their properties and algebra and so on and so forth. You may not have realized where is all of this going to be used.

Well, it is definitely very much going to be used in electrical engineering perhaps in your circuits network theory machines course, you have already made use of complex numbers. The complex numbers not only complex numbers but their geometrical representation as what are called phasors or even vectors and sometimes they are called rotating vectors to distinguish them, from quantities like force or velocity, which are true physical vectors. So complex numbers we cannot share a waveform and therefore, roots of a polynomial may be complex, well that is fine and of course, thanks to a formula which goes back to Euler, one can go from complex to real and real to complex without too much of problem and this is something again which one has to learn in electrical engineering and today, students are lucky because your calculator has this provision.

So you do not have to really think much about it, what is, it that I am referring to problems like this. A complex number is given in this form, let us say 2 plus j 3 or in mathematics it will be probably written as 2 plus 3 I, change this into the so called polar form that is some positive number and some angle or going back to Euler then, this 2 plus j 3 would be represented by some coefficient A which is a positive number multiplied by e raised to j theta, where this A is the modulus of the complex number and theta is the argument of the complex number and this is called a rectangular to polar conversion. From the rectangular form 2 plus j 3 obtain the polar form and conversely, from the polar form go to the rectangular form.

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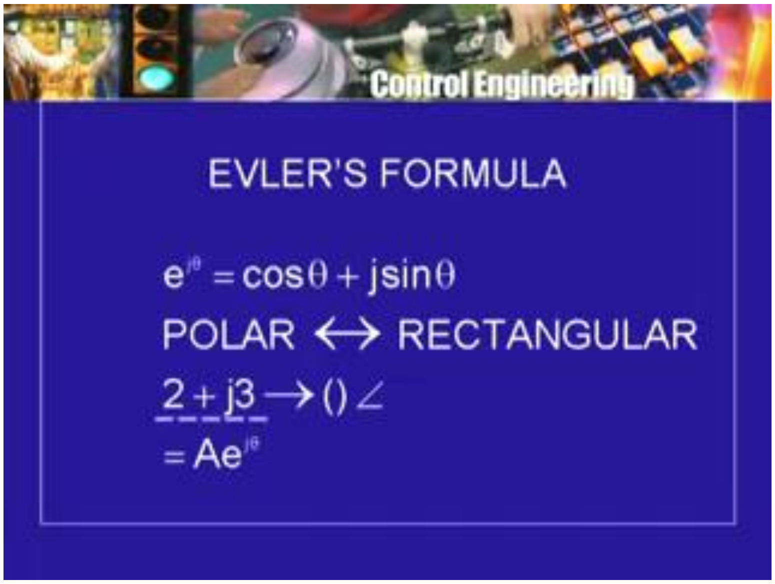

Now this is the function which is provided on most modern calculators, pocket calculators. In my time, we had to do it using slide rule and doing squares in additions square root and what not and looking up sin or sin inverse on the sin on the slide rule or using trigonometric tables. Today, it is all done for you but you have to know how to do it. So make sure you are familiar with this conversion from a rectangular to polar and polar to rectangular form of complex numbers.

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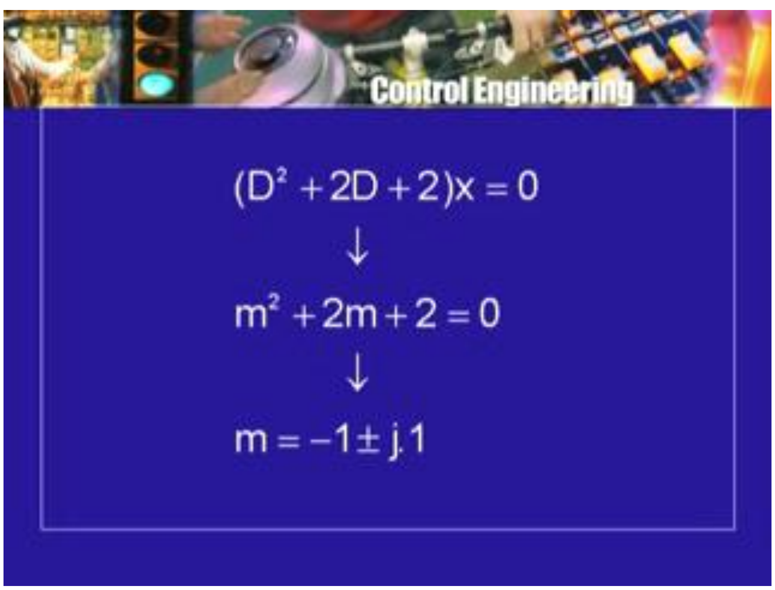

So if we are not afraid of complex numbers then, we can look at factors linear factors which have complex numbers in them. Let us take the example, which I had taken earlier though I did not write down the polynomial. Let us say I have D square plus 2 D plus 2 is the operator. So our differential equation is let us say, D square plus 2 D plus 2 acting on X equals the function 0, find X. This leads to the indicial equation m square plus 2 m plus 2 equal to 0 and the roots of this turn out to be complex, they are minus 1 plus or minus j into 1 or minus 1 plus minus j and if you remember, I had shown them in the complex plane as a pair of crosses like this and I wrote here minus 1 plus j 1 and minus 1 minus j 1. The horizontal axis usually is called the real axis, the vertical axis is called the imaginary axis.

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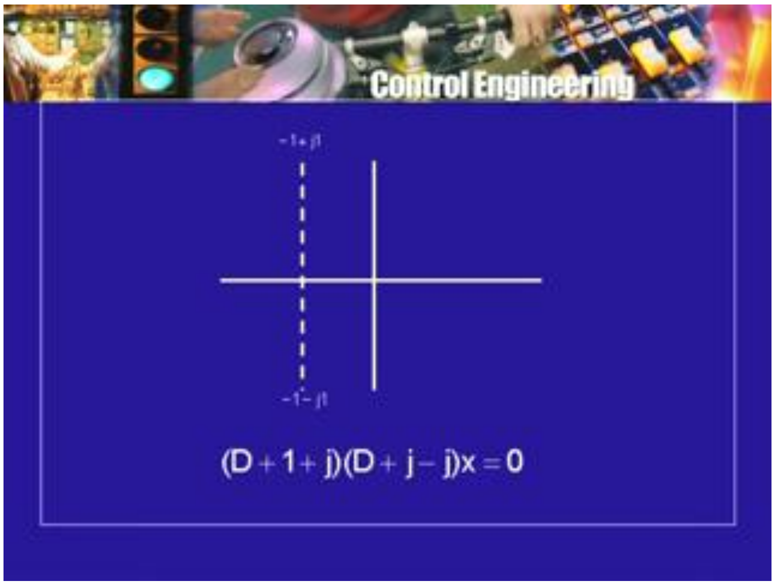

So this is a plot which shows the root of this complex of this polynomial of degree2, but the roots are complex. Now, if the roots are complex when I factorize it, the factors will also will have complex numbers in them and so, what will be the factorization, the factorization will be D plus 1 plus j and D plus 1 minus j and we can check that this these 2, when multiplied out do give you D square plus 2 D plus 2. So, now I have complex numbers inside the factor.

So, corresponding to this factor then the root is minus1 minus j and corresponding to this factors the root is minus1 plus j. But I need not be afraid of it in place of a real number that I had earlier say D plus 2 into D plus 3, I have now complex number. So this X equal to 0, I have to solve it fortunately as electrical engineers we not only live with complex numbers but we live with something more complex than that we would look expressions like e raised to j theta and of course, everybody knows what is e raised to j theta, e raised to j theta is cos theta plus j sin theta and this relationship is usually ascribed to Euler but it goes back to 230 years nearly.

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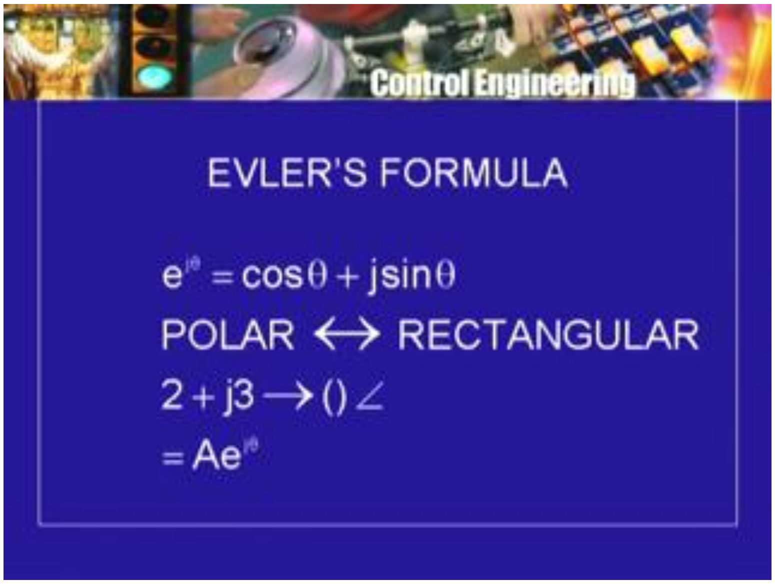

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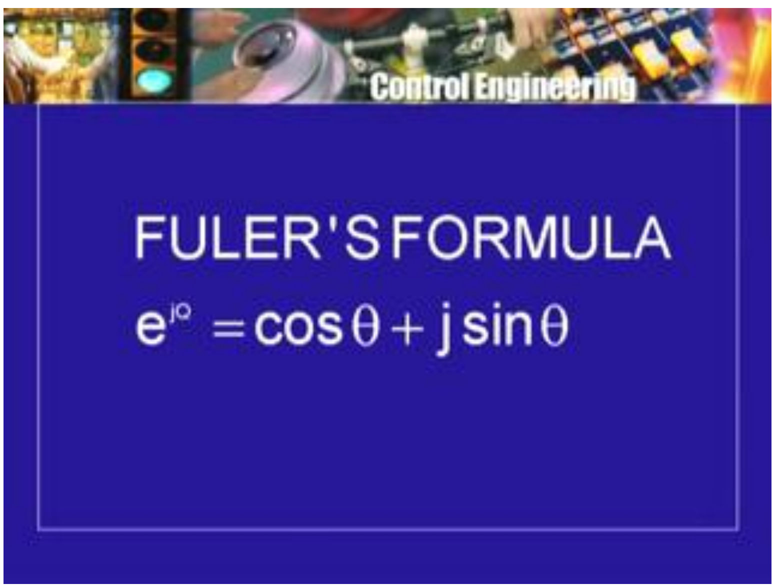

Of course, Euler was a great mathematician, we are not but we need not be afraid of using this relationship. So using this relationship then, when I write e raised to m t, if m is not real but it is complex then, what does it mean. Well, suppose m I replace by what I have here say minus 1 plus j. So it is e raised to m is minus 1 plus j multiplied by t, now this can be written as e raised to separating this into minus t plus j t. Now, again therefore this is e raised to minus t multiplied by e raised to j t by the law of a index or exponents and therefore, this is equal to e raised to minus t.

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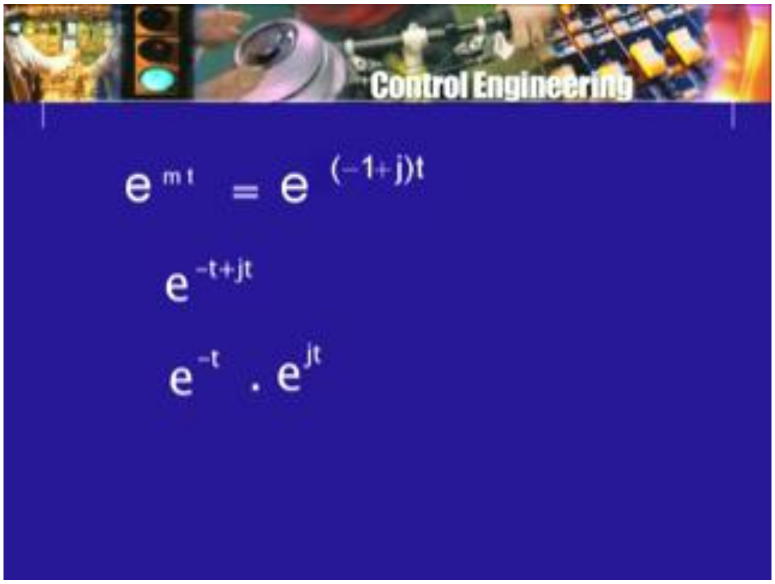

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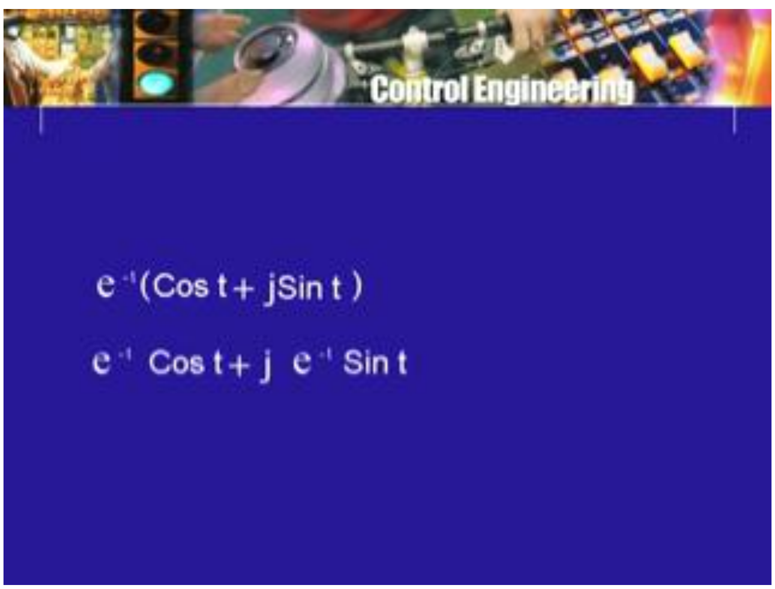

Now, I go back and expand e raised to j t as cos sin t plus j sin t, so this whole thing then is e raised to minus t cos t plus j raised to minus t sin t. So this exponential e raised to m t, when m is complex turns out to be not real but complex. It has a real part and it has an imaginary part then, fortunately what happens is that when you solve the equation, you just do not have this particular m only because I had 2 factors and is one of the factor I had 1 plus j and the other factor had 1 minus j.

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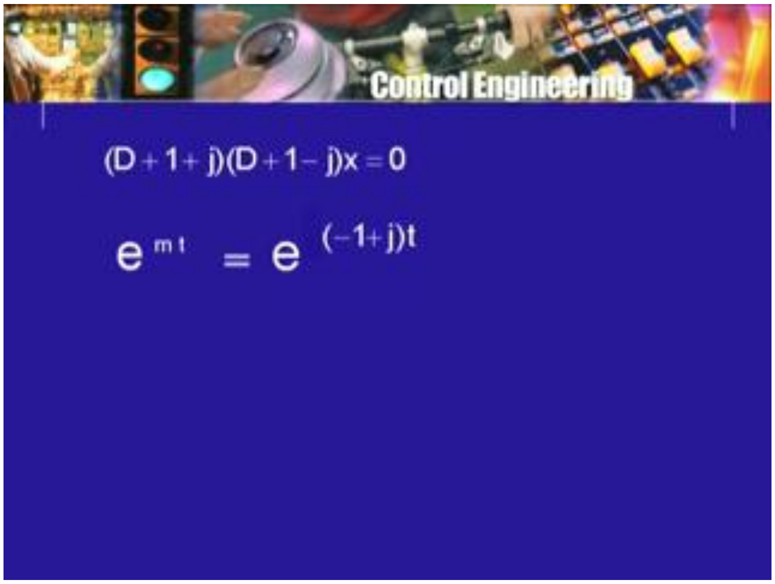

So when you actually solve this equation carry on the computation, you will find that in the end, the answer will be real, there will not be any j remaining in your answer. In fact, if it remains then you a made a mistake somewhere. So although, we will work with complex numbers and complex functions at the end of it, when you get to the final solution. We will find that the solution has only real terms and the terms are of this kind e raised to minus t cos t, e raised to minus t sin t. Now, one should know the graph of these functions, electrical engineers should know what the graph of this these 2 functions look like and what are they called because there is this e raised to minus t.

So it goes to 0 as t increases goes to 0 meaning limit as t increases is 0 within 5 time constants, it will be something like 1 percent or .01 that is e raised to minus t is the exponential decay. But that does not sitting by itself that is multiplying cos sin t, if I has only cos sin t or if I had only sin t then, what is it that I have, I have the simple sustained sinusoidal oscillation and frequency the angular frequency for it will be simply omega equal to 1 because it is cos t not cos omega t, omega equal to 1 and therefore, a corresponding frequency 2 phi f equal to 1, so f equal to1 by 2 phi hertz. But I do not have that I have that multiplied by e raised to minus t.

So this is said to be a sinusoidal function either cosine or sin, a common name for it is sinusoidal function which is exponentially damped. So it looks now, I would like you to draw the graph of these 2 functions e raised to minus t cos t and e raised to minute t sin t carefully by using your graph paper, by choosing the appropriate scale on the time axis
using the appropriate scale on the value axis where you are going to plot the value of this function.

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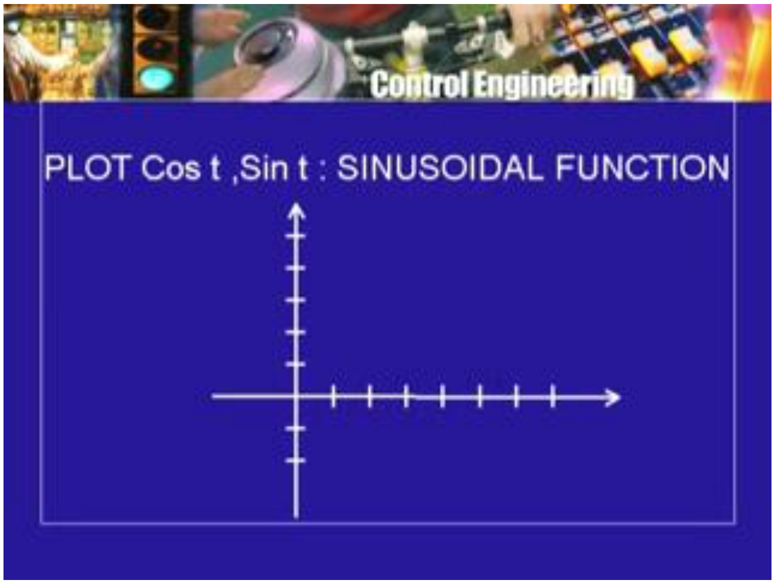

For example, put t equal to 1 use your calculator, we have to find out what e raised to minus t cosine t is, maybe you should choose t equal to .5 also. So choose a number of values for t calculate this expression e raised to minus t cosine t, get enough number of points and then join them by a smooth curve and see what it looks like, what it will look like, I will show you the drawn figure. In fact, most of your books text books on control theory and on circuit theory will have plots of these functions, it is not enough to just see it in the book, you should be able to plot it yourself.

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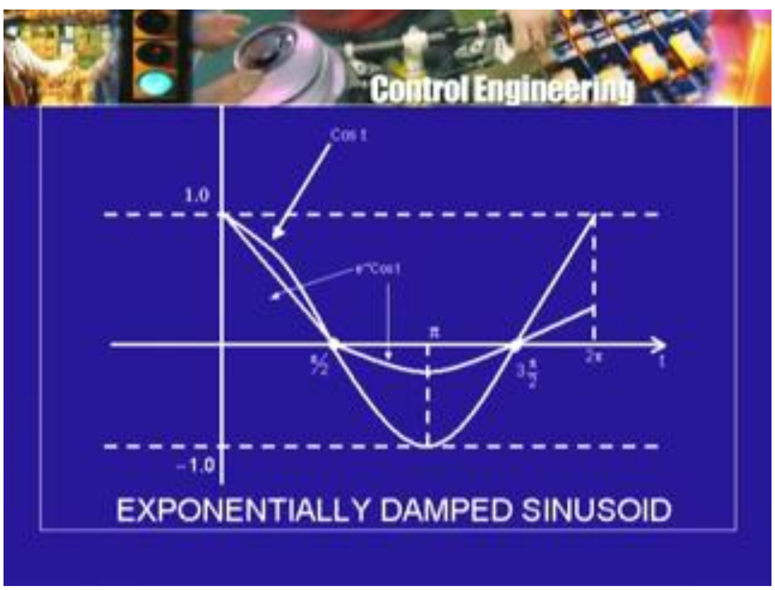

So, let me do it but suppose I want to plot cosine t then, how do I start first I put 2 lines 1 here 1 .0 minus 1 .0 because I know that cosine t lies between 1.0and minus 1.0. It takes both values of course, it takes the value 1 at t equal to 0, t 2 phi t equal to 4 phi and so on. It takes the value minus 1 at t equal to phi t equal to 3 phi and so on. So on the horizontal axis I will put here phi, 2 phi, in fact I will put phi by 2 for a reason you will see. So this is phi this 3 phi by 2 and this is 2 phi. So, if I do if I plotted only the cosine curve what would it look like it would be 1 here, it will be 0 here, it will minus 1 here, it will 0 here and again, it will be 1 here.

So this graph will look something like this as engineers, we have to be good at drawing we need not be very good, we are not going to be artists but we have to be reasonably good at drawing. As I told you my exponential graph should not look like what, it should not look like, should not have corners, it should not behave in some arbitrary way, no that is not acceptable. Similarly, cosine graph should not be just drawn carelessly. I am not saying that my graph is exact here but at least some salient points are there, it is 1 at 0, 0 at phi by 2, minus 1 at phi, 0 at 3 phi by 2 and back again to 1 at 2 phi.

So whatever done, I have put 2 line minus 1 and plus 1 and I have made them reasonably equal distance from the x axis or t axis. On the t axis I have put down 4 points, phi by 2 phi, 3 phi by 2, 2 phi, again I mean little careful and not been absolutely careless. The distances are not exactly equal I have not even measured them, I do not have a graph paper but I have not been careless either and then, I am drawing a smooth and smooth meaning, when one knows that here, for example it short of rounded of like this. Similarly, it begins with a slope which is 0 it ends with slope which is 0. These are some other things which as electrical engineers and as control engineers you must become not only familiar but also reasonably skilled at.

Now this is only the plot of cos sin t, so this is cos t but this is going to be multiplied by e raised to minus t. So what is going to happen at t equal to 0, e raised to minus t is 1. So this is just what the product is but by the time t becomes phi by 2, I have e raised to minus phi by 2. Now e raised to minus phi by 2 is already less than e raised to minus 1 which is about .37 but of course, this is going to be 0, so there is no problem. But at the next point where, the function takes its maximum absolute value minus 1, cos t is minus 1 but e to the minus t now is e to the minus phi that is more than its exponent is larger than minus 3.

So it is less than e raised to minus 3, e raised to minus may be of order of say 1 by 20 so the value of the function will not be minus 1 but it will be 120th of it. By the time, I go to 2 phi cos t is again 1 but e raised to minus t is now e raised to minus 2 phi. So it is even less, so it is going to go through 0 here. It will start decreasing then here, it is already much less, it will go through 0 again here it is much less and so on. So, if I want to show more that1 period of the cosine function and now, because I do not have the time and I do not want to spend the time actually drawing it. It will look something like this, it will have oscillations, the oscillations are because of the cosine but the values will go on decreasing because of e raised to minus t the damping or the decay.

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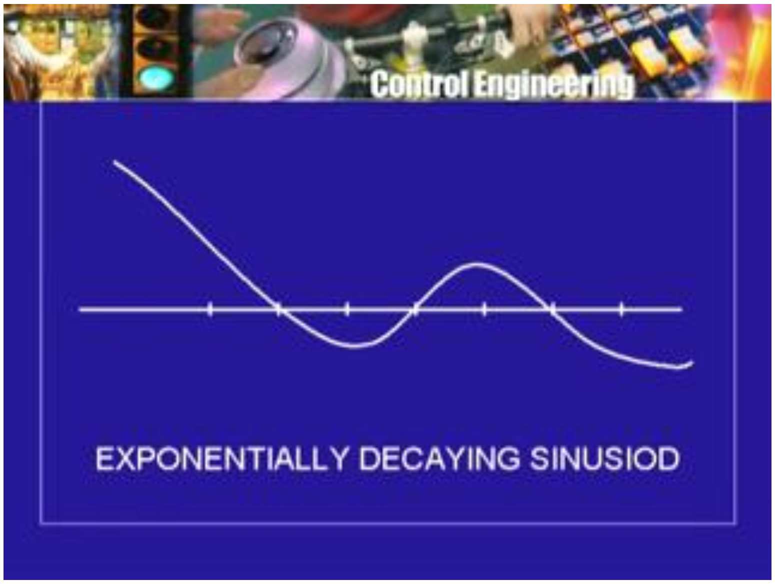

So it will look as a very quick graph something like this, exponentially decaying sinusoidal function that is the name given to it and in control theory, this is a function you come across, you cannot avoid altogether . We not only have the exponential function, we not only have the trigonometric functions sin and cosine, you also have exponentially decaying sinusoidal function not only that we came across something else earlier we had a function like t, e raised to m t, in particular we could have t, e raised to minus t.

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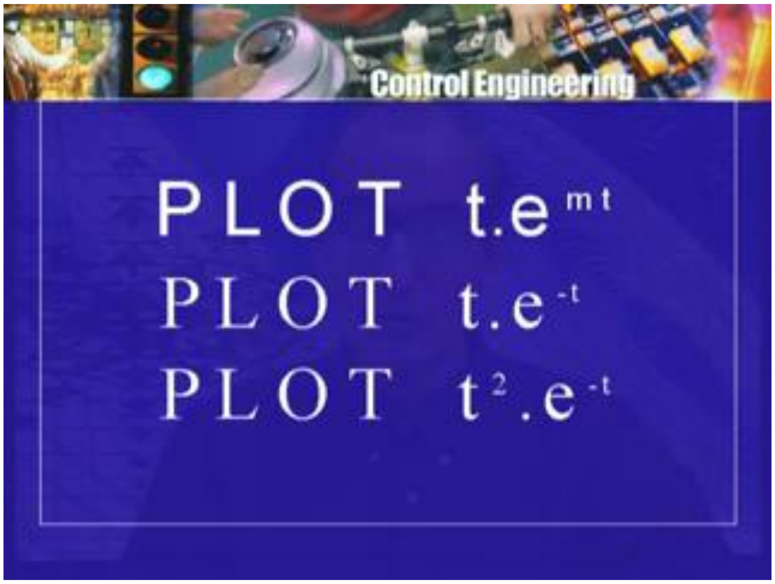

If it was t then, it is something which is going to grow without limit, if it was e raised to minus t it is going to decay to 0 as tends to infinity, what happens to the product, t into e raised to minus

t. So this is another homework, plot the graph of t into e raised to minus t by choosing a number of different values of t, plot it on a graph paper using appropriate scales for the horizontal and for the vertical axis and check for yourself, what this plot is going to look like. It is quite different from this exponentially decaying sinusoidal function because e raised to minus t has oscillations, t has no oscillations, yet it shows something, it shows 1 oscillation at least because at t equal to 0, it is 0, although e to the minus t is 1. So this is an exercise which you must carry out because again this function and sometimes t square into e raised to minus t also may be needed, one should know atleast qualitatively what this behaves like, okay.

So solving higher order LDEWCC with initial conditions is not that difficult for us. We have found out one more method of solving such a differential equation namely the method of factorization of the polynomial differential operation. You can go back to your mathematics course or a mathematic text book which asks you to solve some specific simple looking linear differential equation, equations may be of second degree or third degree with some simple forcing functions like sin t or cosine t or e to the minus 3 t on the right hand side or just unit step that means, 1 on the right hand side and solve them using the 2 methods that I have mentioned so far.

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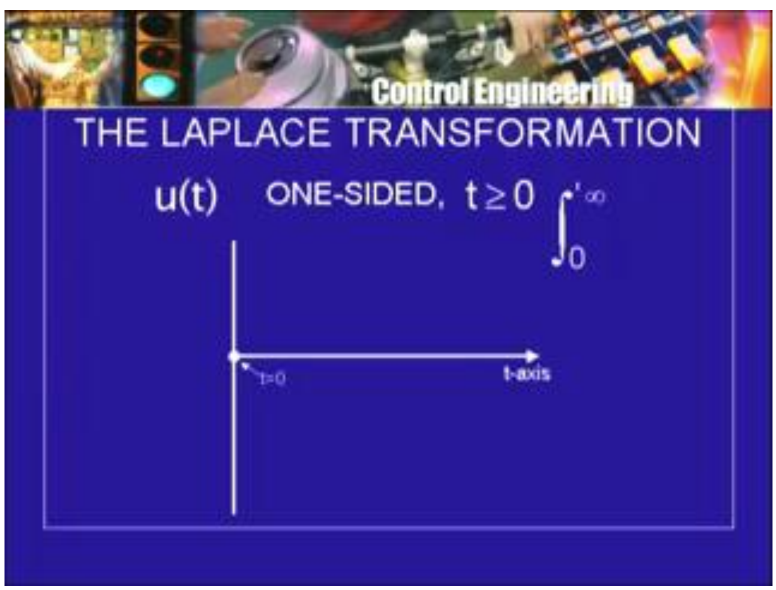

The method which uses the complementary function or the general solution of the homogeneous equations and then from that also getting was is called that impulse response, calculates the 0 state solution, of the solution that depends on the input and then, add up the 2 that is one method. The second was factorization method and now, we are ready to start taking a look at the third method namely, the method of Laplace transformation.

Once again, this is a method which every control engineering person whether from electrical engineering or instrumentation engineering or mechanical or chemical. Today, it is almost imperative, by the by if you have of course a lot program packages, a lot of software which is available which will do a lot of work which earlier we had to by hand or by using the pocket calculator or the slide rule even or even a computer but writing down equations substituting and then, adding subtracting and so on getting the final solution then plotting it.

Today there are program packages which will do all this for you or you have to do is really to enter the data for the problem. In some way, I have to put down the differential equation problem with initial conditions and input and then my package will take over or the software will take over and it will usually give me a nice graph on the oscilloscope or on the monitor and I can take print outs, so I get plots, all that is okay. But that does not mean that I do not know anything as to what the sinusoidal function is, a what is exponential damping or what is exponentially damp sinusoidal function or what is the differential equation or now, what is the Laplace transformation. Programming packages do not mean that you do not have to know anything, it only means that what may take a long time, if you were to do it all by your own.

For example, I can multiply 210 digit numbers without using a calculator but it will take me a long time but if I have a readymade program available then essentially I have to enter the data and then, today the multiplication will be done in perhaps one microsecond or may be one millisecond faster than I can write down the number but does not mean that, we do not learn anything about these things like differential equations and Laplace transformations, what have you because all this, many ideas, designed ideas etcetera, make use of these ideas such as the Laplace transform and so on.

In fact the idea of a transfer function is essentially based on the concept or on the use of the Laplace transformation. So, you have to study the Laplace transformation, its not something very difficult perhaps we already studied in your mathematics courses and may be in your circuit or network theory course, in control theory or control engineering course certainly, one has to know the Laplace transformation and how to use it. So what is this Laplace transformation, how is it to be used, why it is that it can be used for a certain problem and why not for each and every problem.

Now Laplace of course, was a mathematician you can also call him a physicist because you are solving some problems in physics. We have he was interested in astronomy, he was interested in the propagation of heat, you come across his name in many places Laplace’s equation, Laplacian. One more thing that you contributed was not quite the Laplace transformation but what is or what was earlier called the Laplace integral and his motivation was not solving differential equation as such. Although, he was interested in that problem, in fact he was interested in the problem of motion of the planets not just the sun and the earth but when you have 3 bodies the sun, the earth and may be Mars which affects the movement of earth and conversely, the position of the earth, of the movement of the earth affects in movement of Mars. We have a 3 body problem, we can write down differential equations and then, we have to solve them. He was also investigating many other areas like, he was he was on the idea of probability theory and other related subjects.

So he introduced this idea of using some kind of an integral to simplify certain problems that he was trying to solve. Subsequently, in the 20th century which is over now, we are in the 21 century, when people were looking at differential equations in mechanics, they found that this Laplace integral was a useful tool and so in the19, first decade of the 19th, the 20th century 1920 and so on. This method was developed and almost perfected and therefore then, it became known as the Laplace transformation or the Laplace transform method.

Now, what is this method? Now, as I said this method involves Laplace's idea of using an integral. So obviously then, you have know what an integral is but for an electrical engineer that is nothing to be afraid of, we already used integral. But this integral is a little different suppose, we were given some function. Let us say, it is our input function u of t and as I told you as control engineers, our action normally starts at some moment of time which we call the initial time and which we therefore, label usually as 0 and we are interested in what happens there after.

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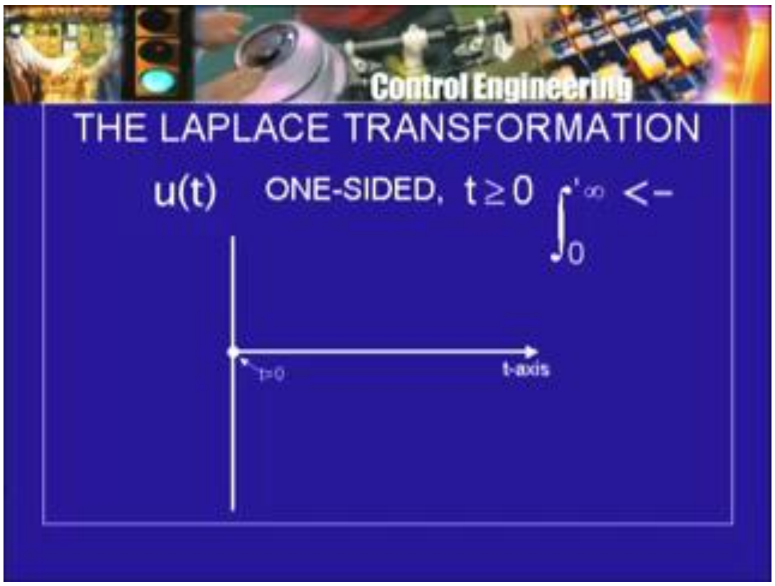

So, we are interested in t greater than or equal to 0 so suppose, we have some functions specified u of t, for t greater than equal to 0, for example u t may be sin t, cosine t, e raised to minus t, it may be t square, it may be 1 a constant function, what not. So, I have a function u which is specified for values of t greater than equal to 0 and greater than equal to 0 literally means, for all values greater than 0. So for t equal to not only one, if t is seconds,1 second, 10 second but may be 1 hour, 10 hours, 1 year, 10 year, million years, for all t and this of course one represents diagrammatically as a t axis.

I am trying to show here, that I start t equal to 0 and I am interested in what happens there after. You do not ask me what happens earlier, I am not interested in that I do not know anything about it and I will not have to know anything about, what is this the state at t equal to 0, tell me what it is and then, I may be able to find out what is going to happen there after. So, we need to know a function for t greater than equal to 0 therefore, the integral that one in talking about is not now just an integral, a definite integral with limits which are numbers like say 0, 2, 3 and so on. The lower limit is 0 but the upper limit on the right the symbol infinity and so, this is what is called an infinite integral and therefore, it is something which has to be handled carefully, it is not integral 0 to 1, u t dt.

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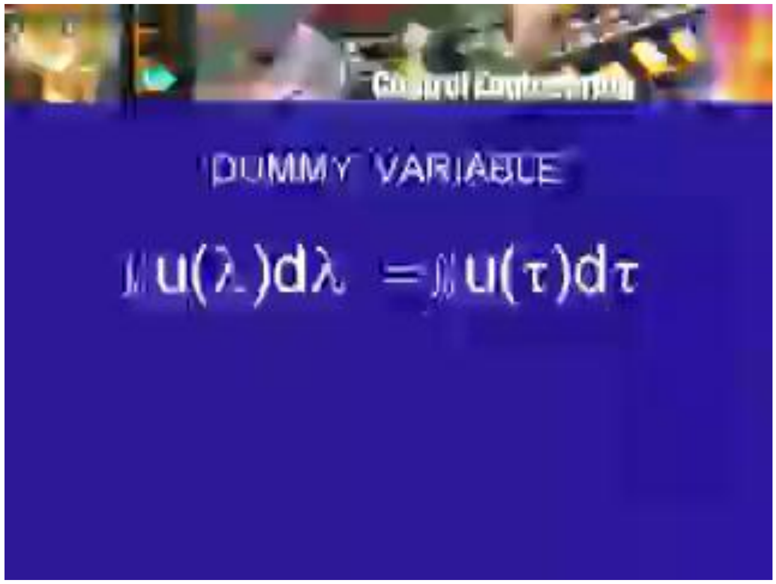

Now, since I mention the idea of a dummy variable this definite integral is exactly the same number as integral 0 to1 u tau, d tau and if, a some third wants to introduce the symbol lambda, let him to do so integral 0 to1, u lambda, d lambda. All of these are one and the same number and depending on what the function u is I can calculated and you all know how to calculate. So the lower limit is 0 the upper limit is 1 here, here the lower limit is 0, the upper limit is the symbol infinity which means I have to integrate function or find the integral of the function not from 0 to some other time but over the entire t axis going to the right of t equal to 0 that is for all future time.

Now, this of course is an idealization because who knows anything about the future beyond may be even a day or a year. But in mathematics we know, for example examination know the function sin t and if, t equal to 0, sin 0 is 0, if t is equal to 10 raised to 20 then, I can find out what sin of 10 raised to 20, there is no bother. I am not saying this is 1 second and that is 10 raised to 20 seconds because by the time we reach 10 raised to 20 seconds may be there will be a another big bang or big collapse or what not that is different although I call it time really, what I mean here is a number t.

So, there is no harm in integrating over the entire interval as it is said from 0 all the way to infinity but infinity is not a particular point on the t axis, it is not 0 to 10, 0 to 1 million or 0 to 1 billion that is not infinity. So that is why, we have to be careful with this symbol infinity and there is one more complication, I take the integral from 0 to infinity but not of u t but of u t multiplied by a factor.

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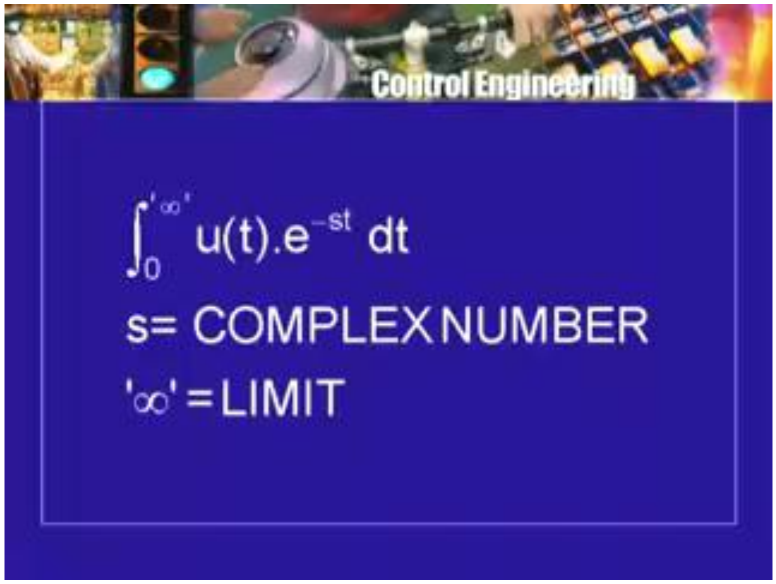

Now this factor is something where many of you probably, I have already made mistakes when you solve problems involving the Laplace transform and those who are learning it, sometimes forget, what this factor is and when I asked students in examination, they are unable to remember it. Unfortunately, is one of the things that you have to remember this factor here is usually written as e. So the exponential function appears here and the power of the exponent is written with a minus sign minus and there is one more symbol that makes the appear namely appearance namely s into t dt. So it is not the function but the it is the function multiplied by an exponential function, whose exponent is minus s. The product is integrated over the entire interval from 0 to infinity that is over the entire right hand or positive time axis, all right in this e is familiar but what is this s, what makes it a little difficult in the beginning is the fact that this s, we are told is not simply perhaps a real number but it could be a complex number.

So it is not necessarily something like say 3 or minus 2 or what have you but it may be minus 1 plus j or it may be 2 plus j 3. So this s then, will stand for a complex number but we know enough integration and Euler's formula is known. So I should be able to calculate this integral, if I know what s is and so here is one more homework problem for you, find out the integral 0 to infinity of, I will choose ut equal to sin t and I will give you the particular value of s. So that you do not have to keep the symbol but you can actually find out.

So let us choose s as what we had earlier say minus 1 plus j t dt. So calculate the integral of sin t multiplied by e raised to minus of minus 1 plus j into t. This product integrated from0 to infinity. Calculate this number, simply involves integration the exponential with that j t, we know how to replace it by cosine and sin using Euler's formula. You will find that you will get an answer which will be not a real number but which will be a complex number and what will be complex number. Well , since of course I know what it is but of course I should also be careful, I should not make any mistakes, I know what the answer is going to be and I am just going to through the steps of the answer but after we have looked at this for a while, you will see what the answer is.

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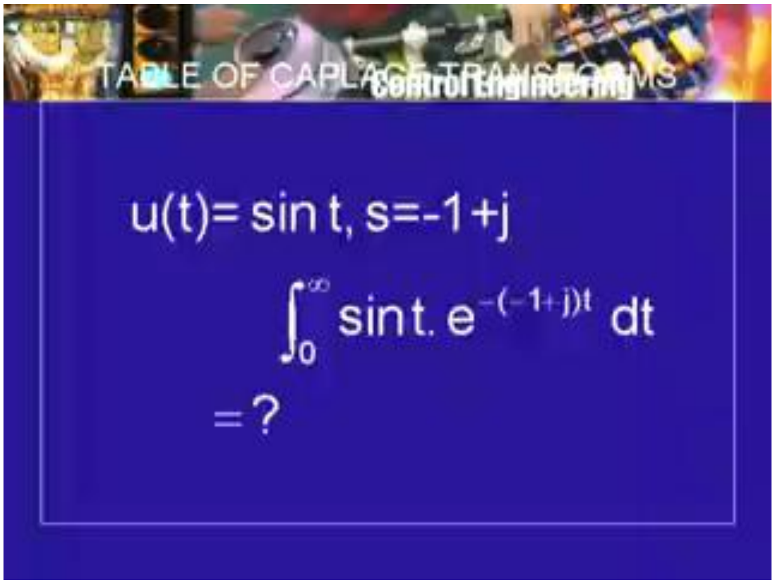

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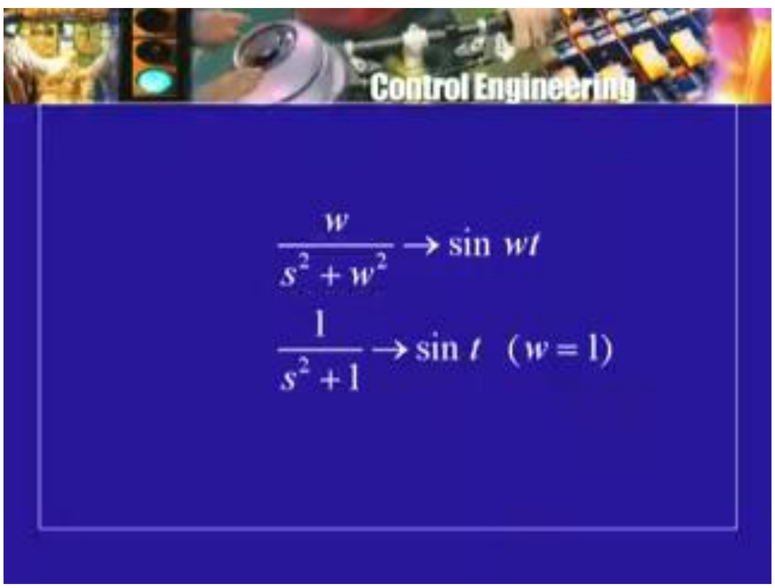

So I see sin t, so then something I remember which looks like omega over s square plus omega square. In fact, we will see that this will be what is called the Laplace transform of sin omega t, the Laplace transform of sin omega t is omega divided by s square plus omega square. One more thing that an electrical engineer and certainly, a control engineer has to remember, not only Laplace transform, what it is but also Laplace transforms of some simple functions like, the sin function here sin omega t, Laplace transform is omega divided by s squared plus omega squared. But, what I have written now is not sin omega t but I have sin t which means I have considering a situation where, omega is 1.

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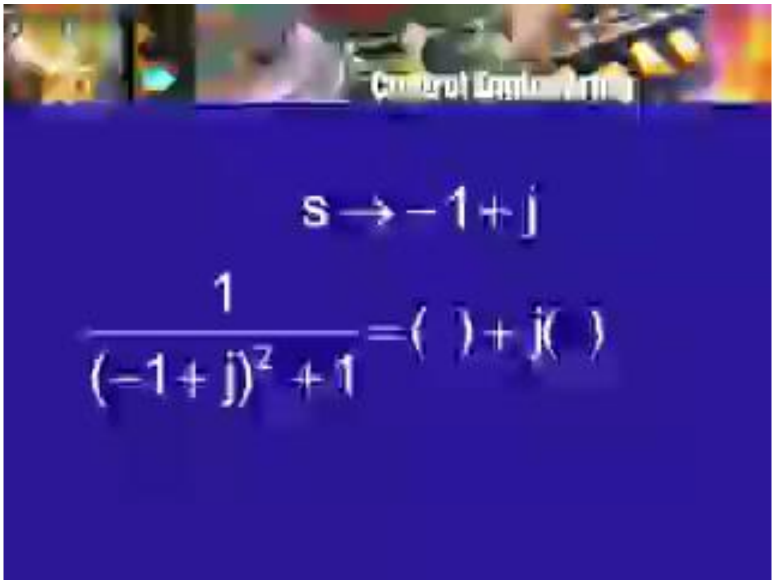

So put omega equal to 1, so I will get 1 divided by s square plus 1 squared but now what I have in the integral is not minus s t but minus of minus 1 plus j. So in other words now s is to be replaced by minus 1 plus j and so, I will get 1 divided by minus 1 plus j square plus 1. Now, of course this some complex arithmetic that I can do square then, put it in the rectangular form convert to the polar form or do rationalization or whatever and I will get it as some real number plus j times some other real number that is I can get it finally, in the rectangular form. Check that, your integral value agrees with mine that 1 over minus 1 plus j square plus 1. Of course, you have to reduce this to the rectangular form and just check that, it tallies okay.

So, we have to look at such an integral but not just for one particular number s. We have to think of all kinds of all complex numbers not just 1. In fact, you normally do not evaluate this integral at all although, what I have done here omega by s square plus omega square replace omega by 1 and replace s by something, this is something which we will have to do, as we will see. But I do not really have to calculate the integral, why because somebody has done that for me already, you already have a readymade tables of Laplace transform and your control engineering book as well as your mathematics book which talks about Laplace transformation, will have such a table, it may have a small table, which may be some 5 or 10 entries but there are more, exhaustive tables, tables running over may be 30 pages or even more, there are all kinds of functions are listed with their transform, people has spent that effort to save you the trouble but still you have to know about what it is.

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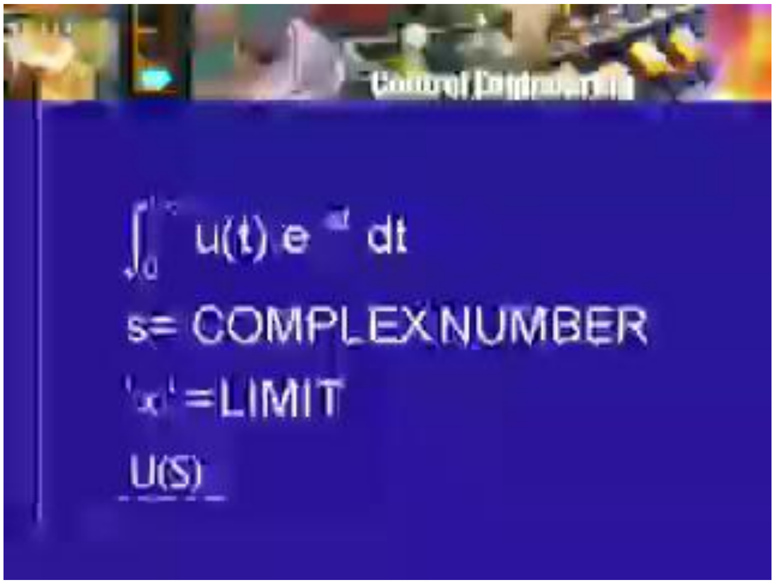

So this integral then, you can think of, the integral which is the complex number for a particular s there is a particular result. Now, this result is then denoted by a symbol which following our convention in control, it is capital U of s, why capital U because I have a small u, which means the function of time. Now this is something else that I am getting and why U of s, why do I write it and talk of it as U of s because for each number s, I will get one value of the integral and I change s for s equals minus1 plus j, I will get one value, I can then change s to say 2 plus j 3 perhaps, I will get another one value.